# Grothendieck inequalities

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This talk is based mainly on a recent joint work with Jan Hamhalter, Antonio M. Peralta and Herman Pfitzner contained in the papers:

[HKPP1] J.Hamhalter, O.Kalenda, A.Peralta and H.Pfitzner: *Measures of weak non-compactness in preduals of von Neumann algebras and JBW*-triples*, arXiv:1901.08056, to appear in J. Funct. Anal.

[HKPP2] J.Hamhalter, O.Kalenda, A.Peralta and H.Pfitzner: *Grothendieck's inequalities for JB*-triples: Proof of the Barton-Friedman conjecture*, arXiv:1903.08931

# Plan

1. Grothendieck inequalities – historical introduction
2. JB*-triples – definitions, examples, properties
3. Grothendieck inequalities for JB*-triples
4. Optimal constants in the Little Grothendieck Theorem
5. Strong* topology and strongly WCG spaces

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# Grothendieck inequalities – the origin

**Theorem [Grothendieck 1956]**

There is $\kappa_G > 0$ such that whenever $V : C(K_1) \times C(K_2) \to \mathbb{F}$ is a bounded bilinear form, there are probability measures $\mu_1, \mu_2$ on $K_1$, $K_2$ such that

$$|V(f,g)| \leq \kappa_G \|V\| \left( \int_{K_1} |f|^2 d\mu_1 \right)^{\frac{1}{2}} \left( \int_{\Omega_2} |g|^2 d\mu_2 \right)^{\frac{1}{2}}$$

for $f \in C(K_1)$, $g \in C(K_2)$.

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for $f \in C(K_1)$, $g \in C(K_2)$.

**Remark**

The best value of $\kappa_G$ is called Grothendieck constant. The exact value is not known.

# The case of C*-algebras

**Theorem [Pisier 1978 & Haagerup 1985]**

$A_1, A_2$ C*-algebras, $V : A_1 \times A_2 \to \mathbb{C}$ a bounded bilinear form $\Rightarrow$ there are states $\varphi_1 \in A_1^*$, $\varphi_2 \in A_2^*$ such that

$$|V(x,y)| \leq 4\,\|V\|\,\varphi_1\left(\frac{xx^* + x^*x}{2}\right)^{\frac{1}{2}} \varphi_2\left(\frac{yy^* + y^*y}{2}\right)^{\frac{1}{2}}$$

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**Remarks**

- $\varphi \in A^*$ is a state if $\varphi \geq 0$ and $\|\varphi\| = 1$.

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### Remarks

- $\varphi \in A^*$ is a state if $\varphi \geq 0$ and $\|\varphi\| = 1$.
- States on $C(K)$ are exactly probability measures.

# An easier version - Little Grothedieck Theorem

**Theorem [Grothendieck 1956]**

There is $k_G > 0$ such that whenever $T : C(K) \to H$ is a bounded linear operator (where $H$ is a Hilbert space), then there is a probability $\mu$ on $K$ such that

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The optimal value of $k_G$ is $\sqrt{\frac{\pi}{2}}$ in the real case and $\frac{2}{\sqrt{\pi}}$ in the complex case.

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**Question:** Is the constant 2 optimal?
**Easy:** The optimal constant is from $[\sqrt{2}, 2]$.

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## A drawback of the argument

The proof of the key step uses GT.

# LGT – a dual version

### Theorem

$M$ von Neumann algebra, $H$ Hilbert space, $T : M \to H$ w*-to-w continuous linear operator $\Rightarrow$ there is a normal state $\varphi \in M_*$ such that

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## Remark

One way of proving LGT is to prove first the dual version and then to apply it to $T^{**} : A^{**} \to H$.

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## Definition

$M$ von Neumann algebra. The strong* topology on $M$ is generated by seminorms $x \mapsto \varphi\left(\frac{x^*x+xx^*}{2}\right)^{1/2}$, $\varphi \in M_*$ state.

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LGT shows that any w*-to-w continuous linear operator $T : M \to H$ is strong*-to-norm continuous in a precise way.

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$\Rightarrow T|_{B_M}$ is strong*-to-norm continuous [Akemann 1967]

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# JB*-triples – definition

A JB*-triple is a complex Banach space $E$ equipped with a continuous triple product $\{\cdot,\cdot,\cdot\} : E^3 \to E$ satisfying the following conditions:

1. $\{\cdot, y, \cdot\}$ is a symmetric bilinear mapping,
2. $\{x, \cdot, z\}$ is conjugate-linear,
3. $\{a, b, \{x, y, z\}\} =$
   $\{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
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**One of the motivations:** Characterization of complex Banach spaces whose unit balls are bounded symmetric domains. [Kaup 1983]

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- a closed subspace of a C*-algebra stable under the above triple product is a JB*-triple;
- $C(K, H_3(\mathbb{O}))$ (and its subtriples);
  ($\mathbb{O}$ = complex octonions, $\dim \mathbb{O} = 8$,
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- any JB*-triple is of the form $E_s \oplus_\infty E_e$, where
  $E_s$ (special) is a subtriple of a C*-algebra
  and $E_e$ (exceptional) is a subtriple of $C(K, H_3(\mathbb{O}))$.

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# JB*-triples and JBW*-triples – properties

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- $E$ JB*-triple $\Rightarrow$ $E^{**}$ is a JBW*-triple, the triple product on $E^{**}$ extends that on $E$.

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- $u$ is complete if $\{u, u, x\} = 0 \Rightarrow x = 0$.

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# Tripotents and partial isometries

## $E$ is a JB*-triple

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# Plan

# Barton-Friedman conjecture

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(LGT) Let $E$ be a JB*-triple, $H$ a Hilbert space $T : E \to H$ a bounded linear operator. Then there is $\varphi \in E^*$, $\|\varphi\| = 1$ with

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Remark

- [Peralta 2001] The proof contains a gap. The proof of (LGT) works only if $T^{**}$ attains its norm. (Similarly for (GT).)
- No counterexample to the statement itself has been found.

# A try to fix it – approximate version

**Theorem** [Peralta & Rodríguez-Palacios 2001]

(LGT) Let $E$ be a JB*-triple, $H$ a Hilbert space $T : E \to H$ a bounded linear operator and $\varepsilon > 0$. Then there are $\varphi_1, \varphi_2 \in E^*$, $\|\varphi_1\| = \|\varphi_2\| = 1$ with

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## The main result [HKPP2]

$K = \sqrt{2}$ works.

## Step 1 – representation of JBW*-triples [Horn 1987], [Horn & Neher 1988], [HKPP1]

Any JBW*-triple is isometrically isomorphic to

$$\Big(\bigoplus_{j\in J} L^\infty(\mu_j, C_j) \oplus N \oplus pV \oplus qW\Big)_{\ell^\infty}$$

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- $V$ and $W$ are von Neumann algebras;
- $p \in V$ is a finite projection;
- $q \in W$ is a properly infinite projection.

# Ingredients of the proof II

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$$\{x \in V_2; 0 \le x \le 1 \& T(x) = T(p_2)\}$$

are only projections.

# Conclusion

**Theorem [HKPP2]**

(LGT) Let $E$ be a JB*-triple, $H$ a Hilbert space $T : E \to H$ a bounded linear operator and $\varepsilon > 0$. Then there is $\varphi \in E^*$, $\|\varphi\| = 1$ with

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(GT) Let $E, F$ be JB*-triples, $V : E \times F \to \mathbb{C}$ a bounded bilinear form and $\varepsilon > 0$. Then there are $\varphi \in E^*$, $\psi \in F^*$, $\|\varphi\| = \|\psi\| = 1$ such that

$$|V(x, y)| \leq (8 + 16\sqrt{2} + \varepsilon) \|V\| \|x\|_\varphi \|y\|_\psi .$$

# Plan

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- **Conjecture:** Optimal constant should be $\sqrt{2} + \varepsilon$.

# LGT for C*-algebras

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# Plan

1. Grothendieck inequalities – historical introduction

2. JB*-triples – definitions, examples, properties

3. Grothendieck inequalities for JB*-triples

4. Optimal constants in the Little Grothendieck Theorem

5. Strong* topology and strongly WCG spaces

# On seminorms generating the strong* topology

**Recall**

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**Proposition [HKPP1]**

Let $M$ be a JBW*-triple.
The topologies generated by $\|\cdot\|_\varphi$, $\varphi \in M_*$, on $B_M$ are upwards $\sigma$-directed.

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**Theorem [Bohata & HKPP 2018]**

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### Example [HKPP1]

$\Gamma$ uncountable $\Rightarrow$ $M = B(\ell^2, \ell^2(\Gamma))$ is a $\sigma$-finite JBW* triple,
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Theorem [Horn 1987], [Horn & Neher 1988], [HKPP1]

Any JBW*-triple is isometrically isomorphic to

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where

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(In this case the summand $qW$ may be omitted.)

Thank you for your attention.