Products of compact lines

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joint work with Grzegorz Plebanek (University of Wrocław)

Workshop on Banach spaces and Banach lattices

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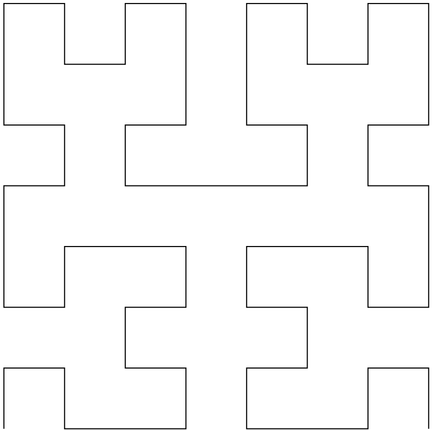
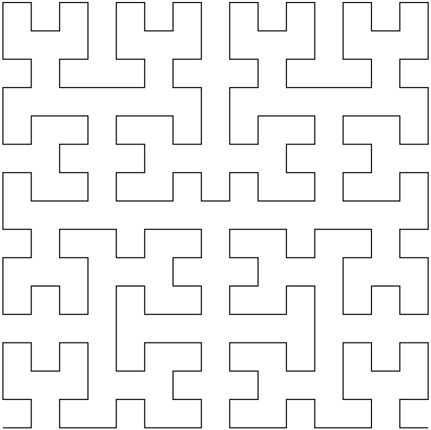
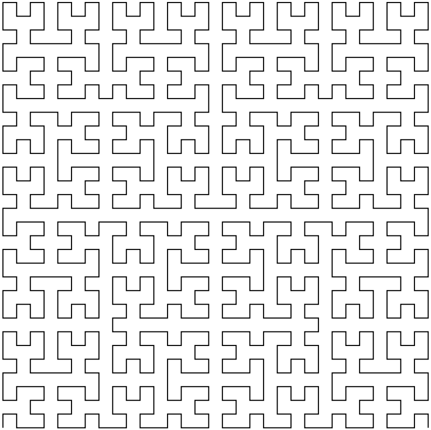
Yes

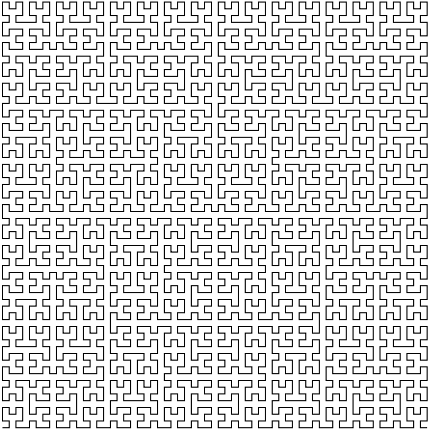
Yes. It's true.

There are curves which fill the plane.









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Moreover, the unit interval can be mapped onto the cube $[0, 1]^3$, onto the **tesseract** $[0, 1]^4$, and even onto the

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In 2001 Mary Ellen Rudin characterized continous images of **compact lines** as compact monotonically normal spaces.

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For example, the unit interval $[0, 1]$, the long interval $[0, \omega_1]$ and the split interval.

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In particular, a nonmetrizable compact line L cannot be mapped onto its square L^2.

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Last year Plebanek and I proved the conjecture!

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Thus, the conjecture follows from the previous points and the following fact:

If $K_1, \ldots, K_d$ are nonmetrizable compact spaces and K_{d+1} is an infinite compact space, then $\text{free-dim}(K_1 \times \ldots \times K_{d+1}) \geq d + 1$.

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Notice that $\text{free-dim}(L_1 \times \ldots \times L_d) \leq d$ whenever $L_1, \ldots, L_d$ are compact lines.

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A basis for the topology of $L_1 \times L_2$ is given by **rectangles**.

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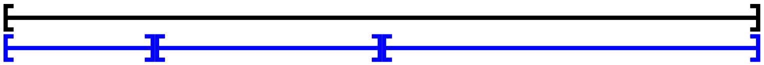

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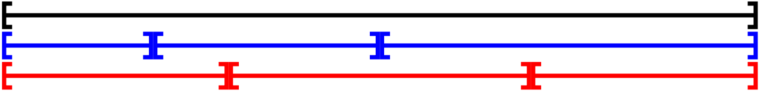

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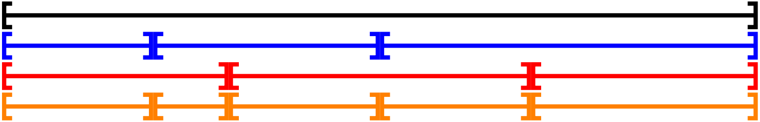

The blue cover has 3 intervals.

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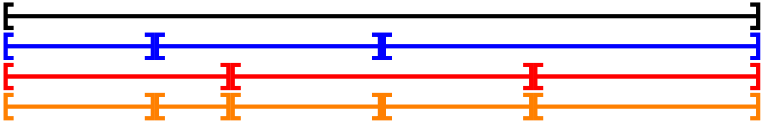

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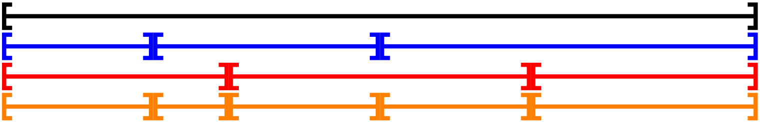

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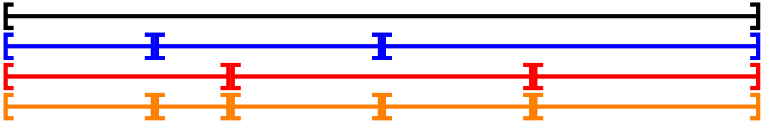

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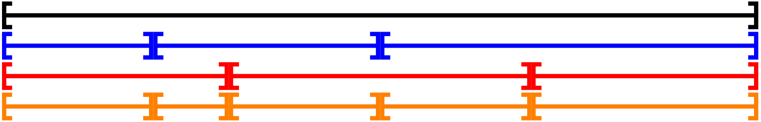

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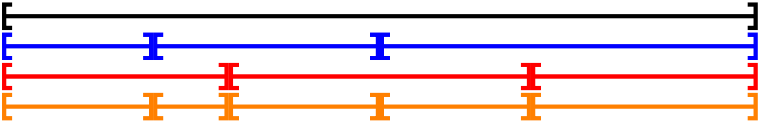

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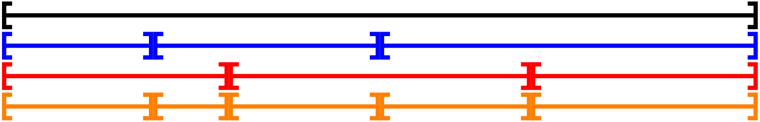

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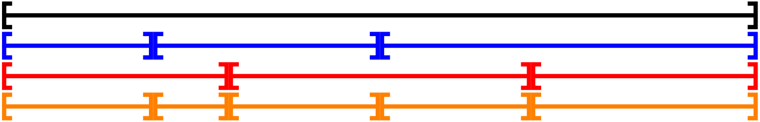

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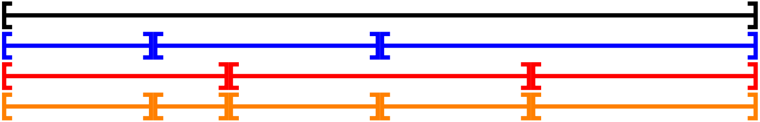

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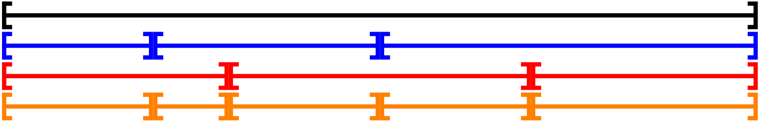

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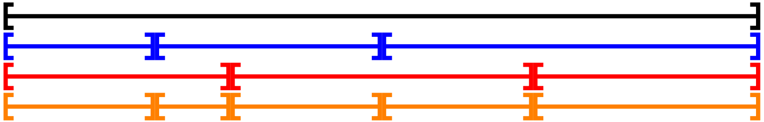

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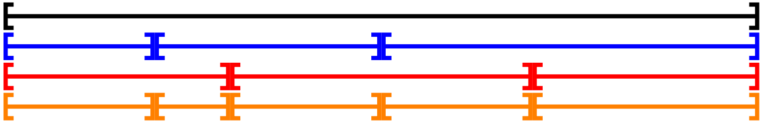

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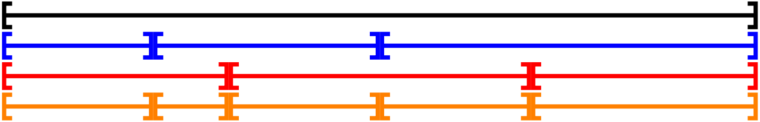

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$$|\mathcal{C}| \leq M \left(\chi(\mathcal{C}_1) + \ldots + \chi(\mathcal{C}_k)\right)^d .$$

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To prove the property stated in the Lemma use Ramsey.

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