Factorization of operators on Banach (function) spaces

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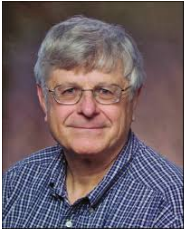
Joint work with Nigel Kalton and Lutz Weis

Euclidean structures

- Project started in the early 2000s
- Understand the role $\mathcal{R}$-boundedness from an abstract operator-theoretic viewpoint
- Connected to completely bounded maps
- Project on hold since Nigel passed away
- Studied and improved/extended parts of the manuscript in 2016
- Joined the project in 2018, rewrote and modernized the manuscript
- Euclidean structures
  - Part I: Representation of operator families on a Hilbert space
  - Part II: Factorization of operator families
  - Part III: Vector-valued function spaces
  - Part IV: Sectorial operators and H^∞-calculus
  - Part V: Counterexamples
- Project to be finished this fall



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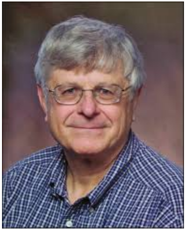

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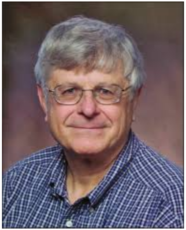

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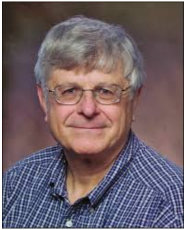

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$\mathcal{R}$-boundedness

Definition

Let X be a Banach space and $\Gamma \subseteq \mathcal{L}(X)$. Then we say that Γ is *$\mathcal{R}$-bounded* if for any finite sequences $(T_k)_{k=1}^n$ in Γ and $(x_k)_{k=1}^n$ in X

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k T_k x_k\right\|_X^2\right)^{1/2} \leq C\left(\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|_X^2\right)^{1/2},$$

where $(\varepsilon_k)_{k=1}^n$ is a sequence of independent Rademacher variables.

- $\mathcal{R}$-boundedness is a strengthening of uniform boundedness.
  - Equivalent to uniform boundedness on Hilbert spaces.
- $\mathcal{R}$-boundedness plays a (key) role in e.g.
  - Schauder multipliers
  - Operator-valued Fourier multiplier theory
  - Functional calculus
  - Maximal regularity of PDE's
  - Stochastic integration in Banach spaces
- $\left(\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|^2\right)^{1/2}$ is a norm on X^n.
  - A Euclidean structure is such a norm with a left and right ideal property.

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Euclidean structures

Definition

Let X be a Banach space. A *Euclidean structure* α is a family of norms $\|\cdot\|_\alpha$ on X^n for all $n \in \mathbb{N}$ such that

$$\begin{aligned} \|(x)\|_\alpha &= \|x\|_X, && x \in X, \\ \|\boldsymbol{A}\boldsymbol{x}\|_\alpha &\leq \|\boldsymbol{A}\|\|\boldsymbol{x}\|_\alpha, && \boldsymbol{x} \in X^n, \quad \boldsymbol{A} \in M_{m,n}(\mathbb{C}), \\ \|(Tx_1, \cdots, Tx_n)\|_\alpha &\leq C\,\|T\|\|\boldsymbol{x}\|_\alpha, && \boldsymbol{x} \in X^n, \quad T \in \mathcal{L}(X), \end{aligned}$$

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Non-example:

- On a Banach space X:

$$\|\mathbf{x}\|_{\mathcal{R}} := \Big(\mathbb{E}\Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_X^2\Big)^{\frac{1}{2}}, \qquad \mathbf{x} \in X^n,$$

  is **not** a Euclidean structure. It fails the right-ideal property.

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Examples:

- On any Banach space X: The *Gaussian structure*

$$\|\mathbf{x}\|_\gamma := \Big(\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_X^2\Big)^{\frac{1}{2}}, \qquad \mathbf{x} \in X^n,$$

where $(\gamma_k)_{k=1}^n$ is a sequence of independent normalized Gaussians.

- On a Banach lattice X: The *ℓ^2-structure*

$$\|\mathbf{x}\|_{\ell^2} := \Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_X, \qquad \mathbf{x} \in X^n.$$

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α-boundedness

Definition

Let X be a Banach space, α an Euclidean structure and $\Gamma \subseteq \mathcal{L}(X)$. Then we say that Γ is *α-bounded* if for any $\boldsymbol{T} = \operatorname{diag}(T_1, \cdots, T_n)$ with $T_1, \cdots, T_n \in \Gamma$

$$\|\boldsymbol{T}\boldsymbol{x}\|_\alpha \leq C\|\boldsymbol{x}\|_\alpha, \qquad x \in X^n.$$

- α-boundedness implies uniform boundedness
- On a Banach space X with finite cotype

$$\|\boldsymbol{x}\|_\gamma = \Big(\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|_X^2\Big)^{\frac{1}{2}} \simeq \Big(\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_X^2\Big)^{\frac{1}{2}} = \|\boldsymbol{x}\|_{\mathcal{R}},$$

  so γ-boundedness is equivalent to $\mathcal{R}$-boundedness
- On a Banach lattice X with finite cotype

$$\|\boldsymbol{x}\|_{\ell^2} = \Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2}\Big\|_X \simeq \Big(\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_X^2\Big)^{\frac{1}{2}} = \|\boldsymbol{x}\|_{\mathcal{R}},$$

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Factorization through a Hilbert space

Theorem (Kwapień '72 and Maurey '74)

Let X and Y be a Banach spaces and $T \in \mathcal{L}(X, Y)$. If X has type 2 and Y cotype 2, then there is a Hilbert space H and operators $S \in \mathcal{L}(X, H)$ and $U \in \mathcal{L}(H, Y)$ s.t.

$$T = US.$$

Corollary (Kwapień '72)

Let X be a Banach space. X has type 2 and cotype 2 if and only if X is isomorphic to a Hilbert space.

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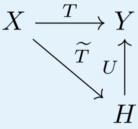
Let X and Y be a Banach spaces, $\Gamma_1 \subseteq \mathcal{L}(X, Y)$. If X has type 2, Y cotype 2 and Γ_1 is γ-bounded, then there is a Hilbert space H, a $\widetilde{T} \in \mathcal{L}(X, H)$ for every $T \in \Gamma_1$ and $U \in \mathcal{L}(H, Y)$ s.t. the following diagram commutes:



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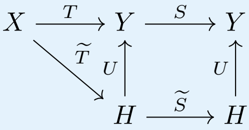
Let X and Y be a Banach spaces, $\Gamma_1 \subseteq \mathcal{L}(X, Y)$ and $\Gamma_2 \subseteq \mathcal{L}(Y)$. If X has type 2, Y cotype 2 and Γ_1 and Γ_2 are γ-bounded, then there is a Hilbert space H, a $\widetilde{T} \in \mathcal{L}(X, H)$ for every $T \in \Gamma_1$, a $\widetilde{S} \in \mathcal{L}(H)$ for every $S \in \Gamma_2$ and $U \in \mathcal{L}(H, Y)$ s.t. the following diagram commutes:



Factorization through weighted L^2

For a measure space (S, Σ, μ) and a weight $w\colon S \to \mathbb{R}_+$ let $L^2(S, w)$ be the space of all measurable $f : S \to \mathbb{C}$ such that

$$\|f\|_{L^2(S,w)} := \Big(\int_S |f|^2 w \,\mathrm{d}\mu\Big)^{1/2} < \infty$$

Theorem (Kalton–L.–Weis '19)

Let X be an order-continuous Banach function space over (S, μ) and let $\Gamma \subseteq \mathcal{L}(X)$. Γ is ℓ^2-bounded if and only if for any $g_0, g_1 \in X$ there is a weight $w\colon S \to \mathbb{R}_+$ s.t.

$$\begin{aligned} \|Tf\|_{L^2(S,w)} &\leq C\,\|f\|_{L^2(S,w)}, \quad f \in X \cap L^2(S,w),\ T \in \Gamma \\ \|g_0\|_{L^2(S,w)} &\leq C\,\|g_0\|_X, \\ \|g_1\|_X &\leq C\,\|g_1\|_{L^2(S,w)}. \end{aligned}$$

The if statement is trivial. Indeed for $f_1, \cdots, f_n \in X$ and $T_1, \cdots, T_n \in \Gamma$ set

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$$\|\boldsymbol{T}\boldsymbol{f}\|_{\ell^2}^2 = \|g_1\|_X^2 \leq C^2 \sum_{k=1}^n \int_S |T_k f_k|^2 w \,\mathrm{d}\mu \leq C^4 \sum_{k=1}^n \int_S |f_k|^2 w \,\mathrm{d}\mu \leq C^6 \|g_0\|_X^2 = \|\boldsymbol{f}\|_{\ell^2}^2$$

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Application: Harmonic analysis

Let $p \in [1,\infty)$. For $f \in L^p(\mathbb{R})$ define the *Hilbert transform*

$$\begin{aligned} Hf(x) &:= \mathrm{p.\,v.}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{x-y} f(y)\,\mathrm{d}y, && x \in \mathbb{R}, \\ &= \mathscr{F}^{-1}\big(\xi \mapsto -i\,\mathrm{sgn}(\xi)\mathscr{F}f(\xi)\big)(x). \end{aligned}$$

For $f \in L^p(\mathbb{R}^d)$ define the *Hardy–Littlewood maximal operator*

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where the supremum is taken in the lattice sense.

- Boundedness of $\widetilde{H}$ is independent of $p \in (1, \infty)$ (Calderón–Benedek–Panzone '62).
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Let X be a Banach space and $p \in (1, \infty)$. The following are equivalent:

(i) X has the UMD property.

(ii) $\widetilde{H}$ is bounded on $L^p(\mathbb{R}; X)$.

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Theorem (Bourgain '84 and Rubio de Francia '86)

Let X be a Banach function space and $p \in (1, \infty)$. The following is equivalent to (i) and (ii)

(iii) $\widetilde{M}$ is bounded on $L^p(\mathbb{R}^d; X)$ and on $L^p(\mathbb{R}^d; X^*)$.

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Application: Banach space geometry

Let X be a Banach space and $p \in (1, \infty)$. Let $(f_k)_{k=0}^n$ be a finite martingale in $L^p(\Omega; X)$ with difference sequence $(df_k)_{k=1}^n$. Then X has the UMD property if and only if for all signs $\epsilon_k = \pm 1$ we have

$$\Big\|\sum_{k=1}^n \epsilon_k df_n\Big\|_{L^p(\Omega;X)} \overset{(1)}{\lesssim} \Big\|\sum_{k=1}^n df_n\Big\|_{L^p(\Omega;X)},$$

if and only if

$$\Big\|\sum_{k=1}^n df_n\Big\|_{L^p(\Omega;X)} \overset{(2)}{\lesssim} \Big\|\sum_{k=1}^n \varepsilon_k df_n\Big\|_{L^p(\Omega\times\Omega';X)} \overset{(3)}{\lesssim} \Big\|\sum_{k=1}^n df_n\Big\|_{L^p(\Omega;X)},$$

where $(\varepsilon)_{k=1}^n$ is a Rademacher sequence on Ω'.

- (2) does not imply (1).
- It is an open question whether (3) implies (1)

Theorem (Kalton–L.–Weis '19)

Let X be a Banach function space. Then (3) implies (1).

- Proof similar to previous slide.

Outlook

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  - More factorization theorems.
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  - Applications to interpolation, function spaces, functional calculus.
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Thank you for your attention!