An update on the classification program of (maximal) ideals of algebras of operators on Banach spaces: *the cases of Tsirelson and Schreier spaces*.

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joint work with K. Beanland & N. J. Laustsen

Overview

$\mathcal{B}(X)$ the Banach algebra of all bdd ops on a B. space X.

Goal: to understand the lattice of **closed** ideals ($\cong$ representations) of $\mathcal{B}(X)$.

This is an isomorphic problem due to Eidelheit's thm (1940).

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- $Z = X_{\mathrm{AH}} \oplus$ suitably constructed subspace (K.–Laustsen).

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This behaviour is rather rare.

$$\mathcal{M}_X = \{T \in \mathcal{B}(X) \colon I_X \neq ATB \ (A, B \in \mathcal{B}(X))\}$$

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- c_0, ℓ_p (here $p = \infty$ is included, btw. $\ell_\infty \cong L_\infty$);
- $L_p[0,1]$ for $p \in [1, \infty]$.
- $c_0(\Gamma), \ell_p(\Gamma)$ for $p \in [1, \infty)$
- ℓ_∞ / c_0, $\ell_\infty^c(\Gamma)$ for any set Γ (but not every $L_\infty(\mu)$ is in this class!)
- c_0- and ℓ_p-sums of ℓ_2^ns or ℓ_∞^ns as well as more general sums.
- Lorentz sequence spaces
  determined by a decreasing, non-summable sequence and $p \in [1, \infty)$.
- certain Orlicz spaces.
- $C[0,1]$, $C[0, \omega^\omega]$, $C[0, \omega_1]$, and the list goes on.

Tsirelson space revisited (Figiel–Johnson)

Put a norm on c_{00}:

$$\|x\|_T = \max\left\{\|x\|_{\ell_\infty}, \frac{1}{2}\sup\sum_i \|N_i x\|_T\right\}$$

where the sup runs over $j \in \mathbb{N}$ and all finite sequences of sets $N_1 < \cdots < N_j$ in $\mathbb{N}$ with $j \leqslant \min N_1$.

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A chain Γ of spatial ideals either stabilises, so that $\overline{\bigcup \Gamma} \in \Gamma$, or the ideal $\overline{\bigcup \Gamma}$ is not spatial. In T, $\mathrm{im}\, P_N \cong \mathrm{im}\, P_M \iff (t_j)_{j\in N}, (t_j)_{j\in M}$ are equivalent.

Tsirelson space is now classical, isn't it?

Theorem (Beanland–K.–Laustsen, 2019+). *Let T be the (dual of the original) Tsirelson space.*

1. *The family of non-trivial spatial ideals of $\mathcal{B}(T)$ is non-empty and has no minimal or maximal elements.*
2. *Let $\mathcal{I} \hookrightarrow \mathcal{J}$ be spatial ideals of $\mathcal{B}(T)$. Then there is a family $\{\Gamma_L : L \in \Delta\}$ such that:*
   - $|\Delta| = \mathfrak{c}$;
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Theorem, ctd. The ideals of compact, strictly singular, and inessential operators on T coincide, and they are equal to the intersection of the non-trivial spatial ideals of $\mathcal{B}(T)$:

$$\mathcal{K}(T) = \mathcal{S}(T) = \mathcal{E}(T) = \bigcap \{\mathcal{I} : \mathcal{I} \text{ is a non-trivial spatial ideal of } \mathcal{B}(T)\}.$$

How to decide if two subsequences of (t_n) are equivalent?

For $M = \{m_1 < m_2 < \cdots\} \in [\mathbb{N}]$ and $J \in [\mathbb{N}]^{<\infty}$, let

$$\sigma(M, J) = \sup\Big\{\sum_{j\in J} \alpha_j : \alpha_j \in [0,1], \Big\|\sum_{j\in J} \alpha_j t_{m_j}\Big\|_T \leqslant 1\Big\}, \qquad \sigma(M, \varnothing) = 0.$$

For $N = \{n_1 < n_2 < \cdots\} \in [\mathbb{N}]$, set $m_0 = n_0 = 0$.

Theorem (Casazza–Johnson–Tzafriri) $(t_j)_{j\in M} \sim (t_j)_{j\in N}$ if and only if

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Key lemma The following conditions are equivalent for infinite $M \subseteq N \subseteq \mathbb{N}$:

1. $P_N \in \overline{\langle P_M \rangle}$;
2. $\langle P_M \rangle = \langle P_N \rangle$;
3. T_N is isomorphic to a complemented subspace of T_M;
4. T_N is isomorphic to T_M;
5. $(t_j)_{j\in M}$ is equivalent to $(t_j)_{j\in N}$;
6. there is a constant $C \geqslant 1$ such that $\sigma(N, J) \leqslant C$ for each interval J in N with $J \cap M = \emptyset$.

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$\mathcal{S}_0 = \big\{\{k\} : k \in \mathbb{N}\big\} \cup \{\emptyset\}$, and for $n \in \mathbb{N}_0$, recursively define

$$\mathcal{S}_{n+1} = \left\{\bigcup_{i=1}^{k} E_i : k \in \mathbb{N},\ E_1, \ldots, E_k \in \mathcal{S}_n \setminus \{\varnothing\},\ k \leqslant \min E_1,\ E_1 < E_2 < \cdots < E_k\right\} \cup \{\varnothing\}.$$

The *Schreier space of order n*, $X[\mathcal{S}_n]$, is the completion of c_{00} w.r.t.

$$\|x\| = \sup\left\{\sum_{j\in E} |\alpha_j| : E \in \mathcal{S}_n \setminus \{\varnothing\}\right\} \qquad \big(x = (\alpha_j)_{j=1}^{\infty} \in c_{00}\big).$$

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$\mathcal{S}_n$ is *spreading*: let $J = \{j_1 < j_2 < \cdots < j_m\}$, $K = \{k_1 < k_2 < \cdots < k_m\} \subset \mathbb{N}$. If K is a *spread* of J; that is, $j_i \leqslant k_i$ for each $i \leqslant m$, then $J \in \mathcal{S}_n \Rightarrow K \in \mathcal{S}_n$.

A word on Schreier spaces

$\mathcal{S}_0 = \big\{\{k\} : k \in \mathbb{N}\big\} \cup \{\emptyset\}$, and for $n \in \mathbb{N}_0$, recursively define

$$\mathcal{S}_{n+1} = \Big\{\bigcup_{i=1}^{k} E_i : k \in \mathbb{N},\ E_1, \ldots, E_k \in \mathcal{S}_n \setminus \{\varnothing\},\ k \leqslant \min E_1,\ E_1 < E_2 < \cdots < E_k\Big\} \cup \{\varnothing\}.$$

The *Schreier space of order n*, $X[\mathcal{S}_n]$, is the completion of c_{00} w.r.t.

$$\|x\| = \sup\Big\{\sum_{j\in E} |\alpha_j| : E \in \mathcal{S}_n \setminus \{\varnothing\}\Big\} \qquad \big(x = (\alpha_j)_{j=1}^{\infty} \in c_{00}\big).$$

$\mathcal{S}_n$ is *spreading*: let $J = \{j_1 < j_2 < \cdots < j_m\}$, $K = \{k_1 < k_2 < \cdots < k_m\} \subset \mathbb{N}$. If K is a *spread* of J; that is, $j_i \leqslant k_i$ for each $i \leqslant m$, then $J \in \mathcal{S}_n \Rightarrow K \in \mathcal{S}_n$.

Theorem (Beanland–K.–Laustsen, 2019+). *Let $n \geqslant 1$.*

1. *The family of non-trivial spatial ideals of $\mathcal{B}(X[\mathcal{S}_n])$ has no min/max el^s.*
2. *Let $\mathcal{I} \hookrightarrow \mathcal{J}$ be spatial ideals of $\mathcal{B}(X[\mathcal{S}_n])$. Then there is $\{\Gamma_L : L \in \Delta\}$ s.t.:*
   - *$|\Delta| = \mathfrak{c}$;*
   - *for each $L \in \Delta$, Γ_L is an uncountable chain of spatial ideals of $\mathcal{B}(X[\mathcal{S}_n])$ such that*
     $$\mathcal{I} \hookrightarrow \mathcal{L} \hookrightarrow \mathcal{J} \qquad (\mathcal{L} \in \Gamma_L),$$
     and $\bigcup \Gamma_L$ is a closed ideal that is not spatial;
   - *$\overline{\mathcal{L} + \mathcal{M}} = \mathcal{J}$ ($\mathcal{L} \in \Gamma_L$ and $\mathcal{M} \in \Gamma_M$, $L, M \in \Delta, L \neq M$).*
3. *The Banach algebra $\mathcal{B}(X[\mathcal{S}_n])$ contains at least $\mathfrak{c}$* ***many maximal ideals****.*

A way to distinguish isomorphism types

Let $X = X[\mathcal{S}_n]$ for some $n \in \mathbb{N}$, and suppose that $M, N \in [\mathbb{N}]$ satisfy $P_M \in \overline{\langle P_N \rangle}$. Then the following conditions are equivalent:

1. $P_N \in \overline{\langle P_M \rangle}$;
2. $\langle P_M \rangle = \langle P_N \rangle$;
3. X_M is isomorphic to X_N;
4. X_N is isomorphic to a subspace of X_M;
5. the n^{th} Gasparis–Leung index $d_n(M, N) = \sup\{\tau_n\big(M(J)\big) : J \in [\mathbb{N}]^{<\infty},\ N(J) \in \mathcal{S}_n\}$ is finite;
6. there is a constant $k \in \mathbb{N}$ such that $\tau_n(N(J)) \leqslant k$ for each set $J \in [\mathbb{N} \cap (k, \infty)]^{<\infty}$,

where

$$\tau_n(J) = \min\Big\{k \in \mathbb{N} : J \subseteq \bigcup_{i=1}^{k} E_i, \text{ where } E_1, \ldots, E_k \in \mathcal{S}_n \text{ and } E_1 < E_2 < \cdots < E_k\Big\}.$$

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Muchas gracias!