Approximation properties in Lipschitz-free spaces over groups

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Lipschitz-free space definition reminder

Let (M, d) be a metric space, $0 \in M$.

Notation

$$Lip_0(M) := \{f : M \to \mathbb{R} | f \text{ is Lipschitz, } f(0) = 0\}$$

is a Banach space when equipped with the norm

$$\|f\|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

For each $x \in M$, consider the evaluation functional $\delta_x \in Lip_0(M)^*$ por $\delta_x f := f(x)$.

Definition/Proposition

$$\mathcal{F}(M) := \overline{span}\{\delta_x | x \in M\}$$

is the *free space over* M, and it is an isometric predual to $Lip_0(M)$.

$\delta : M \to \mathcal{F}(M)$ is an isometry. Moreover:

Linear interpretation property

$\forall\ \varphi : M \to N$ Lipschitz with $\varphi(0_M) = 0_N$ $\exists!$ $\hat{\varphi} : \mathcal{F}(M) \to \mathcal{F}(N)$ *linear* such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle \delta^M} & & \downarrow{\scriptstyle \delta^N} \\ \mathcal{F}(M) & \xrightarrow{\hat{\varphi}} & \mathcal{F}(N) \end{array}$$

Also, $\|\hat{\varphi}\| = L(\varphi)$.

On the w^* topology of $Lip_0(M)$

On bounded subsets of $Lip_0(M)$, w^* and the topology of pointwise convergence coincide.

Metric framework: topological groups equipped with invariant, compatible metrics. Such metrics are plentyful:

• A topological group G is left-invariant metrizable whenever: **(1)** G is T_0 and e admits a countable open basis, **(2)** G is locally countably compact and $\{e\}$ is a countable intersection of open sets, or **(3)** G is compact and $\{e\}$ is a countable intersection of open sets.

• If G is compact with Haar measure $d\lambda$ admitting a compatible metric d, we can construct a compatible and bi-invariant one by taking:

$$d'(x,y) \doteq \int \int d(zxw, zyw) d\lambda(z)\, d\lambda(w).$$

General goal: study free spaces over such metrics, in particular concerning *approximation properties*.

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Different equivalent metrics can lead to very different free spaces:
On $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, let d be the usual metric inherited from $\mathbb{R}$ and d^α ($0 < \alpha < 1$) be its snowflaked version. Then $\mathcal{F}(\mathbb{T}, d) \simeq L_1$ and $\mathcal{F}(\mathbb{T}, d^\alpha)$ is isomorphic to a dual space.

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Approximation properties

Let $\lambda \geq 1$. A Banach space X has the λ-**bounded approximation property (**λ**-BAP)** if it satisfies one of the following equivalent properties:

• $\forall \epsilon > 0, \forall K \subset X$ compact, there is a λ-bounded finite rank operator on X such that $\|Tx - x\| < \epsilon$, $\forall x \in K$;

• There is a λ-bounded net of finite rank operators (T_α) on X such that $T_\alpha \xrightarrow{WOT} Id_X$.

X has the **metric approximation property (MAP)** if it has the 1-BAP.

X has λ-**FDD** if there is a sequence P_n of commuting projections with increasing range such that $\dim(P_n - P_{n-1})X < \infty$ $(P_0 = 0)$, $\|P_n\| \leq \lambda$ and $P_n \xrightarrow{SOT} Id$. If $\dim(P_n - P_{n-1})X = 1$, X has a λ-**Schauder basis**.

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Approximation properties

Problem: if G is a compact group with a left invariant compatible metric, does $\mathcal{F}(M)$ have the MAP?

Keep in mind:

Godefroy, Ozawa 2012

If X is a separable Banach space and $C \subset X$ is convex with $\overline{span} C = X$, then X has an 1-complemented isometric copy in $\mathcal{F}(C)$.

Corollary

There exists a compact metric K with $\mathcal{F}(K)$ failing AP.

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Let X be a Banach space. If for all $\delta > 0$ X has the $(\lambda + \delta)$-BAP, it follows that X hast the λ-BAP.

Proof. Fix a compact set $K \subset X$ and $\epsilon > 0$, and take δ small enough so that $M\delta(\lambda + \delta)/\lambda < \epsilon/2$, where $M = \sup_{x \in K} \|x\|$. Let T be a finite rank, $(\lambda + \delta)$-bounded operator on X such that $\|Tx - x\| < \epsilon/2$, for all $x \in K$. Then the λ-bounded operator $S = \lambda T/\|T\|$ satisfies, for each $x \in K$,

$$\begin{aligned}\|Sx - x\| &\leq \|Sx - Tx\| + \|Tx - x\| \\ &= \left\| \left(\frac{\|T\|}{\lambda} - 1 \right) Tx \right\| + \epsilon/2 \\ &\leq M\delta(\lambda + \delta)/\lambda + \epsilon/2 \\ &< \epsilon. \square\end{aligned}$$

Harmonic analysis cannon

Let G be a locally compact group equipped with Haar measure λ. A *D"-sequence* in G is a sequence $\{U_n, V_n\}$ of pairs of Borel subsets of finite measure in G such that

1. $U_1 \supset U_2 \supset \ldots,$
2. there is an $A > 0$ such that $0 < \lambda(U_n U_n^{-1}) < A\lambda(U_n)$, for all n,
3. every neighborhood of e contains some U_n, and
4. $V_n^{-1} V_n \subset U_n$ and there exists $B > 0$ such that $\lambda(U_n) \leq B\lambda(V_n)$.

We shall say that a locally compact group G is *summability-friendly* if it admits a D"-sequence U_n with the property that each U_n is invariant under inner automorphisms of G ($\forall g \in G$, $gU_n g^{-1} \in U_n$).

Examples of summability-friendly groups: closed subgroups of finite dimensional unitary groups.

Harmonic analysis cannon

Thm 44.25, Hewitt & Ross Abstract Harmonic Analysis Vol. 2

Suppose that G is a summability-friendly compact group equipped with left-invariant Haar measure λ. Then there exists a sequence F_n of positive functions on G satisfying

1. each F_n is a positive definite *central* (commutes under convolution with any L_1 function) trigonometric polynomial,
2. $F_n(g^{-1}) = F_n(g)$, $g \in G$, for each n,
3. for each n, $\int F_n \, d\lambda = 1$, and
4. $f * F_n(x) \to f(x)$ λ-almost everywhere for every $f \in L_p(G)$, $1 \leq p < \infty$.

Key points:

- if P is a trigonometric polynomial, the continuous operator $f \in C(G) \mapsto f * P \in C(G)$ is of finite rank;
- G equipped with left-invariant metric $\Rightarrow \|f * g\|_{Lip} \leq \|f\|_{L_1} \|g\|_{Lip}$.

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Aim and shoot

Characterization of BAP for Lipschitz-free spaces over compact metric spaces

Let K be a compact metric space and $\lambda \geq 1$. The following assertions are equivalent:

1. $\mathcal{F}(K)$ has the λ-BAP;
2. for each $\epsilon > 0$ there is a net T_α of bounded operators on $C(K)$ such that
    1. T_α are of finite rank,
    2. T_α maps Lipschitz functions to Lipschitz functions,
    3. $\|T_\alpha f\|_{Lip} \leq (\lambda + \epsilon)\|f\|_{Lip}$ for each α and each Lipschitz f, and
    4. $T_\alpha f \overset{pt}{\to} f$, for each Lipschitz f.

Proof. ((1)⇒(2)) Let $T_\alpha : \mathcal{F}(K) \to \mathcal{F}(K)$ be λ-bounded, finite rank, $T_\alpha \overset{WOT}{\to} Id_{\mathcal{F}(M)}$. Then $T_\alpha^* : Lip_0(K) \to Lip_0(K)$ extend to operators on $C(K)$ satisfying (1)–(4).

((2)⇒(1)) Chose some $0 \in K$. The operators S_α defined on $C(K)$ by

$$S_\alpha(f)(x) = T_\alpha(f)(x) - T_\alpha(f)(0)$$

are bounded, and satisfy (1)–(4) as T_α and are $(\lambda + \epsilon)$-bounded operators on $Lip_0(K)$. Lemma ⇒ ∃ R_α finite-rank, $(\lambda + \epsilon)$-bounded operators on $\mathcal{F}(K)$ with $R_\alpha^* = S_\alpha$, that is:

$$\langle S_\alpha f, \gamma \rangle = \langle f, R_\alpha \gamma \rangle, \forall f \in Lip_0(K), \forall \gamma \in \mathcal{F}(K).$$

Now since $S_\alpha f$ is a bounded net in $Lip_0(K)$ converging pointwise to f, it must converge w^*, thus R_α converges to $Id_{\mathcal{F}(M)}$ in the weak operator topology. □

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Summability methods are great for preserving constants

Another method: Godefroy 2016, "*BAP characterization for free spaces via near-extensions*"

Let M_n be finite ϵ_n-dense subsets of a compact metric space M with $\epsilon_n \searrow 0$. TFAE:

1. $\mathcal{F}(M)$ has the λ-BAP;
2. There are λ-bounded linear operators E_n from $(Lip_0(M_n), \|\cdot\|_{Lip})$ into $(Lip_0(M), \|\cdot\|_{Lip})$ such that

$$\sup_{\|f\|_{Lip_0(M_n)} \leq 1} \|E_n(f)|_{M_n} - f\|_\infty \xrightarrow{n} 0.$$

If moreover $M_1 \subset M_2 \subset ...$ and (2) is satisfied with the $\|\cdot\|_\infty$ norm involved being always 0 (that is, E_n are actual extensions), we have that $\mathcal{F}(M)$ has the λ-FDD.

Adapting from [Lancien, Pernecká 2013], we get 1-FDD for $\ell_1^n/\mathbb{Z}^n$. But not for $\ell_p^n/\mathbb{Z}^n$!

Summability methods are great for preserving constants

Yet another method. With the toolkit:

Lang, Plaut 2001 *(euclidean embedding tool)*

If a compact metric space is locally bilipschitz embeddable in some $\mathbb{R}^N$, then it is bilipschitz embeddable in some $\mathbb{R}^M$.

Lee, Naor 2005 *(Lipschitz extension tool)*

There is a universal constant $C > 0$ such that, for any $N \in \mathbb{N}$ and any subset $F \subset \mathbb{R}^N$ (with the euclidean norm), there exists a $C\sqrt{N}$-bounded and w^* continuous linear extension operator from $Lip_0(F)$ into $Lip_0(\mathbb{R}^N)$.

Godefroy, Kalton 2003

X finite dimensional $\Rightarrow \mathcal{F}(X)$ has the MAP.

we can show that $(\mathbb{R}^n, \|\cdot\|)/\mathbb{Z}^n$ has λ-BAP, for any norm $\|\cdot\|$, but λ is sensitive to the choice of $\|\cdot\|$.

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Thank you! Gracias! Obrigado!