A characterization of Random Variables

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Some examples

- **Real numbers.**

Let F be a totally ordered field which is Dedekind complete (i.e., each nonempty upper bounded subset S admits the least upper bound $\sup(S)$).

Then $F = \mathbf{R}$, up to field isomorphism.

- (*Kakutani, 1941*) **Continuous functions over a compact** $\mathcal{C}(K)$.

Let E be a Banach lattice (i.e., a complete normed vector lattice) such that:

1. there exists a unit e, i.e., $E = \bigcup_{n\geq 1}[-ne, ne]$;
2. $\|x \vee y\| = \max(\|x\|, \|y\|)$ for all $x, y \geq 0$.

Then there exists a compact space K such that $E = \mathcal{C}(K)$, up to lattice isometry.

In addition, K is unique, up to homeomorphism.

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Bounded measurable functions $\mathsf{L}^\infty(\mathrm{P})$

• (*Abramovich et al., 1994*) **Bounded random variables.**

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4. E admits a strictly positive order continuous linear functional.

Then there exists a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ such that $\mathsf{E} = \mathsf{L}^\infty(\mathrm{P})$, up to lattice isometry.

Corollary. Each $\mathsf{L}^\infty(\mathrm{P})$ can be regarded as a space $\mathcal{C}(\mathsf{K})$.

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Measurable functions $L^0(\mu)$

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Let E be an Archimedean vector lattice. Then there exists a σ-finite measure space (X, Σ, μ) such that $E = L^0(\mu)$, up to lattice isomorphism, if and only if:

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- Here, $\Gamma(E)$, is the set of equivalence classes of order continuous linear functionals defined on order dense ideals of E, where two functionals are identified whenever they agree on an order dense ideal of E.

- It is known that $\Gamma(E)$ is separating on E if and only if there exists an order dense ideal I of E such that the order continuous dual of I is separating on I. Moreover, in that case, there exists an order dense ideal which admits a strictly positive order continuous linear functional.

QUESTION: Can we provide a more "concrete" characterization?

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Main result

Characterization of (equivalence classes of) random variables $L^0(P)$

Let E be a Dedekind complete vector lattice with weak order unit $e > 0$ (i.e., $0 \le x \wedge e = 0$ implies $x = 0$). Then the following are equivalent:

1. There exists a probability space $(\Omega, \mathscr{F}, P)$ such that $E = L^0(P)$, up to lattice isomorphism;
2. There exists a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$, where $E_e := \bigcup_{n \ge 1}[-ne, ne]$, for which the induced metric

$$d_\varphi : E \times E \to \mathbf{R} : (x, y) \mapsto \varphi(|x - y| \wedge e)$$

   is complete on E.
3. There exists a strictly positive order continuous linear functional $\psi : E_e \to \mathbf{R}$ and E is laterally complete (i.e., the supremum of every disjoint subset of E^+ exists in E).

Main result

Moreover, in such case, we have:

1. $E_e = L^\infty(\mathrm{P})$, up to lattice isomorphism;
2. The metrics d_φ and d_ψ are topologically equivalent; and
3. E has the countable sup property (i.e., the least upper bound of subsets S can be attained through sequences in S, provided it exists).

As a corollary:

Characterization of $L^0(\mathrm{P})$ in Archimedean Riesz spaces

Let E be an Archimedean vector lattice. Then $E = L^0(\mathrm{P})$ for some probability space $(\Omega, \mathscr{F}, \mathrm{P})$, up to lattice isomorphism, if and only if E is Dedekind complete, laterally complete, and admits a strictly positive order continuous linear functional on E_e.

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1. $\mathsf{E}_{\mathsf{e}} = \mathsf{L}^{\infty}(\mathrm{P})$, up to lattice isomorphism;
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Characterization of $\mathsf{L}^0(\mathrm{P})$ in Archimedean Riesz spaces

Let E be an Archimedean vector lattice. Then $\mathsf{E} = \mathsf{L}^0(\mathrm{P})$ for some probability space $(\Omega, \mathscr{F}, \mathrm{P})$, up to lattice isomorphism, if and only if E is Dedekind complete, laterally complete, and admits a strictly positive order continuous linear functional on E_{e}.

An application: Predicatable stochastic processes

• Given a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n \in \mathbf{N}}, \mathrm{P})$, then a stochastic process $(X_n)_{n \in \mathbf{N}}$ is said to be *predictable* if X_{n+1} is $\mathscr{F}_n$-measurable for all $n \in \mathbf{N}$.

• Note the collection $\mathbb{L}$ of (equivalence classes of) predicatable stochastic processes is a proper subset of $L_0(\Omega, \mathscr{F}, \mathrm{P})^{\mathbf{N}}$.

• It follows by our main result that

$$\mathbb{L} = L^0(X, \Sigma, \mu),$$

up to lattice isomorphism, for some probability space (X, Σ, μ).

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Another application: f-algebras

• A Riesz algebra E is a f-**algebra** if $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$ for all $a, b, c \geq 0$ such that $a \wedge b = 0$.

• A f-algebra E is a **Stonean algebra** if it is Dedekind complete and admits a non-zero multiplicative unit e. The following facts are well known:

- ◇ The multiplication is commutative;
- ◇ $x^2 := x \cdot x \geq 0$ for all $x \in E$; in particular, $e > 0$, and
- ◇ e is a weak order unit; in particular, $(x \wedge ne)_{n \geq 1} \uparrow x$ for all $x \geq 0$.

• A Stonean algebra E is a f-**algebra of** $\mathcal{L}^0$ **type** whenever the principal ideal E_e is an Arens algebra (i.e., a real commutative Banach algebra such that $\|e\| = 1$ and $\|a\|^2 \leq \|a^2 + b^2\|$ for all $a, b \in E_e$) and there exists a strictly positive order continuous linear functional φ on E_e such that the metric d_φ is complete.

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Thank you!

References:

[1] Y. A. Abramovich, C. D. Aliprantis, and W. R. Zame, *A representation theorem for Riesz spaces and its applications to economics*, Econom. Theory **5** (1995), no. 3, 527–535.

[2] S. Cerreia-Vioglio, M. Kupper, F. Maccheroni, M. Marinacci, and N. Vogelpoth, *Conditional L_p-spaces and the duality of modules over f-algebras*, J. Math. Anal. Appl. **444** (2016), no. 2, 1045–1070.

[3] S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. (2) **42** (1941), 523–537.

[4] J. J. Masterson, *A characterization of the Riesz space of measurable functions*, Trans. Amer. Math. Soc. **135** (1969), 193–197.