Factorization of operators on Banach (function) spaces

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Joint work with Nigel Kalton and Lutz Weis
Euclidean structures

- Project started in the early 2000s
- Understand the role $R$-boundedness from an abstract operator-theoretic viewpoint
- Connected to completely bounded maps
- Project on hold since Nigel passed away
- Studied and improved/extended parts of the manuscript in 2016
- Joined the project in 2018, rewrote and modernized the manuscript
- Euclidean structures
  - Part I: Representation of operator families on a Hilbert space
  - Part II: Factorization of operator families
  - Part III: Vector-valued function spaces
  - Part IV: Sectorial operators and $H^\infty$-calculus
  - Part V: Counterexamples
- Project to be finished this fall
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\( \mathcal{R} \)-boundedness

**Definition**

Let \( X \) be a Banach space and \( \Gamma \subseteq \mathcal{L}(X) \). Then we say that \( \Gamma \) is \( \mathcal{R} \)-bounded if for any finite sequences \((T_k)_{k=1}^n \) in \( \Gamma \) and \((x_k)_{k=1}^n \) in \( X \)

\[
\left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k T_k x_k \right\|^2_X \right)^{1/2} \leq C \left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|^2_X \right)^{1/2},
\]

where \((\varepsilon_k)_{k=1}^n \) is a sequence of independent Rademacher variables.

- \( \mathcal{R} \)-boundedness is a strengthening of uniform boundedness.
- Equivalent to uniform boundedness on Hilbert spaces.

- \( \mathcal{R} \)-boundedness plays a (key) role in e.g.
  - Schauder multipliers
  - Operator-valued Fourier multiplier theory
  - Functional calculus
  - Maximal regularity of PDE’s
  - Stochastic integration in Banach spaces

- \( \left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|^2 \right)^{1/2} \) is a norm on \( X^n \).
  - A Euclidean structure is such a norm with a left and right ideal property.
**$\mathcal{R}$-boundedness**

**Definition**

Let $X$ be a Banach space and $\Gamma \subseteq \mathcal{L}(X)$. Then we say that $\Gamma$ is $\mathcal{R}$-bounded if for any finite sequences $(T_k)_{k=1}^n$ in $\Gamma$ and $(x_k)_{k=1}^n$ in $X$

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- $(\mathbb{E}\left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X^2)^{1/2}$ is a norm on $X^n$.
  - A Euclidean structure is such a norm with a left and right ideal property.
**Definition**

Let $X$ be a Banach space and $\Gamma \subseteq \mathcal{L}(X)$. Then we say that $\Gamma$ is $R$-bounded if for any finite sequences $(T_k)_{k=1}^n$ in $\Gamma$ and $(x_k)_{k=1}^n$ in $X$

$$
\left( \mathbb{E} \| \sum_{k=1}^n \varepsilon_k T_k x_k \|_X^2 \right)^{1/2} \leq C \left( \mathbb{E} \| \sum_{k=1}^n \varepsilon_k x_k \|_X^2 \right)^{1/2},
$$

where $(\varepsilon_k)_{k=1}^n$ is a sequence of independent Rademacher variables.

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- $(\mathbb{E} \| \sum_{k=1}^n \varepsilon_k x_k \|_X^2)^{1/2}$ is a norm on $X^n$.
  - A Euclidean structure is such a norm with a left and right ideal property.
Euclidean structures

Definition

Let $X$ be a Banach space. A Euclidean structure $\alpha$ is a family of norms $\| \cdot \|_\alpha$ on $X^n$ for all $n \in \mathbb{N}$ such that

$$
\| (x) \|_\alpha = \| x \|_X, \quad x \in X,
$$

$$
\| Ax \|_\alpha \leq \| A \| \| x \|_\alpha, \quad x \in X^n, \quad A \in M_{m,n}(\mathbb{C}),
$$

$$
\| (Tx_1, \cdots, Tx_n) \|_\alpha \leq C \| T \| \| x \|_\alpha, \quad x \in X^n, \quad T \in \mathcal{L}(X),
$$

A Euclidean structure induces a norm on the finite rank operators from $\ell^2$ to $X$. 
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\|(Tx_1, \ldots, Tx_n)\|_\alpha \leq C \|T\|\|x\|_\alpha, \quad x \in X^n, \quad T \in \mathcal{L}(X),
\]

A Euclidean structure induces a norm on the finite rank operators from $\ell^2$ to $X$.

Non-example:

- On a Banach space $X$:

\[
\|x\|_R := \left( \mathbb{E} \left[ \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|^2 \right] X \right)^{1/2}, \quad x \in X^n,
\]

is not a Euclidean structure. It fails the right-ideal property.
Euclidean structures

Definition

Let $X$ be a Banach space. A *Euclidean structure* $\alpha$ is a family of norms $\|\cdot\|_\alpha$ on $X^n$ for all $n \in \mathbb{N}$ such that

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$$

A Euclidean structure induces a norm on the finite rank operators from $\ell^2$ to $X$.

Examples:

- On any Banach space $X$: The *Gaussian structure*

  $$
  \| x \|_\gamma := \left( \mathbb{E} \left\| \sum_{k=1}^n \gamma_k x_k \right\|_X^2 \right)^{1/2}, \quad x \in X^n,
  $$

  where $(\gamma_k)_{k=1}^n$ is a sequence of independent normalized Gaussians.

- On a Banach lattice $X$: The *\ell^2-structure*

  $$
  \| x \|_{\ell^2} := \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\|_X, \quad x \in X^n.
  $$
Euclidean structures

Definition

Let $X$ be a Banach space. A Euclidean structure $\alpha$ is a family of norms $\| \cdot \|_\alpha$ on $X^n$ for all $n \in \mathbb{N}$ such that

\[
\begin{align*}
\|(x)\|_\alpha &= \|x\|_X, & x &\in X, \\
\|Ax\|_\alpha &\leq \|A\| \|x\|_\alpha, & x &\in X^n, \ A \in M_{m,n}(\mathbb{C}), \\
\|(Tx_1, \cdots, Tx_n)\|_\alpha &\leq C \|T\| \|x\|_\alpha, & x &\in X^n, \ T \in \mathcal{L}(X),
\end{align*}
\]

A Euclidean structure induces a norm on the finite rank operators from $\ell^2$ to $X$.

Examples:

- On any Banach space $X$: The Gaussian structure
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  \|x\|_\gamma := \left(\mathbb{E}\left\|\sum_{k=1}^n \gamma_k x_k\right\|_X^2\right)^{1/2}, \quad x \in X^n,
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- On a Banach lattice $X$: The $\ell^2$-structure
  \[
  \|x\|_{\ell^2} := \left\|\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}\right\|_X, \quad x \in X^n.
  \]
\(\alpha\)-boundedness

**Definition**

Let \(X\) be a Banach space, \(\alpha\) an Euclidean structure and \(\Gamma \subseteq \mathcal{L}(X)\). Then we say that \(\Gamma\) is **\(\alpha\)-bounded** if for any \(T = \text{diag}(T_1, \cdots, T_n)\) with \(T_1, \cdots, T_n \in \Gamma\)

\[
\| Tx \|_\alpha \leq C \| x \|_\alpha, \quad x \in X^n.
\]

- \(\alpha\)-boundedness implies uniform boundedness
- On a Banach space \(X\) with finite cotype

\[
\| x \|_\gamma = \left( \mathbb{E} \left\| \sum_{k=1}^n \gamma_k x_k \right\|_X^2 \right)^{1/2} \approx \left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X^2 \right)^{1/2} = \| x \|_\mathcal{R},
\]

so \(\gamma\)-boundedness is equivalent to \(\mathcal{R}\)-boundedness
- On a Banach lattice \(X\) with finite cotype

\[
\| x \|_{\ell^2} = \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\|_X \approx \left( \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X^2 \right)^{1/2} = \| x \|_\mathcal{R},
\]

so \(\ell^2\)-boundedness is equivalent to \(\mathcal{R}\)-boundedness.
**α-boundedness**

**Definition**

Let $X$ be a Banach space, $α$ an Euclidean structure and $Γ ⊆ ℒ(X)$. Then we say that $Γ$ is $α$-bounded if for any $T = \text{diag}(T_1, \cdots, T_n)$ with $T_1, \cdots, T_n ∈ Γ$

$$\|Tx\|_α ≤ C\|x\|_α, \quad x ∈ X^n.$$  

- $α$-boundedness implies uniform boundedness
- On a Banach space $X$ with finite cotype
  
  $$\|x\|_γ = \left(Ε\left\|\sum_{k=1}^{n} γ_k x_k \right\|^2_X \right)^{1/2} ≃ \left(Ε\left\|\sum_{k=1}^{n} ε_k x_k \right\|^2_X \right)^{1/2} = \|x\|_R,$$

  so $γ$-boundedness is equivalent to $R$-boundedness

- On a Banach lattice $X$ with finite cotype
  
  $$\|x\|_{\ell^2} = \left\|\left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}\right\|_X ≃ \left(Ε\left\|\sum_{k=1}^{n} ε_k x_k \right\|^2_X \right)^{1/2} = \|x\|_R,$$

  so $ℓ^2$-boundedness is equivalent to $R$-boundedness.
\(\alpha\)-boundedness

**Definition**

Let \(X\) be a Banach space, \(\alpha\) an Euclidean structure and \(\Gamma \subseteq \mathcal{L}(X)\). Then we say that \(\Gamma\) is \(\alpha\)-bounded if for any \(T = \text{diag}(T_1, \cdots, T_n)\) with \(T_1, \cdots, T_n \in \Gamma\)

\[
\| Tx \|_{\alpha} \leq C \| x \|_{\alpha}, \quad x \in X^n.
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- \(\alpha\)-boundedness implies uniform boundedness
- On a Banach space \(X\) with finite cotype
  \[
  \| x \|_{\gamma} = \left( \mathbb{E} \left( \sum_{k=1}^{n} \gamma_k x_k \right)^2 \right)^{\frac{1}{2}} \approx \left( \mathbb{E} \left( \sum_{k=1}^{n} \varepsilon_k x_k \right)^2 \right)^{\frac{1}{2}} = \| x \|_{\mathcal{R}},
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  \]
  so \(\ell^2\)-boundedness is equivalent to \(\mathcal{R}\)-boundedness.
Factorization through a Hilbert space

Theorem (Kwapięń ’72 and Maurey ’74)

Let $X$ and $Y$ be a Banach spaces and $T \in \mathcal{L}(X, Y)$. If $X$ has type 2 and $Y$ cotype 2, then there is a Hilbert space $H$ and operators $S \in \mathcal{L}(X, H)$ and $U \in \mathcal{L}(H, Y)$ s.t.

$$T = US.$$ 

Corollary (Kwapięń ’72)

Let $X$ be a Banach space. $X$ has type 2 and cotype 2 if and only if $X$ is isomorphic to a Hilbert space.
Factorization through a Hilbert space

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Corollary (Kwapień '72)

Let $X$ be a Banach space. $X$ has type 2 and cotype 2 if and only if $X$ is isomorphic to a Hilbert space.

Theorem (Kalton–L.–Weis '19)

Let $X$ and $Y$ be a Banach spaces, $\Gamma_1 \subseteq \mathcal{L}(X, Y)$. If $X$ has type 2, $Y$ cotype 2 and $\Gamma_1$ is $\gamma$-bounded, then there is a Hilbert space $H$, a $\tilde{T} \in \mathcal{L}(X, H)$ for every $T \in \Gamma_1$ and $U \in \mathcal{L}(H, Y)$ s.t. the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{\tilde{T}} & & \downarrow{U} \\
H & & \\
\end{array}
\]
Factorization through a Hilbert space

Theorem (Kwapień ’72 and Maurey ’74)

Let $X$ and $Y$ be a Banach spaces and $T \in \mathcal{L}(X, Y)$. If $X$ has type 2 and $Y$ cotype 2, then there is a Hilbert space $H$ and operators $S \in \mathcal{L}(X, H)$ and $U \in \mathcal{L}(H, Y)$ s.t. $T = US$.

Corollary (Kwapień ’72)

Let $X$ be a Banach space. $X$ has type 2 and cotype 2 if and only if $X$ is isomorphic to a Hilbert space.

Theorem (Kalton–L.–Weis ’19)

Let $X$ and $Y$ be a Banach spaces, $\Gamma_1 \subseteq \mathcal{L}(X, Y)$ and $\Gamma_2 \subseteq \mathcal{L}(Y)$. If $X$ has type 2, $Y$ cotype 2 and $\Gamma_1$ and $\Gamma_2$ are $\gamma$-bounded, then there is a Hilbert space $H$, a $\widetilde{T} \in \mathcal{L}(X, H)$ for every $T \in \Gamma_1$, a $\widetilde{S} \in \mathcal{L}(H)$ for every $S \in \Gamma_2$ and $U \in \mathcal{L}(H, Y)$ s.t. the following diagram commutes:

\[\begin{array}{c c c c c c c}
X & \xrightarrow{T} & Y & \xrightarrow{S} & Y \\
\searrow & & \swarrow & & \searrow & & \swarrow \\
H & \xrightarrow{\widetilde{T}} & \widetilde{S} & \xrightarrow{\widetilde{S}} & H \\
\end{array}\]
Factorization through weighted $L^2$

For a measure space $(S, \Sigma, \mu)$ and a weight $w: S \to \mathbb{R}_+$ let $L^2(S, w)$ be the space of all measurable $f : S \to \mathbb{C}$ such that

$$\|f\|_{L^2(S, w)} := \left( \int_S |f|^2 w \, d\mu \right)^{1/2} < \infty$$

Theorem (Kalton–L.–Weis ’19)

Let $X$ be an order-continuous Banach function space over $(S, \mu)$ and let $\Gamma \subseteq \mathcal{L}(X)$. $\Gamma$ is $\ell^2$-bounded if and only if for any $g_0, g_1 \in X$ there is a weight $w: S \to \mathbb{R}_+$ s.t.

$$\|Tf\|_{L^2(S, w)} \leq C \|f\|_{L^2(S, w)}, \quad f \in X \cap L^2(S, w), \quad T \in \Gamma$$

$$\|g_0\|_{L^2(S, w)} \leq C \|g_0\|_X,$n

$$\|g_1\|_X \leq C \|g_1\|_{L^2(S, w)}.$$

The if statement is trivial. Indeed for $f_1, \ldots, f_n \in X$ and $T_1, \ldots, T_n \in \Gamma$ set

$$g_0 := \left( \sum_{k=1}^n |f_k|^2 \right)^{1/2}, \quad g_1 := \left( \sum_{k=1}^n |T_k f_k|^2 \right)^{1/2}$$

Then we have

$$\|Tf\|_{\ell^2}^2 = \|g_1\|_X^2 \leq C^2 \sum_{k=1}^n \int_S |T_k f_k|^2 w \, d\mu \leq C^4 \sum_{k=1}^n \int_S |f_k|^2 w \, d\mu \leq C^6 \|g_0\|_X^2 = \|f\|_{\ell^2}^2$$
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**Theorem (Kalton–L.–Weis '19)**

Let $X$ be an order-continuous Banach function space over $(S, \mu)$ and let $\Gamma \subseteq L(X)$. $\Gamma$ is $\ell^2$-bounded if and only if for any $g_0, g_1 \in X$ there is a weight $w : S \to \mathbb{R}_+$ s.t.

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Application: Harmonic analysis

Let $p \in [1, \infty)$. For $f \in L^p(\mathbb{R})$ define the Hilbert transform

$$Hf(x) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-y} f(y) \, dy,$$

for $x \in \mathbb{R}$,

$$= \mathcal{F}^{-1} (\xi \mapsto -i \text{sgn}(\xi) \mathcal{F} f(\xi))(x).$$

For $f \in L^p(\mathbb{R}^d)$ define the Hardy–Littlewood maximal operator

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{\mathbb{R}^d} |f(y)| \, dy,$$

for $x \in \mathbb{R}^d$.

- Very important operators in harmonic analysis.
- Both $H$ and $M$ are bounded on $L^p$ for $p \in (1, \infty)$.
- We are interested in the tensor extensions of $H$ and $M$ on the Bochner space $L^p(\mathbb{R}^d; X)$.
- These have numerous applications in both harmonic analysis and PDE.
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- We are interested in the tensor extensions of $H$ and $M$ on the Bochner space $L^p(\mathbb{R}^d; X)$.
  - These have numerous applications in both harmonic analysis and PDE.
Application: Harmonic analysis

Let $p \in [1, \infty)$. For $f \in L^p(\mathbb{R})$ define the Hilbert transform

$$Hf(x) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x - y} f(y) \, dy, \quad x \in \mathbb{R},$$

$$= \mathcal{F}^{-1}(\xi \mapsto -i \text{sgn}(\xi) \mathcal{F} f(\xi))(x).$$

For $f \in L^p(\mathbb{R}^d)$ define the Hardy–Littlewood maximal operator

$$Mf(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{\mathbb{R}^d} |f(y)| \, dy, \quad x \in \mathbb{R}^d.$$

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Let $p \in [1, \infty)$. For $f \in L^p(\mathbb{R})$ define the **Hilbert transform**

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Application: Harmonic analysis

Let \( p \in [1, \infty) \) and let \( X \) be a Banach space. For \( f \in L^p(\mathbb{R}; X) \) define the vector-valued Hilbert transform

\[
\tilde{H}f(x) := \text{p.v.} \int_{\mathbb{R}} \frac{1}{x-y} f(y) \, dy,
\]

where the integral is interpreted in the Bochner sense.

Let \( X \) be a Banach function space. For \( f \in L^p(\mathbb{R}^d; X) \) define the lattice Hardy–Littlewood maximal operator

\[
\tilde{M}f(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{\mathbb{R}^d} |f(y)| \, dy,
\]

where the supremum is taken in the lattice sense.

- Boundedness of \( \tilde{H} \) is independent of \( p \in (1, \infty) \) (Calderón–Benedek–Panzone '62).
- Boundedness of \( \tilde{M} \) is independent of \( p \in (1, \infty) \) and \( d \in \mathbb{N} \) (García–Cuerva–Macías–Torrea '93).
- Boundedness of \( \tilde{H} \) and \( \tilde{M} \) depends on the geometry of \( X \).
Application: Harmonic analysis

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Let \( p \in [1, \infty) \) and let \( X \) be a Banach space. For \( f \in L^p(\mathbb{R}; X) \) define the \textit{vector-valued Hilbert transform}

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\tilde{H}f(x) := \text{p. v.} \int_{\mathbb{R}} \frac{1}{x - y} f(y) \, dy, \quad x \in \mathbb{R}
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where the integral is interpreted in the Bochner sense.

Let \( X \) be a Banach \textit{function} space. For \( f \in L^p(\mathbb{R}^d; X) \) define the \textit{lattice Hardy–Littlewood maximal operator}

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Theorem (Burkholder '83 and Bourgain '83)

Let $X$ be a Banach space and $p \in (1, \infty)$. The following are equivalent:

(i) $X$ has the UMD property.

(ii) $\tilde{H}$ is bounded on $L^p(\mathbb{R}; X)$.

- $X$ has the UMD property if any finite martingale $(f_k)_{k=0}^n$ in $L^p(\Omega; X)$ has Unconditional Martingale Differences $(df_k)_{k=1}^n$.

Theorem (Bourgain '84 and Rubio de Francia '86)

Let $X$ be a Banach function space and $p \in (1, \infty)$. The following is equivalent to (i) and (ii)

(iii) $\tilde{M}$ is bounded on $L^p(\mathbb{R}^d; X)$ and on $L^p(\mathbb{R}^d; X^*)$.

- (iii) $\Rightarrow$ (ii) is “easy” and quantitative.
- (ii) $\Rightarrow$ (iii) is very involved and technical.
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- That is, if for some $p \in (1, \infty)$ and all signs $\epsilon_k = \pm 1$ we have
  \[ \left\| \sum_{k=1}^n \epsilon_k df_n \right\|_{L^p(\Omega; X)} \lesssim \left\| \sum_{k=1}^n df_n \right\|_{L^p(\Omega; X)}, \]

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- Reflexive Lebesgue, Lorentz, (Musielak)-Orlicz, Sobolev, Besov spaces and Schatten classes have the UMD property.

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Application: Harmonic analysis

Theorem (Kalton–L.–Weis ’19)

Let $X$ be a Banach function space on $(S, \mu)$ and $p \in (1, \infty)$. If $\tilde{H}$ is bounded on $L^p(\mathbb{R}; X)$, then $\tilde{M}$ is bounded on $L^p(\mathbb{R}; X)$ with

$$\|\tilde{M}\|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)} \leq C \|\tilde{H}\|^2_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)}$$

- Apply factorization theorem with $\Gamma = \{\tilde{H}\}$ and $g_1 = \tilde{M}g_0$ to transfer the question from $L^p(\mathbb{R}; X)$ to $L^2(\mathbb{R} \times S, w)$:
  For any $g_0 \in L^p(\mathbb{R}; X)$ there is a weight $w : \mathbb{R} \times S \to \mathbb{R}_+$ such that
  $$\|\tilde{H}f\|_{L^2(\mathbb{R} \times S, w)} \leq C \|f\|_{L^2(\mathbb{R} \times S, w)}, \quad f \in L^p(\mathbb{R}; X) \cap L^2(\mathbb{R} \times S, w)$$
  $$\|g_0\|_{L^2(\mathbb{R} \times S, w)} \leq C \|g_0\|_{L^p(\mathbb{R}; X)}$$
  $$\|\tilde{M}g_0\|_{L^p(\mathbb{R}; X)} \leq C \|\tilde{M}g_0\|_{L^2(\mathbb{R} \times S, w)}.$$

- By Fubini it suffices to show the boundedness of $M$ on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$.

- For a weight $v : \mathbb{R} \to \mathbb{R}_+$, $M$ is bounded on $L^2(\mathbb{R}, v)$ if and only if $H$ is bounded on $L^2(\mathbb{R}, v)$ (Muckenhoupt '72,...).

- As $H$ is bounded on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$, $M$ is as well.

- Thus $\|\tilde{M}g_0\|_{L^p(\mathbb{R}; X)} \lesssim \|g_0\|_{L^p(\mathbb{R}; X)}$. 

□
Theorem (Kalton–L.–Weis ’19)

Let $X$ be a Banach function space on $(S, \mu)$ and $p \in (1, \infty)$. If $\widetilde{H}$ is bounded on $L^p(\mathbb{R}; X)$, then $\widetilde{M}$ is bounded on $L^p(\mathbb{R}; X)$ with

$$
\|\widetilde{M}\|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)} \leq C \|\widetilde{H}\|^2_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)}
$$

- Apply factorization theorem with $\Gamma = \{\widetilde{H}\}$ and $g_1 = \widetilde{M}g_0$ to transfer the question from $L^p(\mathbb{R}; X)$ to $L^2(\mathbb{R} \times S, w)$:
  For any $g_0 \in L^p(\mathbb{R}; X)$ there is a weight $w : \mathbb{R} \times S \to \mathbb{R}_+$ such that

  $$
  \|\widetilde{H}f\|_{L^2(\mathbb{R} \times S, w)} \leq C \|f\|_{L^2(\mathbb{R} \times S, w)}, \quad f \in L^p(\mathbb{R}; X) \cap L^2(\mathbb{R} \times S, w)
  $$

  $$
  \|g_0\|_{L^2(\mathbb{R} \times S, w)} \leq C \|g_0\|_{L^p(\mathbb{R}; X)},
  $$

  $$
  \|\widetilde{M}g_0\|_{L^p(\mathbb{R}; X)} \leq C \|\widetilde{M}g_0\|_{L^2(\mathbb{R} \times S, w)}.
  $$

- By Fubini it suffices to show the boundedness of $M$ on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$.

- For a weight $v : \mathbb{R} \to \mathbb{R}_+$, $M$ is bounded on $L^2(\mathbb{R}, v)$ if and only if $H$ is bounded on $L^2(\mathbb{R}, v)$ (Muckenhoupt ’72,...).

- As $H$ is bounded on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$, $M$ is as well.

- Thus $\|\widetilde{M}g_0\|_{L^p(\mathbb{R}; X)} \lesssim \|g_0\|_{L^p(\mathbb{R}; X)}$.
Application: Harmonic analysis

Theorem (Kalton–L.–Weis ’19)

Let $X$ be a Banach function space on $(S, \mu)$ and $p \in (1, \infty)$. If $\tilde{H}$ is bounded on $L^p(\mathbb{R}; X)$, then $\tilde{M}$ is bounded on $L^p(\mathbb{R}; X)$ with

$$\|\tilde{M}\|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)} \leq C \|\tilde{H}\|^2_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)}$$

- Apply factorization theorem with $\Gamma = \{\tilde{H}\}$ and $g_1 = \tilde{M}g_0$ to transfer the question from $L^p(\mathbb{R}; X)$ to $L^2(\mathbb{R} \times S, w)$:
  
  For any $g_0 \in L^p(\mathbb{R}; X)$ there is a weight $w : \mathbb{R} \times S \to \mathbb{R}_+$ such that

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  $$\|g_0\|_{L^2(\mathbb{R} \times S, w)} \leq C \|g_0\|_{L^p(\mathbb{R}; X)}$$

  $$\|\tilde{M}g_0\|_{L^p(\mathbb{R}; X)} \leq C \|\tilde{M}g_0\|_{L^2(\mathbb{R} \times S, w)}.$$  

- By Fubini it suffices to show the boundedness of $M$ on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$.

  - For a weight $\nu : \mathbb{R} \to \mathbb{R}_+$, $M$ is bounded on $L^2(\mathbb{R}, \nu)$ if and only if $H$ is bounded on $L^2(\mathbb{R}, \nu)$ (Muckenhoupt ’72,...).

  - As $H$ is bounded on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$, $M$ is as well.

  - Thus $\|\tilde{M}g_0\|_{L^p(\mathbb{R}; X)} \lesssim \|g_0\|_{L^p(\mathbb{R}; X)}$. 

### Application: Harmonic analysis

#### Theorem (Kalton–L.–Weis ’19)

Let $X$ be a Banach function space on $(S, \mu)$ and $p \in (1, \infty)$. If $\tilde{H}$ is bounded on $L^p(\mathbb{R}; X)$, then $\tilde{M}$ is bounded on $L^p(\mathbb{R}; X)$ with

$$
\|\tilde{M}\|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)} \leq C \|\tilde{H}\|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)}^2
$$

- Apply factorization theorem with $\Gamma = \{\tilde{H}\}$ and $g_1 = \tilde{M}g_0$ to transfer the question from $L^p(\mathbb{R}; X)$ to $L^2(\mathbb{R} \times S, w)$:
  
  For any $g_0 \in L^p(\mathbb{R}; X)$ there is a weight $w : \mathbb{R} \times S \rightarrow \mathbb{R}^+$ such that
  
  $$
  \|\tilde{H}f\|_{L^2(\mathbb{R} \times S, w)} \leq C \|f\|_{L^2(\mathbb{R} \times S, w)}, \quad f \in L^p(\mathbb{R}; X) \cap L^2(\mathbb{R} \times S, w)
  $$
  
  $$
  \|g_0\|_{L^2(\mathbb{R} \times S, w)} \leq C \|g_0\|_{L^p(\mathbb{R}; X)},
  $$
  
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  $$

- By Fubini it suffices to show the boundedness of $M$ on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$.

- For a weight $v : \mathbb{R} \rightarrow \mathbb{R}^+$, $M$ is bounded on $L^2(\mathbb{R}, v)$ if and only if $H$ is bounded on $L^2(\mathbb{R}, v)$ (Muckenhoupt ’72, ...).

- As $H$ is bounded on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$, $M$ is as well.

- Thus $\|\tilde{M}g_0\|_{L^p(\mathbb{R}; X)} \lesssim \|g_0\|_{L^p(\mathbb{R}; X)}$.
Theorem (Kalton–L.–Weis ’19)

Let $X$ be a Banach function space on $(S, \mu)$ and $p \in (1, \infty)$. If $\tilde{H}$ is bounded on $L^p(\mathbb{R}; X)$, then $\tilde{M}$ is bounded on $L^p(\mathbb{R}; X)$ with

$$
\| \tilde{M} \|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)} \leq C \| \tilde{H} \|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)}^2
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- Apply factorization theorem with $\Gamma = \{ \tilde{H} \}$ and $g_1 = \tilde{M}g_0$ to transfer the question from $L^p(\mathbb{R}; X)$ to $L^2(\mathbb{R} \times S, w)$:
  
  For any $g_0 \in L^p(\mathbb{R}; X)$ there is a weight $w : \mathbb{R} \times S \to \mathbb{R}^+$ such that

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\| \tilde{H}f \|_{L^2(\mathbb{R} \times S, w)} \leq C \| f \|_{L^2(\mathbb{R} \times S, w)}, \quad f \in L^p(\mathbb{R}; X) \cap L^2(\mathbb{R} \times S, w)
$$

$$
\| g_0 \|_{L^2(\mathbb{R} \times S, w)} \leq C \| g_0 \|_{L^p(\mathbb{R}; X)},
$$

$$
\| \tilde{M}g_0 \|_{L^p(\mathbb{R}; X)} \leq C \| \tilde{M}g_0 \|_{L^2(\mathbb{R} \times S, w)}.
$$

- By Fubini it suffices to show the boundedness of $M$ on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$.

- For a weight $v : \mathbb{R} \to \mathbb{R}^+$, $M$ is bounded on $L^2(\mathbb{R}, v)$ if and only if $H$ is bounded on $L^2(\mathbb{R}, v)$ (Muckenhoupt ’72,...).

- As $H$ is bounded on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$, $M$ is as well.

$$
\| \tilde{M}g_0 \|_{L^p(\mathbb{R}; X)} \lesssim \| g_0 \|_{L^p(\mathbb{R}; X)}
$$
Application: Harmonic analysis

Theorem (Kalton–L.–Weis ’19)

Let $X$ be a Banach function space on $(S, \mu)$ and $p \in (1, \infty)$. If $\tilde{H}$ is bounded on $L^p(\mathbb{R}; X)$, then $\tilde{M}$ is bounded on $L^p(\mathbb{R}; X)$ with

\[
\|\tilde{M}\|_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)} \leq C \|\tilde{H}\|^2_{L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)}
\]

- Apply factorization theorem with $\Gamma = \{\tilde{H}\}$ and $g_1 = \tilde{M}g_0$ to transfer the question from $L^p(\mathbb{R}; X)$ to $L^2(\mathbb{R} \times S, w)$:

For any $g_0 \in L^p(\mathbb{R}; X)$ there is a weight $w : \mathbb{R} \times S \to \mathbb{R}^+$ such that

\[
\|\tilde{H}f\|_{L^2(\mathbb{R} \times S, w)} \leq C \|f\|_{L^2(\mathbb{R} \times S, w)}, \quad f \in L^p(\mathbb{R}; X) \cap L^2(\mathbb{R} \times S, w)
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\[
\|g_0\|_{L^2(\mathbb{R} \times S, w)} \leq C \|g_0\|_{L^p(\mathbb{R}; X)}
\]

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\]

- By Fubini it suffices to show the boundedness of $M$ on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$.

- For a weight $v : \mathbb{R} \to \mathbb{R}^+$, $M$ is bounded on $L^2(\mathbb{R}, v)$ if and only if $H$ is bounded on $L^2(\mathbb{R}, v)$ (Muckenhoupt ’72, ...).

- As $H$ is bounded on $L^2(\mathbb{R}, w(\cdot, s))$ for $s \in S$, $M$ is as well.

- Thus $\|\tilde{M}g_0\|_{L^p(\mathbb{R}; X)} \lesssim \|g_0\|_{L^p(\mathbb{R}; X)}$
Application: Banach space geometry

Let $X$ be a Banach space and $p \in (1, \infty)$. Let $(f_k)_{k=0}^n$ be a finite martingale in $L^p(\Omega; X)$ with difference sequence $(df_k)_{k=1}^n$. Then $X$ has the UMD property if and only if for all signs $\epsilon_k = \pm 1$ we have

$$\left\| \sum_{k=1}^n \epsilon_k df_n \right\|_{L^p(\Omega; X)} \lesssim \left\| \sum_{k=1}^n df_n \right\|_{L^p(\Omega; X)},$$

if and only if

$$\left\| \sum_{k=1}^n df_n \right\|_{L^p(\Omega; X)} \lesssim \left\| \sum_{k=1}^n \epsilon_k df_n \right\|_{L^p(\Omega \times \Omega'; X)} \lesssim \left\| \sum_{k=1}^n df_n \right\|_{L^p(\Omega; X)},$$

where $(\epsilon)_{k=1}^n$ is a Rademacher sequence on $\Omega'$.

- (2) does not imply (1).
- It is an open question whether (3) implies (1)

**Theorem (Kalton–L.–Weis ’19)**

Let $X$ be a Banach function space. Then (3) implies (1).

- Proof similar to previous slide.
Outlook

• In the Euclidean structures manuscript:
  • More factorization theorems.
  • Representation of an $\alpha$-bounded family of operators on a Hilbert space.
  • Applications to interpolation, function spaces, functional calculus.

• More applications of factorization through weighted $L^2$:
  • Show the necessity of UMD for the $\mathcal{R}$-boundedness of certain operators
  • Potentially many more applications!

• To appear on arXiv before Christmas!
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Thank you for your attention!