Order and Topology

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Based on joint work with Niushan Gao, Cosimo Munari, Made Tantrawan and Foivos Xanthos

Background and notation

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Special case: Banach function space (BFS).
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Examples: L^p, Orlicz space, Lorentz space, rearrangement invariant (r.i.) function space.

Order convergence

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Examples: $(L^\infty)_n^\sim = L^1$, $(L^\varphi)_n^\sim = L^\psi$, where ψ is the conjugate Orlicz function to φ.
$X_n^\sim = X^*$ if and only if X has order continuous norm.

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Main problem: Study order closure and order closedness of a convex set and relation to closure with respect to some topologies, particularly $\sigma(X, X_n^\sim)$.

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We can ask specifically:
For which Banach lattice X is it true that two of these sets coincide for all convex sets $C \subseteq X$.

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Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Consider $X = \ell^\infty(\mathbb{N} \times \mathbb{N})$. Set

$$Y = \{y = (y_{mn}) \in X : \lim_{n \to \mathcal{U}} y_{mn} = m y_{m1} \text{ for all } m \in \mathbb{N}\}.$$

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Y is a sublattice of X.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots \\ 1 & 0 & \cdots & 0 & 2 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & m & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \xrightarrow{o} \begin{pmatrix} 1 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} := e_m \text{ as } n \to \infty$$

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$$e_m \xrightarrow{o} e = \begin{pmatrix} 1 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 1 & 0 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Hence $e \in \overline{\overline{Y}^o}^o$.

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Thus $\overline{Y}^o$ is not order closed.

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Corollary

Let X be a Banach lattice that contains a lattice isomorphic copy of ℓ^∞. There is a closed sublattice Y of X so that $\overline{Y}^o$ is not order closed.
In particular, if X is a countably order complete Banach lattice, then $\overline{Y}^o$ is order closed for every closed sublattice Y of X if and only if X has order continuous norm.

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Proposition

Let Y be a sublattice of a Banach lattice X and let I be an ideal of $X_n^\sim$ that separates points of X. Then

$$\overline{Y}^o \subseteq \overline{Y}^{uo} \subseteq \overline{\overline{Y}^o}^o \subseteq \overline{Y}^{\sigma(X,I)}$$

and $\overline{Y}^{\sigma(X,I)}$ is order closed.
If X has the countable sup property, then $\overline{Y}^{uo} = \overline{Y}^{\sigma(X,I)}$ is the order closed envelope of Y.

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Theorem

Let X be a countably order complete Banach lattice. TFAE.

1. X is order continuous.
2. $\overline{Y}^o = \overline{Y}^{\sigma(X, X_n^\sim)}$ for any sublattice Y of X.
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For any subspace Z of ℓ^∞, define $Z_1 = \overline{Z}^{\text{seq-}w^*}$, $Z_{\beta+1} = (Z_\beta)_1$ and $Z_\beta = \bigcup_{\alpha<\beta} Z_\alpha$ if β is a limit ordinal. A result of Ostrovskij shows that for any countable ordinal α, there is a subspace Z of ℓ^∞ so that $Z_\beta, \beta < \alpha$, are all distinct.

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Proposition

Let X be countably order complete. Then $\overline{C}^o = \overline{C}^{\sigma(X,X_n^\sim)}$ for any convex set C if and only if X is order continuous.

Order closed but not $\sigma(X, X_n^\sim)$-closed

Example: A convex set C that is order closed but not $\sigma(X, X_n^\sim)$-closed in $X = \ell^\infty \oplus \ell^1 = \ell^\infty \oplus (\oplus \ell^1)_1$.

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Observations:
(i) $X_n^\sim = \ell^1 \oplus (\oplus \ell^\infty)_\infty$. Hence $\sigma(X, X_n^\sim) = w^* \oplus w$.

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Example: A convex set C that is order closed but not $\sigma(X, X_n^\sim)$-closed in $X = \ell^\infty \oplus \ell^1 = \ell^\infty \oplus (\oplus \ell^1)_1$. I.e., $\overline{C}^o \subsetneq \overline{C}^{\sigma(X, X_n^\sim)}$.

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$S = \{x_{k,j} : k, j \in \mathbb{N}\}$, where

$$x_{k,j} = (0, \dots, 0, \overset{j\text{th coord}}{2^k}, 2^k, \dots) \oplus (0, \dots, 0, \overset{j\text{th coord}}{\frac{e_j}{2^k}}, 0, \dots),$$

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Suppose that $y = u \oplus (v_1, v_2, \ldots) \in \ell^1 \oplus (\oplus \ell^\infty)_{\ell^\infty}$, where $u = (u_i) \in \ell^1$ and (v_i) is a bounded sequence in ℓ^∞.

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This shows that $0 \in \overline{S}^{w^* \oplus w}$. So S is not $w^* \oplus w = \sigma(X, X_n^\sim)$-closed.

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Thus S is order closed.

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The example can be ported over to a Banach lattice X containing a set $S = \{x_n, y_n : n \in \mathbb{N}\}$ so that

- The elements in S are positive pairwise disjoint and normalized.
- (x_n) is order bounded in X.
- There exists $y^* \in X_n^\sim$ so that $\inf_n y^*(y_n) > 0$.

The final condition says that (y_n) is a disjoint ℓ^1-sequence in X and its "ℓ^1-ness" is witnessed by an element in $X_n^\sim$.

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Theorem

Let X be an order complete Banach lattice so that $X_n^\sim$ isomorphically norms X. If X has property P1, then either X or $X_n^\sim$ is order continuous.

$\widehat{C} = \overline{C}^{\sigma(X, X_n^\sim)}$

Summary: P1 = "any convex set is order closed $\iff$ $\sigma(X, X_n^\sim)$-closed".

$$X \text{ order continuous} \implies P1 \overset{X_n^\sim \text{ norming}}{\implies} X \text{ or } X_n^\sim \text{ order continuous.}$$

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Natural question: $X_n^\sim$ order continuous $\implies$ P1?

[To p.27]

Krein-Smulyan property

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In the example above, the convex set C has the property that $C \cap B$ is $\sigma(X, X_n^\sim)$-closed for any convex, norm bounded, $\sigma(X, X_n^\sim)$-closed set B; however, C is not $\sigma(X, X_n^\sim)$-closed.

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Say that $\sigma(X, X_n^\sim)$ has the Krein-Smulyan property (KS) if for any convex set C in X so that $C \cap B$ is $\sigma(X, X_n^\sim)$-closed for any convex, norm bounded, $\sigma(X, X_n^\sim)$-closed set B, C is $\sigma(X, X_n^\sim)$-closed.

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Example: Suppose that $X = (X_n^\sim)_n^\sim$ (canonically) and $X_n^\sim$ is order continuous. Then $X = (X_n^\sim)^*$. Thus $\sigma(X, X_n^\sim)$ is the weak* topology. Therefore, $\sigma(X, X_n^\sim)$ has KS by the Krein-Smulyan Theorem.

Krein-Smulyan property

Theorem

Suppose that $X = (X_n^\sim)_n^\sim$. Then $\sigma(X, X_n^\sim)$ has KS if and only if X or $X_n^\sim$ is order continuous.

[To p.31]

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Proof.

X order continuous $\implies X_n^\sim = X^*$. So $\sigma(X, X_n^\sim) =$ weak topology and thus has KS.

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Converse follows from construction of set C above. $\square$

Krein-Smulyan property

Corollary

Suppose that $X = (X_n^\sim)_n^\sim$ and that $X_n^\sim$ is order continuous. Then X has P1 if and only if every *norm bounded* order closed convex set in X is $\sigma(X, X_n^\sim)$-closed.

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If B is convex, norm bounded and $\sigma(X, X_n^\sim)$-closed, then it is order closed.

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Hence $C \cap B$ is norm bounded order closed and convex. So $C \cap B$ is $\sigma(X, X_n^\sim)$-closed.

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Hence $C \cap B$ is norm bounded order closed and convex. So $C \cap B$ is $\sigma(X, X_n^\sim)$-closed.

By Theorem above, $\sigma(X, X_n^\sim)$ has KS. Hence C is $\sigma(X, X_n^\sim)$-closed. $\square$

Properties P2 and P3

Motivated by the Corollary, let's define:

(P2) Every norm bounded order closed convex set in X is $\sigma(X, X_n^\sim)$-closed, i.e., $\widehat{C} = \overline{C}^{\sigma(X, X_n^\sim)}$ for every norm bounded convex set C.

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We have

$$X \text{ order continuous} \implies P1, P2, P3.$$

If $X_n^\sim$ is order continuous, then

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Below, we look at some properties motivated by the proof of said result in Orlicz spaces.

Disjoint order continuity property (DOCP)

Say that X has DOCP if for any norm bounded *disjoint* sequence (f_n) in X_+, $f_n \to 0$ $\sigma(X, X_n^{\sim}) \implies f_n \to 0$ weakly.

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Remark. Reason for the name is that if X is countably order complete, then X is order continuous if and only if for any norm bounded sequence (f_n) in X_+, $f_n \to 0$ $\sigma(X, X_n^\sim) \implies f_n \to 0$ weakly.

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Relevance of DOCP to P2 is based on the following concrete situation.

Disjoint order continuity property (DOCP)

Lemma

Let (f_n) be a normalized disjoint ℓ^1 sequence in a Banach lattice X. Then $\overline{\mathrm{co}(f_n)}^o$ is order closed and

$$\overline{\mathrm{co}(f_n)}^o = \{\sum_{n=1}^{\infty} a_n f_n : a_n \geq 0, \sum a_n = 1\}(:= C).$$

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$C =$ norm closure of $\mathrm{co}(f_n) \subseteq \overline{\mathrm{co}(f_n)}^o$.

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Let (f_n) be a normalized disjoint ℓ^1 sequence in a Banach lattice X. Then $\overline{\mathrm{co}(f_n)}^o$ is order closed and

$$\overline{\mathrm{co}(f_n)}^o = \{\sum_{n=1}^{\infty} a_n f_n : a_n \geq 0, \sum a_n = 1\}(:= C).$$

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C = norm closure of $\mathrm{co}(f_n) \subseteq \overline{\mathrm{co}(f_n)}^o$.
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Since $0 \notin C$, (f_n) does not converge to 0 wrt $\sigma(X, X_n^\sim)$. $\square$

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Example. If X is an Orlicz space, then X/X_a is an AM-space, hence X has DOCP.

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So $g_n \to 0$ weakly. But (g_n) is an ℓ^1-sequence. A contradiction.

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Order subsequence splitting property (OSSP)

Say that X has OSSP if for every norm bounded sequence (f_n) in X_+ that uo-converges to 0, there is a subsequence (f_{n_k}) with a splitting

$$f_{n_k} = x_k + y_k,$$

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An order continuous Banach lattice with a weak unit 1 has SSP if every norm bounded sequence (f_n) has a subsequence (f_{n_k}) with a splitting

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Weis showed (among other things) that SSP $\iff$ some special kinds of $\ell^\infty(n)$'s do not uniformly lattice embed into X.

OSSP and P3 ($\overline{C}^{o} = \overline{C}^{\sigma(X, X_n^{\sim})}$ for every norm bounded convex C)

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Proposition

Suppose that $X_n^\sim$ is order continuous, $(X_n^\sim)_+$ contains a strictly positive functional φ and X has OSSP. X has P3 $\iff$ X has DOCP.

[To p.31]

OSSP and P3

Proof

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There is a convex combination of (y_n) that is order bounded.

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Since φ is strictly positive and $X_n^\sim$ is order continuous, $X_n^\sim$ is the closed ideal generated by φ. So $\varphi(|x_n|) \to 0 \implies |x_n| \to 0$ $\sigma(X, X_n^\sim)$.
Using OSSP, WLOG, split $|x_n| = y_n + z_n$, where $y_n, z_n \geq 0$, (y_n) is disjoint and (z_n) is order bounded.
Then $y_n \to 0$ $\sigma(X, X_n^\sim)$. By DOCP, $y_n \to 0$ weakly.
There is a convex combination of (y_n) that is order bounded.
So WLOG, (x_n) is order bounded and $x_n \to 0$ uo. Therefore, $x_n \xrightarrow{o} 0$.

OSSP and P3

Proof

Previously, we know that P3 $\implies$ P2 $\implies$ DOCP.

Conversely, suppose that X is as above and has DOCP.
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So WLOG, (x_n) is order bounded and $x_n \to 0$ uo. Therefore, $x_n \xrightarrow{o} 0$.
Thus $0 \in \overline{C}^{o}$.

Characterizing P1 ($\widehat{C} = \overline{C}^{\sigma(X, X_n^\sim)}$ for any convex C)

Theorem

Suppose that $(X_n^\sim)_+$ contains a strictly positive functional, $X = (X_n^\sim)_n^\sim$ and X has OSSP. TFAE

1. X has P1.
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This proves (2) $\implies$ (4). (4) $\implies$ (1) is easy.

Spaces with a special modular

Spaces with a special modular

A special modular on a Banach lattice X is a functional $\rho : X \to [0, \infty]$ so that

1. $f \in X_+$, $\rho(f) < \infty$, $f_n \in [0, f]$, $f_n \xrightarrow{o} 0 \implies \rho(f_n) \to 0$.
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More generally, if X is the Orlicz-Lorentz space $\Lambda_{\varphi,w}$, then

$$\rho(f) = \int_0^\infty \varphi(f^*) w(t)\, dt$$

is a special modular.

Proposition

Let ρ be a special modular on a Banach lattice X.

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Theorem

Suppose that $(X_n^\sim)_+$ contains a strictly positive functional, $X = (X_n^\sim)_n^\sim$ and there is a special modular on X. Then X has P1 if and only if either X or X^* is order continuous.
If, in addition, X_a is order dense in X, then the above occurs if and only if either X or $X_n^\sim$ is order continuous if and only if $\sigma(X, X_n^\sim)$ has KS property.

Thank You

Muchas Gracias

Order and Topology
Part 2. Miscellaneous topics

Denny H. Leung

National University of Singapore

Workshop on Banach spaces and Banach lattices
ICMAT
September 2019

Based on joint work with Niushan Gao and Made Tantrawan

A. Law invariant sets

$L^0(\mu) =$ space of all measurable functions on a measure space (Ω, Σ, μ).

A BFS is a subspace X of $L^0(\mu)$ endowed with a complete norm $\|\cdot\|$ so that if $|f| \leq |g|$ and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$.

X is rearrangement invariant (r.i.) if $g \overset{\text{dist}}{=} f \in X \implies g \in X$ and $\|g\| = \|f\|$.

Examples: L^p, Orlicz-Lorentz spaces.

Order closure and order closed set

Let $C \subseteq X$, X BFS.

$f \in \overline{C}^o$ if and only if there is a sequence (f_n) in C and $g \in X$ so that $|f_n| \leq g$ for all n and $f_n \to f$ a.e.

C is order closed if $\overline{C}^o = C$.

The space of order continuous linear functionals on X, $X_n^\sim$, is given by

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Now we consider this problem if X is r.i. and C is a law invariant ($=$ r.i.) subset of X.

Main result

Let X be an r.i. space. A subset C of X is law invariant if $g \stackrel{\text{dist}}{=} f \in C \implies g \in C$.

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$\sigma(X, X_n^\sim)$-closed always implies order closed.

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Therefore $f \in C$ by 2.

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Easy to see that $f_n \xrightarrow{o} \mathbb{E}[f|\pi]$.

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Then $\mathbb{E}[f|\pi_n] \xrightarrow{o} f$ and thus $f \in C$.

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Theorem (Delbaen and Owari)

Let $L^\varphi(\Omega, \Sigma, \mu)$ be an Orlicz space defined on a finite measure space. Assume that $(L^\varphi)^*$ has order continuous norm. (Equivalently, the conjugate Orlicz function φ^* is Δ_2 at infinity.) If (f_n) is a norm bounded sequence in L^φ, then there are a subsequence (f_{n_k}) and $f \in L^\varphi$ such that for any further subsequence (g_k) of (f_{n_k}), a *subsequence* of the averages $(\frac{1}{m}\sum_{k=1}^m g_k)$ order converges to f.

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Definition

A Banach lattice X has the (weak) order Banach-Saks property ((w)oBS) if any (weakly null) norm bounded sequence (x_n) in X has a subsequence (x_{n_k}) so that the averages $(\frac{1}{m}\sum_{k=1}^m x_{n_k})$ order converges to an element $x \in X$. (In the case of woBS, x must be 0.)

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L^p has BS if $1 < p < \infty$ and L^1 has wBS. What about "o" versions?

L^1 fails woBS

Suppose that f is a positive function in L^1 and (f_n) is an independent sequence so that $f_n = f$ in distribution for all n.

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Realize f_k, $1 \leq k \leq n$, as functions $f_k(x_1, \ldots, x_n) = f(x_k)$ defined in $[0,1]^n$.

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Since any subsequence of (f_n) has the same joint distribution as the whole sequence, $(\frac{1}{m} \sum_{k=1}^{m} f_{n_k})_m$ cannot be order bounded for any subsequence (f_{n_k}).

Orlicz space $X = L^\varphi$ has woBS implies X^* order continuous

The same idea can be used to prove that if $L^\varphi(\mu)$ has woBS for a finite nonatomic measure μ, then φ^* satisfies Δ_2 at ∞.

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Let $r = \min\{p, 2\}$. Then for any finitely supported (a_n)

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Let $\ell_k = \frac{1}{k(k+1)} \sum_{n=1}^{k} f_n$.
By the above, $\|\ell_k\| \le \frac{Ck^{1/r}}{k(k+1)}$. So $\sum |\ell_k|$ converges in L^p. Since

$$\frac{1}{n} \sum_{k=1}^{n} f_k = g_n - \sum_{k=1}^{n-1} \ell_k,$$

we have

$$|\frac{1}{n} \sum_{k=1}^{n} f_k| \le g + \sum_{k=1}^{\infty} |\ell_k| \text{ for all } n.$$

Recall that $(\frac{1}{n} \sum_{k=1}^{n} f_k)_n$ converges pointwise to some f. LDCT shows that the convergence is also in L^p-norm.
Since (f_n) is weakly null, $f = 0$ and $\frac{1}{n} \sum_{k=1}^{n} f_k \xrightarrow{o} 0$.

From the argument to a general result

The same argument works provided:

1. X is a separable r.i. space – so that the Haar functions is a basis for X. [R.i. in the sense of Lindenstrauss and Tzafriri.]
2. The upper Boyd index $q_X < \infty$ – Johnson-Schechtman proved BDG inequality holds in X.
3. X is p-convex for some $p > 1$.

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Theorem

Let X be a separable r.i. space on $[0, 1]$ that is p-convex for some $p > 1$ and whose upper Boyd index $q_X < \infty$. Then X has woBS.

In particular, if X is a reflexive separable r.i. space on $[0, 1]$, then X has oBS.

Orlicz spaces

For L^φ, define property (H): every weakly null sequence in H^φ has a subsequence whose Cesaro means are order bounded *in* L^φ.

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Question: Is (H) equivalent to φ^* being Δ_2?

Thank You
Muchas Gracias

Local convexity in L^0

Denny H. Leung

National University of Singapore

Workshop on Banach spaces and Banach lattices
ICMAT
September 2019

Based on joint work with Niushan Gao and Foivos Xanthos

The setting

$(\Omega, \Sigma, \mathbb{P})$: atomless probability space.
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General Question: What does local convexity on a subset do?

Local convexity for subsets of $L^0(\mathbb{P})$

- A subset S of $L^0(\mathbb{P})$ is *bounded in probability* if it is a bounded subset of the TVS $L^0(\mathbb{P})$. Same as

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1. All FCCs of (f_n) converges to f in probability.
2. The $L^0(\mathbb{P})$-topology is locally convex on $\mathrm{co}((f_n) \cup \{f\})$.
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Example. Let $K = \{f \in L^1_+(\mathbb{P}) : \int f \, d\mathbb{P} = 1\}$. Then K satisfies (2) but not (3).

[To p.14] [p.16][p.18]

Nonpositive sets & "de-switching"

[Branath-Schachermayer. LNM 1999] Let K be a convex set in $L^0_+(\mathbb{P})$ that is bounded in probability. Then there exists $\mathbb{Q} \sim \mathbb{P}$ so that K is a bounded set in $L^1(\mathbb{Q})$.

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We generalize the questions above to *bounded* convex sets in $L^1(\mathbb{P})$.

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Proposition. Let K be a convex bounded set in $L^1(\mathbb{P})$. Consider the following conditions.

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3. There exists $\mathbb{Q} \sim \mathbb{P}$ such that K is $\mathbb{Q}$-uniform integrable.
4. For any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $K_A = \{f\chi_A : f \in K\}$ is $\mathbb{P}$-uniformly integrable.

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Then (1) $\iff$ (2) and (3) $\iff$ (4).
Remark. To get (2), it suffices to obtain the following:
For any measurable A with $\mathbb{P}(A) > 0$, there exists measurable $B \subseteq A$ with $\mathbb{P}(B) > 0$ so that $\|(f_n - f)\chi_B\|_{L^1(\mathbb{P})} \to 0$ for any $f_n, f \in K$ so that $f_n \to f$ in probability.

An application of Komlos' Theorem

Theorem. [Komlos. Acta MAS Hung. 1967] Let (f_n) be a bounded sequence in $L^1(\mathbb{P})$, there is a subsequence (f_{n_k}) and a function $f \in L^1(\mathbb{P})$ so that $(\frac{1}{m}\sum_{k=1}^{m} f_{n_k})_m$ converges a.e. to f.

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In particular, "Yes" for Q2.

[To p.4]

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By choice of A, $(\frac{1}{n}\sum_{k=1}^{n} f_k\chi_A)_n$ must be norm convergent. Contradiction.

Locally convex solid

Aim: To characterize the condition that there exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K, where K is convex bounded in $L^1(\mathbb{P})$.

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Definition. Let S be a nonempty subset of K. We say that the $L^0(\mathbb{P})$-topology is *uniformly locally convex solid* on S if for each $L^0(\mathbb{P})$-neighborhood U of 0, there is a convex solid set $W \subseteq U$ such that for each $f \in S$, $(f + W) \cap K$ is a neighborhood of f for the restriction of the $L^0(\mathbb{P})$-topology to K.

A separation theorem and its consequence

Theorem. Let K be a convex bounded set in $L^1(\mathbb{P})$ and let S be a nonempty subset of K. Assume that the $L^0(\mathbb{P})$-topology is uniformly locally convex solid on S. If A is a measurable set with $\mathbb{P}(A) > 0$, then there exists $0 \neq g \in L^\infty_+(\mathbb{P})$, $\operatorname{supp} g \subseteq A$ such that

$$\int |f_n - f| g \, d\mathbb{P} \to 0 \text{ if } f_n, f \in K \text{ and } f_n \to f \text{ in probability.}$$

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Idea: Find a sequence of convex solid sets W_k and $r > 0$ so that

1. For each $f \in S$, $(f + W_k) \cap K$ is a a neighborhood of f for the restriction of the $L^0(\mathbb{P})$-topology to K.
2. g is a linear functional that separates $rB_{L^1(\mathbb{P})}$ and kW_k on one side and χ_A on the other.

A local Hahn-Banach Theorem

Theorem. Let (X, τ) be a real Hausdorff TVS. Let K be a convex circled set in X. Suppose that the restriction of τ to K is locally convex (at 0). The set of all linear functionals on X that are τ-continuous on K separates points of K.

A characterization

Theorem. Let K be a bounded convex set in $L^1(\mathbb{P})$. TFAE

1. The $L^0(\mathbb{P})$-topology is uniformly locally convex solid on K.
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Remark. If K is also circled, then the $L^0(\mathbb{P})$-topology is uniformly locally convex solid on K if and only if it is locally convex solid at 0.

Corollary. Let K be convex solid and bounded in $L^1(\mathbb{P})$. TFAE.

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Since $V \cap K$ is solid, $W = \text{co}(V \cap K)$ is a solid convex set contained in $C \subseteq U$ and $W \cap K$ is a neigborhood of 0 in K.

A counterexample

Example. (Based on [Pryce, P Edin MS, 1972]) There is a bounded convex circled set K in $L^1(0,1)$ that is L^0-compact so that the L^0-topology is locally convex on K, but there does not exists $\mathbb{Q} \sim \lambda$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K.

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Set $Y_n = F_n(X_n)$ and

$$K = \{\sum a_n Y_n : \sum |a_n| \leq 1\}.$$

Back to the positive case

Theorem. [Kardaras-Zitkovic. PAMS 2013] Let $f_n, f \in L^0_+(\mathbb{P})$, where (f_n) converges to f in probability. TFAE

1. All FCCs of (f_n) converges to f in probability.
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Corollary. Let K be a bounded convex set in $L^1_+(\mathbb{P})$. Assume that the $L^0(\mathbb{P})$-topology is locally convex on K. Then for any $f \in K$ and any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $\|(f_n - f)\chi_A\|_{L^1(\mathbb{P})} \to 0$ for any sequence (f_n) in K that converges to f in probability.

Corollary. Let K be a bounded convex set in $L^1_+(\mathbb{P})$. Assume that the $L^0(\mathbb{P})$-topology is locally convex on K. Let S be a countable set in K. Then for any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $\|(f_n - f)\chi_A\|_{L^1(\mathbb{P})} \to 0$ for any sequence (f_n) in K that converges to some $f \in S$ in probability.

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Proposition. Let (f_n) be a bounded sequence in $L^1_+(\mathbb{P})$ and let $K = \mathrm{co}(f_n)$. If the $L^0(\mathbb{P})$-topology is locally convex on K, then there exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K.

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This is a special case of Q1. [p.4]

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Thus $\|(g_k - g)\chi_A\|_{L^1(\mathbb{P})} \to 0$.

A counterexample to Q1 in the positive case

Let Γ be an uncountable set. Let $\mathbb{P}$ be the product measure on $2^{\Gamma\times\mathbb{N}}$. There exists a convex norm bounded set K in $L^1_+(\mathbb{P})$ so that the $L^0(\mathbb{P})$ topology on K is locally convex, but there does not exist any $\mathbb{Q}\sim\mathbb{P}$ so that the $L^0(\mathbb{Q})$- and $L^0(\mathbb{P})$-topologies agree on K.

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Q1 is still open if K is assumed to be $L^0(\mathbb{P})$-closed or if $\mathbb{P}$ is a separable probability measure.

[To p.4]

Thank You