Algebra of convolution type operators with continuous data on Banach function spaces

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Banach function norm I

Let

- L^0 be the set of all measurable complex-valued functions on $\mathbb{R}$,
- L^0_+ be the subset of functions in L^0 whose values lie in $[0, \infty]$,
- χ_E be the characteristic function of a measurable set $E \subset \mathbb{R}$.

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A mapping

$$\rho : L^0_+ \to [0, \infty]$$

is called a Banach function norm if,

- for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$,
- for all constants $a \geq 0$,
- for all measurable subsets E of $\mathbb{R}$,

the following properties hold:

Banach function norm II (after W. Luxemburg, 1955)

$$\begin{aligned}
&\text{(A1)}\ \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.},\\
&\qquad\ \rho(af) = a\rho(f),\\
&\qquad\ \rho(f+g) \le \rho(f) + \rho(g),\\
&\text{(A2)}\ 0 \le g \le f \text{ a.e.} \ \Rightarrow\ \rho(g) \le \rho(f) \quad \text{(the lattice property)},\\
&\text{(A3)}\ 0 \le f_n \uparrow f \text{ a.e.} \ \Rightarrow\ \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)},\\
&\text{(A4)}\ \rho(\chi_E) < \infty,\\
&\text{(A5)}\ \int_E f(x)\,dx \le C_E\rho(f)
\end{aligned}$$

with the constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f.

Banach function spaces

When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R})$ of all functions $f \in L^0$ for which

$$\rho(|f|) < \infty$$

is called a Banach function space. For each $f \in X(\mathbb{R})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R})} := \rho(|f|).$$

Associate space

If ρ is a Banach function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{R}} f(x)g(x)\,dx \;:\; f \in L^0_+,\ \rho(f) \le 1\right\}, \quad g \in L^0_+.$$

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Lemma (W. Luxemburg, 1955)

If ρ is a Banach function norm, then ρ' is itself a Banach function norm.

The Banach function space $X'(\mathbb{R})$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{R})$. The associate space $X'(\mathbb{R})$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathbb{R})]^*$.

Density of nice functions in Banach function spaces

Let $C_0^\infty(\mathbb{R})$ denote the set of all infinitely differentiable compactly supported functions on $\mathbb{R}$.

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Lemma

If $X(\mathbb{R})$ is a separable Banach function space, then the sets $C_0^\infty(\mathbb{R})$ and $L^2(\mathbb{R}) \cap X(\mathbb{R})$ are dense in $X(\mathbb{R})$.

Fourier convolution operators

Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform

$$(Ff)(x) := \widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{itx}\,dt, \quad x \in \mathbb{R},$$

and let $F^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of F,

$$(F^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}\,dx, \quad t \in \mathbb{R}.$$

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It is well known that the Fourier convolution operator

$$W^0(a) := F^{-1}aF$$

is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$.

Fourier multipliers on Banach function spaces

Let $X(\mathbb{R})$ be a separable Banach function space. A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the convolution operator

$$W^0(a) := F^{-1} a F$$

maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function a is called the symbol of the Fourier convolution operator $W^0(a)$.

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The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \left\| W^0(a) \right\|_{\mathcal{B}(X(\mathbb{R}))},$$

where $\mathcal{B}(X(\mathbb{R}))$ denotes the Banach algebra of all bounded linear operators on the space $X(\mathbb{R})$.

The Hardy-Littlewood maximal operator

The (non-centered) Hardy-Littlewood maximal function Mf of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is defined by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ of finite length containing x.

The Hardy-Littlewood maximal operator M defined by the rule

$$f \mapsto Mf$$

is a sublinear operator.

Functions of finite total variation

Suppose that $a : \mathbb{R} \to \mathbb{C}$ is a function of finite total variation $V(a)$ given by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all partitions of $\mathbb{R}$ of the form

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty$$

with $n \in \mathbb{N}$.

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with $n \in \mathbb{N}$.

The set $V(\mathbb{R})$ of all functions of finite total variation on $\mathbb{R}$ with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a)$$

is a unital non-separable Banach algebra.

Stechkin's inequality for Banach function spaces

Theorem (O.K., 2015)

Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If a function $a : \mathbb{R} \to \mathbb{C}$ has a finite total variation $V(a)$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$ and

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le c_X(\|a\|_{L^\infty(\mathbb{R})} + V(a))$$

where c_X is a positive constant depending only on $X(\mathbb{R})$.

Continuous and piecewise continuous functions

- Let $C(\dot{\mathbb{R}})$ denote the C^*-algebra of continuous functions on the one-point compactification $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the real line.

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- Let $PC(\dot{\mathbb{R}})$ denote the C^*-algebra of piecewise continuous functions on $\dot{\mathbb{R}}$, that is, the algebra of functions $a \in L^\infty(\mathbb{R})$ such that finite one-sided limits

$$a(x_0 - 0) = \lim_{x \to x_0 - 0} a(x), \quad a(x_0 + 0) = \lim_{x \to x_0 + 0} a(x)$$

exist for each $x_0 \in \dot{\mathbb{R}}$.

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It is well known that

$$V(\mathbb{R}) \subset PC(\dot{\mathbb{R}}).$$

Continuous and piecewise continuous Fourier multipliers

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  In particular, the function

  $$s(x) = \operatorname{sign}(x)$$

  belongs to $PC_X(\dot{\mathbb{R}})$. This function is the symbol of the of the Hilbert transform

  $$(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - x} dt$$

  and $S = W^0(s)$.

Algebras $\mathcal{C}_{X(\mathbb{R})}$ and $\mathcal{PC}_{X(\mathbb{R})}$

Consider the smallest Banach subalgebras

$$\mathcal{C}_{X(\mathbb{R})} = \mathrm{alg}\{aI, W^0(b) \ : \ a \in C(\dot{\mathbb{R}}), \ b \in C_X(\dot{\mathbb{R}})\}$$

$$\mathcal{PC}_{X(\mathbb{R})} = \mathrm{alg}\{aI, W^0(b) \ : \ a \in PC(\dot{\mathbb{R}}), \ b \in PC_X(\dot{\mathbb{R}})\}$$

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of the algebra $\mathcal{B}(X(\mathbb{R}))$ that contain

- all operators of multiplication aI by functions $a \in C(\dot{\mathbb{R}})$ (resp. $a \in PC(\dot{\mathbb{R}})$);
- all Fourier convolution operators $W^0(b)$ with symbols $b \in C_X(\dot{\mathbb{R}})$ (resp. $b \in PC_X(\dot{\mathbb{R}})$).

Fredholm operators and the Calkin algebra

Recall that an operator $T \in \mathcal{B}(X(\mathbb{R}))$ is said to be Fredholm if

- its image is closed;
- $\dim \operatorname{Ker} T < \infty$ and $\dim \operatorname{Ker} T^* < \infty$.

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Equivalently, T is Fredholm if and only if

$T+\mathcal{K}(X(\mathbb{R}))$ is invertible in the Calkin algebra $\mathcal{B}(X(\mathbb{R}))/\mathcal{K}(X(\mathbb{R}))$.

Our motivation:
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- $L^p(\mathbb{R})$, $1 < p < \infty$ (Roland Duduchava, 1970s).
- $L^p(\mathbb{R}, w)$, $1 < p < \infty$ and w is a Muckenhoupt weight, (Albrecht Böttcher and Ilya Spitkovsky, 1994).

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- What about Orlicz spaces, rearrangement-invariant spaces, or Nakano spaces?

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- What about Orlicz spaces, rearrangement-invariant spaces, or Nakano spaces?
- What about general Banach function spaces?

Main result

Theorem

Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ is contained in the Banach algebra $\mathcal{C}_{X(\mathbb{R})}$.

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Why is it important? Because the following subalgebras of the Calkin algebra

$$\mathcal{C}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R})) \subset \mathcal{PC}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R})) \subset \mathcal{B}(X(\mathbb{R}))/\mathcal{K}(X(\mathbb{R}))$$

are correctly defined.

Open questions

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2. Under which condition on a Banach function space $X(\mathbb{R})$, is the algebra $\mathcal{C}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$ commutative?

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   We know the answer for some interesting spaces (for instance, for Orlicz spaces, rearrangement-invariant spaces, Nakano spaces).

   The commutativity of the algebra $\mathcal{C}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$ is important in the study of invertbility in the non-commutative algebra $\mathcal{PC}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$ by means of the Allan local principle (a non-commutative extension of the Gelfand theory).

Wavelet bases in Banach function spaces

Recall that a function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if the family

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R}, \quad j,k \in \mathbb{Z},$$

forms an orthonormal basis in $L^2(\mathbb{R})$.

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Theorem (O.K., 2019)

Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that ψ is an orthonormal C^1-wavelet with compact support. Then the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an unconditional basis in $X(\mathbb{R})$ and the wavelet expansion

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \quad \left(\langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_{j,k}(x)}dx\right)$$

holds for every $f \in X(\mathbb{R})$, where the convergence is unconditional.

About the approximation property in Banach function spaces

The previous theorem was proved to justify that a separable Banach function space $X(\mathbb{R})$ such that the Hardy-Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$ admits a Schauder basis, and thus, has the approximation property.

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This workshop is a right place to ask the following:

Is it known that separable Banach function spaces have the approximation property?

About an one-dimensional operator

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Lemma

Suppose $X(\mathbb{R})$ is a separable Banach function space. Let $a, b \in C_0(\mathbb{R})$ and an one-dimensional operator T_1 be defined on the space $X(\mathbb{R})$ by

$$(T_1 f)(x) = a(x) \int_{\mathbb{R}} b(y) f(y)\, dy.$$

Then there exists a function $c \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ such that

$$T_1 = aW^0(c)bI.$$

Proof of the main result

1. $X(\mathbb{R})$ has the approximation property, that is, every compact operator can be approximated in the norm by finite rank operators.

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2. The Banach space dual $[X(\mathbb{R})]^*$ of the space $X(\mathbb{R})$ is canonically isometrically isomorphic to the associate space $X'(\mathbb{R})$.

Proof of the main result

1. $X(\mathbb{R})$ has the approximation property, that is, every compact operator can be approximated in the norm by finite rank operators.

2. The Banach space dual $[X(\mathbb{R})]^*$ of the space $X(\mathbb{R})$ is canonically isometrically isomorphic to the associate space $X'(\mathbb{R})$.

3. Hence a finite rank operator on $X(\mathbb{R})$ is of the form

$$(T_m f)(x) = \sum_{j=1}^{m} a_j(x) \int_{\mathbb{R}} b_j(y) f(y)\, dy, \quad x \in \mathbb{R},$$

   where $a_j \in X(\mathbb{R})$ and $b_j \in X'(\mathbb{R})$ for $j \in \{1, \ldots, m\}$ and some $m \in \mathbb{N}$.

Proof of the main result (continued)

4. Since the space $X(\mathbb{R})$ is reflexive, the set $C_0(\mathbb{R})$ of all continuous compactly supported functions is dense in $X(\mathbb{R})$ and in $X'(\mathbb{R})$.

Proof of the main result (continued)

4. Since the space $X(\mathbb{R})$ is reflexive, the set $C_0(\mathbb{R})$ of all continuous compactly supported functions is dense in $X(\mathbb{R})$ and in $X'(\mathbb{R})$.

5. For every $\varepsilon \in (0,1)$ and every $j \in \{1, \dots, m\}$, there exist $a_{j,\varepsilon}, b_{j,\varepsilon} \in C_0(\mathbb{R})$ such that

$$\left| \|a_j\|_{X(\mathbb{R})} - \|a_{j,\varepsilon}\|_{X(\mathbb{R})} \right| < 1$$

and

$$\|a_j - a_{j,\varepsilon}\|_{X(\mathbb{R})} < \frac{\varepsilon}{2m(\|b_j\|_{X'(\mathbb{R})} + 1)},$$

$$\|b_j - b_{j,\varepsilon}\|_{X'(\mathbb{R})} < \frac{\varepsilon}{2m(\|a_j\|_{X(\mathbb{R})} + 1)}.$$

Proof of the main result (continued)

6. Let $T_{m,\varepsilon}$ denote the operator defined by

$$(T_{m,\varepsilon}f)(x) = \sum_{j=1}^{m} a_{j,\varepsilon}(x) \int_{\mathbb{R}} b_{j,\varepsilon}(y) f(y)\, dy, \quad x \in \mathbb{R}.$$

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6. Let $T_{m,\varepsilon}$ denote the operator defined by

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7. By Hölder's inequality $f \in X(\mathbb{R})$,

$$\begin{aligned} &\|T_m f - T_{m,\varepsilon} f\|_{X(\mathbb{R})} \\ &\quad \le \sum_{j=1}^{m} \|a_j - a_{j,\varepsilon}\|_{X(\mathbb{R})} \|b_j\|_{X'(\mathbb{R})} \|f\|_{X(\mathbb{R})} \\ &\qquad + \sum_{j=1}^{m} \|a_{j,\varepsilon}\|_{X(\mathbb{R})} \|b_j - b_{j,\varepsilon}\|_{X'(\mathbb{R})} \|f\|_{X(\mathbb{R})} \\ &\quad < \varepsilon \|f\|_{X(\mathbb{R})}, \end{aligned}$$

whence $\|T_m - T_{m,\varepsilon}\| \le \varepsilon$.

Proof of the main result (continued)

8. Therefore, each compact operator on the space $X(\mathbb{R})$ can be approximated in the operator norm by a finite sum of rank one operators T_1 of the form

$$(T_1 f)(x) = a(x) \int_{\mathbb{R}} b(y) f(y)\, dy$$

with $a, b \in C_0(\mathbb{R})$.

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9. By the previous lemma, each such operator can be written in the form

$$T_1 = aW^0(c)bI$$

with $c \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$. Hence $T_1 \in \mathcal{C}_{X(\mathbb{R})}$, which completes the proof.

Thank you very much!

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