Ordered Banach spaces: embeddings and disjointness preserving operators

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Disjointness preserving operators - motivation

In **Banach lattices**:

Theorem (Huijsmans, de Pagter, 1994, Koldunov 1995)

If X is a Banach lattice and $T\colon X \to X$ is a disjointness preserving bijection, then T^{-1} is disjointness preserving.

Theorem (Arendt, 1986)

Let X be a Banach lattice and let $T\colon [0,\infty) \to \mathcal{L}(X)$ be a C_0-semigroup with generator $A\colon X \supseteq \mathcal{D}(A) \to X$ such that for every $t \in [0,\infty)$ the operator $T(t)$ is disjointness preserving. Then A is band preserving.

In **ordered Banach spaces**: Only some first (partial) results on disjointness preserving operators, so far. Active field of research!

Joint work with Onno van Gaans, Universiteit Leiden, Netherlands

Outline

Partially ordered vector spaces

Let X be a (real) vector space. A partial order $\leq$ on X is called a vector space order if

(a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

(b) $x \in X$, $0 \leq x$ and $\lambda \in [0, \infty)$ imply $0 \leq \lambda x$.

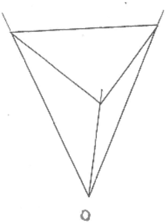
The set $K := \{x \in X; 0 \leq x\}$ is then a cone in X, i.e. $x, y \in K$, $\lambda \in [0, \infty)$ imply $\lambda x + y \in K$, and $K \cap (-K) = \{0\}$.

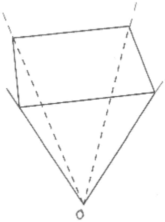
We denote a partially ordered vector space (povs) by (X, K).

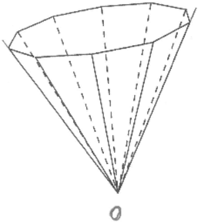
(X, K) is called Archimedean if for every $x, y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$.
(X, K) is directed if and only if K is generating, i.e. $X = K - K$.

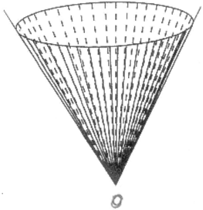
For $M \subseteq X$, denote by M^{u} the set of upper bounds of M, and by M^{l} the set of lower bounds of M.

Cones in $\mathbb{R}^3$ – from lattice to anti-lattice



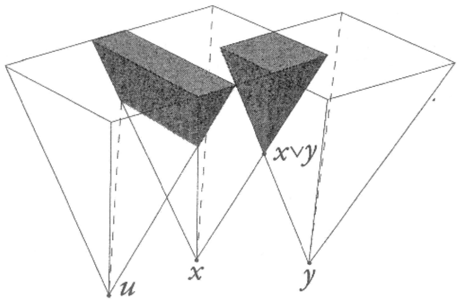






Four-ray cone in $\mathbb{R}^3$

The supremum of x and y exists, the supremum of x and u does not exist.



Disjointness - motivation

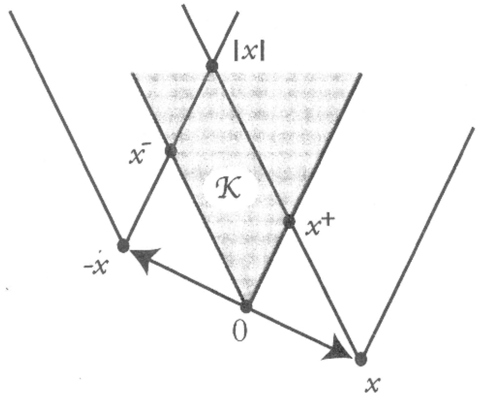
If X is a vector lattice, then $x, y \in X$ are called disjoint ($x \perp y$) if $|x| \wedge |y| = 0$, which is equivalent to $|x + y| = |x - y|$.

In function spaces (that are not vector lattices, in general, e.g. $\mathrm{C}^1[0, 1]$), we want that two functions are disjoint if and only if they have disjoint supports.

In the literature there were suggestions for disjointness in povs, e.g. (for x, y positive) $x \perp y$ if $[0, x] \cap [0, y] = \{0\}$, but then disjoint complements are not convex, in general.

We need another idea ..

Replace modulus $|x|$ by $\{x, -x\}^{\mathrm{u}}$



Structures in partially ordered vector spaces

Let (X, X_+) be a povs.

Disjointness: [van Gaans, K. 2006]

$$x \perp y \quad :\Longleftrightarrow \quad \{x+y, -(x+y)\}^{\mathrm{u}} = \{x-y, -(x-y)\}^{\mathrm{u}}$$

$M \subseteq X$ is called

- solid, if $\forall x \in X$, $m \in M$ with $\{x, -x\}^{\mathrm{u}} \supseteq \{m, -m\}^{\mathrm{u}}$ one has $x \in M$ [van Gaans, 1999]
- ideal, if M is a solid subspace of X.
- band, if $M = M^{\mathrm{dd}}$.

To prove properties of these structures, one needs an appropriate embedding of the povs into a vector lattice (and a theory how the structures in the povs and the ambient vector lattice are related).

Embeddings for povs

We discuss two classical embeddings for povs into vector lattices,

- the Dedekind completion for Archimedean directed povs, and
- the functional representation for order unit spaces,

and the generalized framework

- vector lattice cover and Riesz completion for pre-Riesz spaces.

Dedekind completion

Let (X, K) be a povs and let

$$X^\delta := \{A \subseteq X;\ A^{\mathrm{ul}} = A\} \setminus \{\emptyset, X\}$$

be ordered by inclusion.
If (X, K) is directed, then X^δ is a Dedekind complete lattice. The map

$$J\colon X \to X^\delta, \quad x \mapsto \{x\}^{\mathrm{l}}$$

is bipositive and $J[X]$ is order dense in X^δ, i.e.

$$\forall y \in X^\delta\colon\ y = \inf\{J(x);\, x \in X,\ J(x) \geq y\}.$$

For $A, B \in X^\delta$ and $\lambda \in \mathbb{R}$ define $A \oplus B = (A + B)^{\mathrm{ul}}$, $\ominus A = -A^{\mathrm{u}}$,

$$\lambda * A = \begin{cases} \lambda A, & \lambda > 0, \\ \{0\}^{\mathrm{l}}, & \lambda = 0, \\ \lambda A^{\mathrm{u}}, & \lambda < 0. \end{cases}$$

Lemma

For every $A \in X^\delta$ we have

$$A \oplus \ominus A = \{0\}^{\mathrm{l}}$$

if and only if X is Archimedean.

In this case, X^δ endowed with addition $\oplus$, scalar multiplication $*$ and partial order $\subseteq$ is a Dedekind complete vector lattice, and the embedding map $J\colon X \to X^\delta$ is linear.

For which povs does an order dense embedding into a (not necessarily Dedekind complete) vector lattice exist?

$$X^{\rho} := \left\{ \bigvee_{i=1}^{m} \{a_i\}^{\mathrm{l}} \oplus \ominus \bigvee_{j=1}^{n} \{b_j\}^{\mathrm{l}} :\ a_1, \ldots, a_m, b_1, \ldots, b_n \in X,\ m, n \in \mathbb{N} \right\}.$$

Proposition (van Haandel, 1993)

Let X be a directed partially ordered vector space. For every $a_1, \ldots, a_m \in X$ and $A = \bigvee_{i=1}^{m} \{a_i\}^{\mathrm{l}}$ the identity

$$A \oplus \ominus A = \{0\}^{\mathrm{l}}$$

holds true if and only if X is a pre-Riesz space.

Pre-Riesz spaces

Definition (van Haandel, 1993)

A povs X is called a pre-Riesz space if $\forall x, y, z \in X$ with $\{x+y, x+z\}^{\mathrm{u}} \subseteq \{y, z\}^{\mathrm{u}}$ one has $x \geq 0$.

Every vector lattice is a pre-Riesz space.

Proposition

- *Every Archimedean directed povs is a pre-Riesz space.*
- *Every pre-Riesz space is directed.*

Examples of pre-Riesz spaces:

- ordered Banach spaces with closed generating cones
- order unit spaces

Theorem (van Haandel, 1993)

Let X be a partially ordered vector space. The following statements are equivalent:

(i) *X is a pre-Riesz space.*

(ii) *There exist a Riesz space Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y.*

(iii) *There exist a Riesz space Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y and generates Y as a vector lattice, i.e.*

$$Y = \left\{ \bigvee_{i=1}^{m} i(a_i) - \bigvee_{j=1}^{n} i(b_j)\colon \; a_1, \ldots, a_m, b_1, \ldots, b_n \in X, \; m, n \in \mathbb{N} \right\}.$$

Moreover, all Riesz spaces Y as in (iii) are isomorphic as Riesz spaces.

Vector lattice covers and the Riesz completion

In van Haandel's theorem,

- a pair (Y, i) as in (ii) is called a vector lattice cover of X, and
- a pair (Y, i) as in (iii) the Riesz completion of X.

In particular, if X is Archimedean and directed, then X^δ is a vector lattice cover of X, and X^ρ is a representation of the Riesz completion (but in examples there are usually more convenient ones).

Order unit spaces

An element $u > 0$ is called an order unit if for every $x \in X$ there is a $\lambda \in (0, \infty)$ such that $x \in [-\lambda u, \lambda u]$.

If X contains an order unit, then X is directed.

For a fixed order unit u in X, a seminorm on X is introduced by

$$\|\cdot\|_u : X \to [0, \infty), \quad x \mapsto \|x\|_u := \inf\{\lambda \in (0, \infty);\ -\lambda u \le x \le \lambda u\}.$$

If (X, K) is an Archimedean povs with order unit u, then $\|\cdot\|_u$ is a norm. Such spaces are called order unit spaces.

Examples of order unit spaces

- $X = \mathrm{C}(\Omega)$ with point-wise order, where Ω is a compact Hausdorff space.
- If X is a finite-dimensional povs with closed generating cone, then X is an order unit space.
- For a directed povs (X, K) let $L_+(X) := \{T \in L(X); T[K] \subseteq K\}$.
  Then $(L(X), L_+(X))$ is a povs.
  If X is a finite-dimensional order unit space, then $L(X)$ is an order unit space as well.
- Lorentz cone in $\mathbb{R} \times H$, where H is a Hilbert space

Two classical embeddings of a povs into a vector lattice:

1. **Dedekind completion**:
   If (X, K) is an Archimedean directed povs, then there is a Dedekind complete vector lattice Y and a linear bipositive map $J \colon X \to Y$ such that $J[X]$ is order dense in Y.
2. **Functional representation** (Kadison, 1951):
   If (X, K) is an order unit space, then there is a compact Hausdorff space Ω and a linear bipositive map $\Phi \colon X \to \mathrm{C}(\Omega)$.

Is the functional representation of X a vector lattice cover?

Construction of the functional representation

Let (X, K) be an Archimedean povs with order unit u, equipped with the u-norm $\|\cdot\|_u$. X' denotes the (norm) dual space of X and $K' := \{\varphi \in X';\ \varphi[K] \subseteq [0, \infty)\}$ the dual cone. The set

$$\Sigma = \{\varphi \in K';\ \varphi(u) = 1\}$$

is a base of K' (i.e. Σ is convex and every $\psi \in K'$ has a unique representation $\psi = \lambda\varphi$ with $\varphi \in \Sigma$ and $\lambda \in [0, \infty)$).

- By the Banach-Alaoglu theorem, the closed unit ball B' of X' is weakly-$*$ compact.
- As Σ is a weakly-$*$ closed subset of B', Σ is weakly-$*$ compact in X', i.e. Σ equals the weak-$*$ closure of the convex hull of the extreme points of Σ by the Krein-Milman theorem.

Denote the set of all extreme points of Σ by

$$\Lambda := \mathrm{ext}(\Sigma).$$

(In general, Λ need not be weakly-$*$ closed, not even if X is finite dimensional.)
Denote by $\overline{\Lambda}$ the weak-$*$ closure of Λ in Σ, hence $\overline{\Lambda}$ is a compact Hausdorff space. Define

$$\Phi\colon X \to \mathrm{C}(\overline{\Lambda}), \qquad x \mapsto (\varphi \mapsto \varphi(x)).$$

- Φ is linear and maps u to the constant-1 function
- Let $x \in X$. Then $x \in K$ if and only if for every $\varphi \in \Lambda$ one has $\varphi(x) \geq 0$.
  Consequently, Φ is bipositive.

Denote by $\mathrm{L}(X,K)$ the Riesz subspace generated by $\Phi[X]$ inside $\mathrm{C}(\overline{\Lambda})$, i.e.

$$\mathrm{L}(X,K)=\left\{\bigvee_{j=1}^{m}x_j-\bigvee_{j=1}^{n}y_j;\ x_1,\ldots,x_m,y_1,\ldots,y_n\in\Phi[X]\right\}.$$

The Stone-Weierstrass theorem yields that $\mathrm{L}(X,K)$ is norm dense in $\mathrm{C}(\overline{\Lambda})$.

Functional representation as a vector lattice cover

Theorem (Lemmens, van Gaans, K., 2014)

If (X, K) is an Archimedean povs with an order unit u, then $\Phi[X]$ is order dense in $\mathrm{C}(\overline{\Lambda})$, i.e. $(\mathrm{C}(\overline{\Lambda}), \Phi)$ is a vector lattice cover of X and $\mathrm{L}(X, K)$ is the Riesz completion of (X, K).

Note: This is convenient in examples, since the ambient vector lattice has the standard point-wise order!

Example: Polyhedral cone

Let $X = \mathbb{R}^n$ and let $x^{(1)}, \ldots, x^{(r)} \in X$ be such that

$$K := \operatorname{pos}\{x^{(1)}, \ldots, x^{(r)}\}$$

is generating. K is closed and hence Archimedean.
Fix $u \in \operatorname{int}(K)$, then u is an order unit.
$\Sigma = \{f \in K';\ f(u) = 1\}$ has finitely many extreme points $f^{(1)}, \ldots, f^{(k)}$, where $k \geq n$, i.e.

$$\overline{\Lambda} = \Lambda = \{f^{(1)}, \ldots, f^{(k)}\},$$

$$\Phi\colon \mathbb{R}^n \to \mathbb{R}^k, \quad x \mapsto (f^{(1)}(x), \ldots, f^{(k)}(x))^{\mathrm{T}}.$$

The cone K has the representation

$$K = \{x \in \mathbb{R}^n;\ \forall i \in \{1, \ldots, k\}\colon f^{(i)}(x) \geq 0\}\,.$$

Example: Lorentz cone

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $X = \mathbb{R} \times H$ be ordered by means of the Lorentz cone

$$\mathrm{L}_H = \{(r,z) \in \mathbb{R} \times H;\ r^2 - \langle z,z\rangle \geq 0 \text{ and } r \geq 0\}$$

and endowed with the inner product $\langle x|y\rangle = rs + \langle z,v\rangle$, where $x = (r,z)$ and $y = (s,v)$, which turns X into a Hilbert space.
The point $u = (1,0)$ is an interior point of L_H, i.e. u is an order unit.
As L_H is self-dual we get

$$\Sigma = \{(1,z) \in \mathrm{L}_H\},$$

$$\overline{\Lambda} = \Lambda = \{(1,z) \in \mathrm{L}_H;\ \|z\| = 1\}$$

and the order dense embedding $\Phi\colon (X,\mathrm{L}_H) \to \mathrm{C}(\Lambda)$.

Riesz* homomorphisms

For vector lattices X and Y, Riesz homomorphisms $h\colon X \to Y$ are considered, i.e. linear maps with $h(x \vee y) = h(x) \vee h(y)$ for all $x, y \in X$. There are two generalizations:

Let X, Y be povs. A linear map $h\colon X \to Y$ is called a

- Riesz homomorphism if for every $x, y \in X$ one has

  $$h\left[\{x,y\}^{\mathrm{u}}\right]^{\mathrm{l}} = h[\{x,y\}]^{\mathrm{ul}}$$

  [Buskes, van Rooij, 1993], and a
- Riesz* homomorphism, if for every nonempty finite subset F of X one has

  $$h\left[F^{\mathrm{ul}}\right] \subseteq h[F]^{\mathrm{ul}}$$

  [van Handel, 1993].

- Every Riesz homomorphism is a Riesz* homomorphism, and every Riesz* homomorphism is positive.
- If X and Y are Riesz spaces, then the notion of Riesz homomorphism coincides with that of vector lattice theory, and also with the notion of Riesz* homomorphism.

Theorem (van Haandel, 1993)

Let X_1 and X_2 be pre-Riesz spaces, let (Y_1, i_1) be the Riesz completion of X_1 and let (Y_2, i_2) be a vector lattice cover of X_2. For a linear map $h\colon X_1 \to X_2$, the following statements are equivalent:

(i) *h is a Riesz* homomorphism.*

(ii) *There exists a Riesz homomorphism $\hat{h}\colon Y_1 \to Y_2$ with $i_2 \circ h = \hat{h} \circ i_1$.*

We consider again an order unit space X with order unit u and Σ, Λ defined as above.

Proposition

Let (X, K) be an Archimedean povs with order unit u, and let $\varphi \in \Sigma$.

(i) *One has $\varphi \in \Lambda$ if and only if φ is a Riesz homomorphism. [Hayes's theorem]*

(ii) *One has $\varphi \in \overline{\Lambda}$ if and only if φ is a Riesz* homomorphism. [van Haandel, 1993]*

Disjointness, ideals and bands under embedding

- Under which conditions can disjointness, ideals and bands be extended or restricted, respectively, between a pre-Riesz space X and its vector lattice cover Y?
- Which properties do ideals and bands have in povs? What are the differences to the classical vector lattice theory?

Disjointness under embedding

Proposition (van Gaans, K., 2006)

Let X be a pre-Riesz space and (Y, i) a vector lattice cover of X. Then one has for every $x, y \in X$

$$x \perp y \iff i(x) \perp i(y).$$

Order denseness is needed for '$\Longrightarrow$'.

Proposition

For $M \subseteq X$ one has $M^{\mathrm{d}} = \left[i[M]^{\mathrm{d}}\right] i$

Ideals and bands under embedding

(R) If $J \subseteq Y$ is an ideal (band, ...), is then also $[J]i := \{x \in X;\ i(x) \in J\}$ an ideal (band, ...)?

(E) If $I \subseteq X$ is an ideal (band, ...), does there exist an ideal (band, ...) J in Y such that $I = [J]i$?

[van Gaans, K., 2008, 2012]	(R)	(E)
ideal	yes	no
order closed ideal	yes	no
solvex ideal	yes	yes
band	no	yes

Extension and restriction for bands

Let (X, K) be a pre-Riesz space and (Y, i) its Riesz completion.

Extension: If $B \subseteq X$ is a band, then $\hat{B} := i[B]^{dd}$ is the smallest extension band of B, i.e. $\hat{B}$ is the smallest band in Y with $B = \left[\hat{B}\right] i$.

Restriction: X is called pervasive if for every $y \in Y_+ \setminus \{0\}$ there is $x \in X$ such that $0 < i(x) \leq y$.

Theorem (van Gaans, K. 2008)

In a pervasive pre-Riesz space (X, K) the restriction property (R) for bands holds, i.e. for every band $\hat{B}$ in Y the set $\left[\hat{B}\right] i$ is a band in X.

Solvex sets

Let X be a povs. $M \subseteq X$ is called

- solvex, if for every $x \in X$, $x_1, \ldots, x_n \in M$ and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_{k=1}^{n} \lambda_k = 1$ and

$$\{x, -x\}^{\mathrm{u}} \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k;\ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^{\mathrm{u}}$$

one has $x \in M$ [van Gaans, 1999].

Proposition (van Gaans, 1999)

- *Every solvex set is solid and convex.*
- *If X is a vector lattice, then $M \subseteq X$ is solvex $\iff$ M is solid and convex. In particular, every ideal is solvex.*

Theorem (van Gaans, K., 2008)

If (X, K) is a pre-Riesz space, then every band in X is an order closed solvex ideal.

Archimedean vector lattice

$$
\begin{tikzcd}
\boxed{\text{band} = \text{o-closed ideal}} \arrow[d, Rightarrow, shift right] \\
\boxed{\text{ideal} = \text{solvex ideal}} \arrow[u, Rightarrow, shift right, "/" marking]
\end{tikzcd}
$$

pre-Riesz space

$$
\begin{tikzcd}
 & \boxed{\text{band}} \arrow[dl, Rightarrow, shift right] \arrow[dr, Rightarrow, shift left] & \\
\boxed{\text{o-closed ideal}} \arrow[ur, Rightarrow, shift right, "/" marking] \arrow[rr, Rightarrow, shift right, "/" marking] \arrow[dr, Rightarrow, shift left] & & \boxed{\text{solvex ideal}} \arrow[ul, Rightarrow, shift left, "/" marking] \arrow[ll, Rightarrow, shift right, "/" marking] \arrow[dl, Rightarrow, shift right] \\
 & \boxed{\text{ideal}} \arrow[ul, Rightarrow, shift left, "/" marking] \arrow[ur, Rightarrow, shift right, "/" marking] &
\end{tikzcd}
$$

Bands in C(Ω)

[Zaanen 1997]: For an open set $M \subseteq \Omega$ the set

$$I_M := \{x \in C(\Omega);\ \forall \varphi \in \Omega \setminus M\colon\ x(\varphi) = 0\}$$

is an ideal.

Proposition

I_M is a band $\iff$ M is regularly open, i.e. $M = \operatorname{int}(\overline{M})$.

Let X be an order unit space with functional representation

$$\Phi\colon X \to C(\Omega), \qquad \text{where } \Omega := \overline{\Lambda}.$$

Can bands in X be characterized by subsets of Ω ?

Characterization of bands by means of the functional representation

We use the embedding $\Phi\colon X \to \mathrm{C}(\Omega)$, as above.

For $M \subseteq \Omega$ let $\mathrm{Z}(M) = \{x \in X;\ \forall \varphi \in M\colon \varphi(x) = 0\}$,
for $B \subseteq X$ let $\mathrm{N}(B) := \{\varphi \in \Omega;\ \forall b \in B\colon \varphi(b) = 0\}$.

Theorem (Lemmens, van Gaans, K., 2015)

Let X be an Archimedean povs with order unit.

- *If B is a band in X, then $B = \mathrm{Z}(\mathrm{N}(B))$.*
- *Assume that for $B \subseteq X$ one has $B = \mathrm{Z}(\mathrm{N}(B))$.*
  B is a band if and only if $\mathrm{N}(B)$ is bisaturated.

A subset $M \subseteq \Omega$ is called bisaturated, if $M = \mathrm{sat}(\Omega \setminus \mathrm{sat}(\Omega \setminus M))$,
where $\mathrm{sat}(M) = \mathrm{N}(\mathrm{Z}(M)) = \Omega \cap \overline{\mathrm{aff}(M)}$.

In contrast to vector lattices, in povs there can exist non-directed bands! Example: Four-ray cone

Structure preserving operators

Let X and Y be pre-Riesz spaces and $T\colon X \supseteq\to Y$ a linear operator.

- T is called disjointness preserving if for every $x, y \in X$ from $x \perp y$ it follows that $Tx \perp Ty$.
- T is called band preserving (or local) if for every band B in X one has $T(B) \subseteq B$.
  (Equivalent: For every $x, y \in X$ with $x \perp y$ we have $Tx \perp y$.)
- T is called an orthomorphism if T is order bounded and band preserving.

Every band preserving operator is disjointness preserving.

In the theory of C_0-semigroups on a Banach space X, we consider $T\colon X \supseteq \mathcal{D}(T) \to Y$ and call T local if for every band B in X one has $T(B \cap \mathcal{D}(T)) \subseteq B$.

Example of disjointness preserving operators: Riesz* homomorphisms

A linear operator between vector lattices is a Riesz homomorphism if and only if it is positive and disjointness preserving.

Considering operators between pre-Riesz spaces, every Riesz* homomorphism is positive and disjointness preserving.

The converse is not true, in general, see the following example [van Imhoff, 2018]: The operator $T\colon P[0,1] \to P[0,1]$, $x \mapsto (s \mapsto \int_0^s x(t)\mathrm{d}t)$ is positive and disjointness preserving, but not a Riesz* homomorphism.

Riesz* homomorphisms on spaces of continuous functions

Theorem (van Imhoff, 2018)

Let X and Y be order dense subspaces of $\mathrm{C}(P)$ and $\mathrm{C}(Q)$, respectively, and $T\colon X \to Y$ linear.

(i) *If T is a Riesz* homomorphism, then there exist $w\colon Q \to \mathbb{R}_+$ and $\alpha\colon Q \to P$ such that for every $x \in X$ and $q \in Q$ we have*

$$(Tx)(q) = w(q)x(\alpha(q)) \quad (*).$$

If for every $p_1, p_2 \in P$ with $p_1 \neq p_2$ there is $x \in X$ such that $x(p_1) = 0$ and $x(p_2) = 1$, then w can be taken such that w is continuous on Q, the map α is uniquely determined on $\{q \in Q; w(q) > 0\}$, and on this set α is continuous.

(ii) *If for T there exist $w \in \mathrm{C}(Q)$, $w \geq 0$, and $\alpha\colon Q \to P$ continuous on $\{q \in Q; w(q) > 0\}$ such that $(*)$ holds for every $x \in X$ and $q \in Q$, then T is a Riesz* homomorphism.*

Disjointness preserving inverse in vector lattices

Theorem (Huijsmans, de Pagter, 1994, Koldunov 1995)

If X is a Banach lattice and $T\colon X \to X$ is a disjointness preserving bijection, then T^{-1} is disjointness preserving.

There is an example of a disjointness preserving linear bijection on an Archimedean vector lattice whose inverse is not disjointness preserving [Abramovich, Kitover, 2000].

How about disjointness preserving operators on pre-Riesz spaces?

Disjointness preserving inverse

Theorem (Lemmens, van Gaans, K., 2018)

Let (X, K) be a finite-dimensional povs with closed generating cone K. If $T\colon X \to X$ is a disjointness preserving linear bijection, then T^{-1} is disjointness preserving.

The combinatorial proof relies on the upper bound for the number of bands in X.

Theorem (van Imhoff, 2018)

Let X and Y be pre-Riesz spaces with X pervasive. Let $T\colon X \to Y$ be a bijective Riesz homomorphism. Then T^{-1} is a Riesz* homomorphism (and, hence, T is an order isomorphism).*

Disjointness preserving inverse

Theorem (Huijsmans, de Pagter, 1994)

Let X be a uniformly complete vector lattice, Y a normed vector lattice, and $T\colon X \to Y$ an injective and disjointness preserving operator. Then for every $x_1, x_2 \in X$ we have that $Tx_1 \perp Tx_2$ implies $x_1 \perp x_2$.

1. Can we replace Y by a pre-Riesz space?
Idea: Consider the Riesz completion (Y^ρ, i) of Y and apply the above theorem to $i \circ T$. How can we get a Riesz norm on Y^ρ?

2. Can X be replaced by a pre-Riesz space?
Given the Riesz completion X^ρ of X, can T be extended to a disjointness preserving operator on X^ρ? So far, there are a few results with strong additional conditions.

Extensions of norms

We assume the norm to be monotone, i.e. $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$.

Proposition

Let Y be a pre-Riesz space with a monotone norm $\|\cdot\|$ and let (Z, i) be a vector lattice cover of Y. Define $\rho(z) = \inf\{\|y\|; y \in Y, |z| \leq i(y)\}$.

(i) *ρ is a Riesz seminorm on Z.*

(ii) *If Y is pervasive, then ρ is a Riesz norm on Z.*

Theorem (Gaans, Zhang, K. 2018)

Let X be a uniformly complete vector lattice, Y a pervasive pre-Riesz space with a monotone norm, and $T\colon X \to Y$ an injective and disjointness preserving operator. Then for every $x_1, x_2 \in X$ we have that $Tx_1 \perp Tx_2$ implies $x_1 \perp x_2$.

Disjointness preserving C_0-semigroups

We generalize the result of Arendt.

Theorem (van Gaans, K., Zhang, 2018)

Let (X, X_+) be an ordered Banach space with a closed generating cone and a monotone norm and let $T\colon [0,\infty) \to \mathcal{L}(X)$ be a C_0-semigroup with generator $A\colon X \supseteq \mathcal{D}(A) \to X$ such that for every $t \in [0,\infty)$ the operator $T(t)$ is disjointness preserving. Then A is band preserving.

Idea of the proof:

Let (Z, i) be a vector lattice cover of X.
Extend the norm on X to a Riesz seminorm ρ on Z (as above).
For $x, y \in X$ with $\rho(i(x) \wedge i(y)) = 0$ we show that $x \perp y$.
Then we follow the line of reasoning as in the Arendt theorem.

A large part of the theory of disjointness preserving operators and C_0-semigroups in Banach lattices revolves around the following result [Meyer 1976]:

In Archimedean vector lattices, every order bounded disjointness preserving operator has a modulus; in particular, it is regular.

This is not true in Archimedean directed povs!
Counterexample in [van Gaans, K., 2019 (book)]; the space there is not weakly pervasive (and, hence, not pervasive).

Open problems

- Every result on vector lattices or Banach lattices concerning disjointness, bands, ideals and according structure preserving operators (or ideal-irreducible operators etc.) can naturally be translated into (normed) pre-Riesz spaces and yields an open question!
- Structures as solid sets, disjoint complements, and bands appear in results on spaces of operators. If the space of operators is not a vector lattice, analogous questions are not investigated, so far:

Outlook: Spaces of operators

Let (X, X_+), (Y, Y_+) be Archimedean vector lattices.
The space $L(X, Y)$ of all linear operators is a povs with the natural order

$$S \leq T \ :\Longleftrightarrow (T - S)[X_+] \subseteq Y_+$$

for $S, T \in L(X, Y)$, and the cone $L_+(X, Y) := \{T \in L(X, Y) \colon \ T \geq 0\}$.
The space of all regular operators

$$L_r(X, Y) := L_+(X, Y) - L_+(X, Y)$$

is Archimedean and directed, hence a pre-Riesz space.

In the literature, subsets of $L_r(X, Y)$ are investigated in the case that Y is, in addition, Dedekind complete, since then $L_b(X, Y) = L_r(X, Y)$ is a Dedekind complete vector lattice (Riesz-Kantorovich theorem).

Some results in the case that Y is Dedekind complete

1. If Y is Dedekind complete, then the space $L_r^{oc}(X,Y)$ of all order continuous operators in $L_r(X,Y)$ is a band. [Ogasawara]
2. If Y is Dedekind complete, then the set $L_r^{dpo}(X,Y)$ of all disjointness preserving regular operators is solid in $L_r(X,Y)$, moreover the band generated by $L_r^{dpo}(X,Y)$ in $L_r(X,Y)$ and also the disjoint complement $(L_r^{dpo})^d$ can be characterized [Popov, Randrianantoanina, 2013]
3. If Y is Dedekind complete, then the space $\mathrm{Orth}(Y)$ of all orthomorphisms on Y is the band in $L_r(X,Y)$ generated by the identity operator.

Open problem: **Are there similar results in the case that Y is an arbitrary (Archimedean) vector lattice?**

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