Injective dual Banach spaces and operator ideals

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Injective spaces and extension of operators

Definition 1

Given $1 \leq \lambda < \infty$, we say that a Banach space X is λ-*injective* if for every Banach space $Z \supset X$ there is a projection $\pi : Z \to X$ with $\|\pi\| \leq \lambda$.

Examples: ℓ_∞, $\ell_\infty(\Gamma)$, dual $L_\infty(\mu)$-spaces.

Definition 2

Given $1 \leq \lambda < \infty$, we say that a Banach space X has the λ-*extension property* if for all Banach spaces $Y \subset Z$, every operator $T \in \mathcal{L}(Y, X)$ admits an extension $\widetilde{T} \in \mathcal{L}(Z, X)$ with $\left\|\widetilde{T}\right\| \leq \lambda\|T\|$.

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$$X \text{ is } \lambda\text{-injective} \Leftrightarrow X \text{ has the } \lambda\text{-extension property}$$

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Proposition 3

$$X \text{ is } \lambda\text{-injective} \iff X \text{ has the } \lambda\text{-extension property}$$

Extension properties

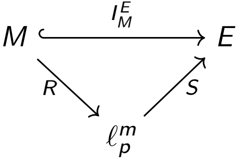
Theorem 4 (Lindenstrauss essentially)

Let X be a Banach space and $1 \leq \lambda < \infty$. TFAE:
(a) *X^{**} is λ-injective.*
(c) *Let $Z \supset X$ and let Y be a dual space. Then every operator $T \in \mathcal{L}(X, Y)$ admits an extension $\widetilde{T} \in \mathcal{L}(Z, Y)$ with $\left\|\widetilde{T}\right\| \leq \lambda \|T\|$.*
(d) *Let $Z \supset Y$ and let $\epsilon > 0$. Then every operator $T \in \mathcal{K}(Y, X)$ admits an extension $\widetilde{T} \in \mathcal{K}(Z, X)$ with $\left\|\widetilde{T}\right\| \leq (\lambda + \epsilon)\|T\|$.*
(f) *If $Z \supset X$, every operator $T \in \mathcal{K}(X, Y)$ admits an extension $\widetilde{T} \in \mathcal{K}(Z, Y)$ with $\left\|\widetilde{T}\right\| \leq \lambda \|T\|$.*
(g) *If $Z \supset X$, every operator $T \in \mathcal{W}(X, Y)$ admits an extension $\widetilde{T} \in \mathcal{W}(Z, Y)$ with $\left\|\widetilde{T}\right\| \leq \lambda \|T\|$.*

$\mathcal{L}_1$ and $\mathcal{L}_\infty$ spaces

Definition 5

Let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. We say that E is an $\mathcal{L}^g_{p,\lambda}$*-space* if for every finite dimensional subspace $M \subset E$ and $\epsilon > 0$ there are operators $R \in \mathcal{L}\left(M, \ell_p^m\right)$ and $S \in \mathcal{L}\left(\ell_p^m, E\right)$ for some $m \in \mathbb{N}$ such that



and $\|S\|\|R\| \leq \lambda + \epsilon$.

Examples:

$\mathcal{L}_1$-spaces: ℓ_1, $\ell_1(\Gamma)$, $L_1(\mu)$.

$\mathcal{L}_\infty$-spaces: ℓ_∞, $\ell_\infty(\Gamma)$, $L_\infty(\mu)$, $C(K)$.

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$$\begin{array}{ccc} M & \xrightarrow{\quad I_M^E \quad} & E \\ & {\scriptstyle R}\searrow \quad \nearrow{\scriptstyle S} & \\ & \ell_p^m & \end{array} \qquad \text{and } \|S\|\|R\| \leq \lambda + \epsilon.$$

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$\mathcal{L}_1$ and $\mathcal{L}_\infty$ spaces

Proposition 6

$$E \text{ is an } \mathcal{L}^g_{1,\lambda}\text{-space} \Leftrightarrow E^* \text{ is } \lambda\text{-injective.}$$

Proposition 7

$$E \text{ is an } \mathcal{L}^g_{p,\lambda}\text{-space} \Leftrightarrow E^* \text{ is an } \mathcal{L}^g_{p',\lambda}\text{-space}$$

where $1/p + 1/p' = 1$.

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$\mathcal{L}_{\mathrm{ir}}$

Definition 8

An operator $T \in \mathcal{L}(E, F)$ *admits an integral representation* if

$$k_F \circ T(x) = \int_{B_{E^*}} x^*(x)\,\mathrm{d}\mathcal{G} \qquad (x \in E)$$

for some weak*-countably additive F^{**}-valued measure $\mathcal{G}$ defined on the Borel sets of B_{E^*} such that the following conditions are satisfied:
(a) $\mathcal{G}(\,\cdot\,)y^*$ is a regular countably additive Borel measure for each $y^* \in F^*$;
(b) the mapping $y^* \mapsto \mathcal{G}(\,\cdot\,)y^*$ of F^* into $C\,(B_{E^*})^*$ is weak*- to weak*-continuous.

$\mathcal{L}_{\text{ir}}$

We denote by $\mathcal{L}_{\text{ir}}(E, F)$ the space of all operators $T \in \mathcal{L}(E, F)$ that admit an integral representation. On this space we define the norm

$$\|T\|_{\text{ir}} = \inf \|\mathcal{G}\|(B_{E^*})$$

where $\|\mathcal{G}\|$ denotes the semivariation of $\mathcal{G}$ and the infimum is taken over all measures $\mathcal{G}$ satisfying Definition 8.

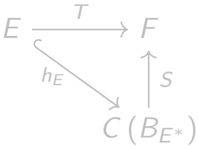
$\mathcal{L}_{\mathrm{ir}}$

Proposition 9

An operator $T \in \mathcal{L}(E, F)$ admits an integral representation if and only if it has an extension

$$S \in \mathcal{L}\left(C\left(B_{E^*}\right), F\right).$$

Moreover, $\|T\|_{\mathrm{ir}} = \inf \|S\|$ *where the infimum is taken over all possible extensions* S *to* $C\left(B_{E^*}\right)$.



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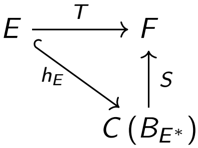
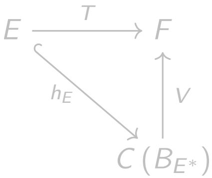


$\mathcal{L}_{\mathrm{ir}}$

Proposition 10

Let F be a finite dimensional Banach space. If $T \in \mathcal{L}_{\mathrm{ir}}(E, F)$, there is $V \in \mathcal{L}(C(B_{E^*}), F)$ such that $\|V\| = \|T\|_{\mathrm{ir}}$ and $T = V \circ h_E$.



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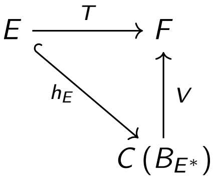

Main theorem I

Theorem 11

Let X be a Banach space and $1 \leq \lambda < \infty$. TFAE:
(1) *X is an $\mathcal{L}^{g}_{\infty,\lambda}$-space.*
(2) *X^{**} is λ-injective.*
(3) *$k_X \in \mathcal{L}_{\mathrm{ir}}(X, X^{**})$ with $\|k_X\|_{\mathrm{ir}} \leq \lambda$.*
(5)-(6) *For every Banach space Y we have $\mathcal{K}(Y, X) \subseteq \mathcal{L}_{\mathrm{ir}}(Y, X)$ (compactly) with (*).*
(7) *For every dual Banach space Y we have $\mathcal{L}(X, Y) = \mathcal{L}_{\mathrm{ir}}(X, Y)$ with (*).*
(8)-(9) *For every Banach space Y we have $\mathcal{K}(X, Y) \subseteq \mathcal{L}_{\mathrm{ir}}(X, Y)$ (compactly) with (*).*
(10)-(11) *For every Banach space Y we have $\mathcal{W}(X, Y) \subseteq \mathcal{L}_{\mathrm{ir}}(X, Y)$ (weakly compactly) with (*).*

(*) $\|T\| \leq \|T\|_{\mathrm{ir}} \leq \lambda \|T\|$ for every T in the ideal under consideration.

Main theorem I (continued)

Theorem (Theorem 11 continued)

Let X be a Banach space and $1 \leq \lambda < \infty$. TFAE:
(1) *X is an $\mathcal{L}^g_{\infty,\lambda}$-space.*
(12) *For every Banach space Y, every $T \in \mathcal{K}(X, Y)$ factors compactly through c_0 with $\|T\| \leq \|T\|_{c_0,\mathcal{K}} \leq \lambda\|T\|$.*
(13) *For every Banach space Y, every $T \in \mathcal{K}(Y, X)$ factors compactly through c_0 with $\|T\| \leq \|T\|_{c_0,\mathcal{K}} \leq \lambda\|T\|$.*

Anthony O'Farrell's question about $\mathcal{L}(X,X) = \mathcal{L}_{\text{ir}}(X,X)$

Proposition 12

Given a Banach space X and $1 \leq \lambda < \infty$, TFAE:
(a) $I_X \in \mathcal{L}_{\text{ir}}(X,X)$ *with* $\|I_X\|_{\text{ir}} \leq \lambda$.
(b) $\mathcal{L}(X,X) = \mathcal{L}_{\text{ir}}(X,X)$ *with (*).*
(c) *For every Banach space Y, we have* $\mathcal{L}(X,Y) = \mathcal{L}_{\text{ir}}(X,Y)$ *with (*).*
(d) *For every Banach space Y, we have* $\mathcal{L}(Y,X) = \mathcal{L}_{\text{ir}}(Y,X)$ *with (*).*
(e) *X is isometrically isomorphic to a λ^+-complemented subspace of $C(B_{X^*})$ (that is, for every $\lambda' > \lambda$ there is a projection with norm $\leq \lambda'$).*
(f) *X is isometrically isomorphic to a λ^+-complemented subspace of a $C(K)$-space.*

(*) $\|T\| \leq \|T\|_{\text{ir}} \leq \lambda\|T\|$ for every T in the space under consideration.

Such a space X is an $\mathcal{L}^{g}_{\infty,\lambda}$-space.

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Proposition 12

Given a Banach space X and $1 \le \lambda < \infty$, TFAE:
(a) $I_X \in \mathcal{L}_{\text{ir}}(X,X)$ *with* $\|I_X\|_{\text{ir}} \le \lambda$.
(b) $\mathcal{L}(X,X) = \mathcal{L}_{\text{ir}}(X,X)$ *with (*).*
(c) *For every Banach space Y, we have* $\mathcal{L}(X,Y) = \mathcal{L}_{\text{ir}}(X,Y)$ *with (*).*
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Such a space X is an $\mathcal{L}^g_{\infty,\lambda}$-space.

Ideal Γ_1

Definition 13

We say that an operator T belongs to $\Gamma_1(X, Y)$ if there are a measure μ and operators $A \in \mathcal{L}(X, L_1(\mu))$ and $B \in \mathcal{L}(L_1(\mu), Y^{**})$ such that $k_Y \circ T = B \circ A$. We endow the space $\Gamma_1(X, Y)$ with the norm $\gamma_1(T) := \inf \|A\| \|B\|$ where the infimum is taken over all such factorizations.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle k_Y} \\ L_1(\mu) & \xrightarrow[B]{} & Y^{**} \end{array}$$

Main theorem II

Theorem 14

Let X be a Banach space and $1 \leq \lambda < \infty$. TFAE:
(a) *X is an $\mathcal{L}^g_{1,\lambda}$-space.*
(b) *X^* is λ-injective.*
(d) *For every Banach space Y, we have $\mathcal{L}(X, Y) = \Gamma_1(X, Y)$ with (*).*
(f) *For every Banach space Y, we have $\mathcal{K}(X, Y) \subseteq \Gamma_1(X, Y)$ with (*).*
(g)-(h) *For every Banach space Y, every $T \in \mathcal{K}(X, Y)$ factors (compactly) through ℓ_1 with (*).*
(j)-(k) *For every Banach space Y, every $T \in \mathcal{K}(Y, X)$ factors (compactly) through ℓ_1 with (*).*
(n) *For every Banach space Y, we have $\mathcal{L}(Y, X) = \Gamma_1(Y, X)$ with (*).*
(v) *For every Banach space Y, every $T \in \mathcal{W}(X, Y)$ factors through ℓ_1 with a weakly compact second factor and (*).*

(*) $\|T\| \leq \gamma_1(T) \leq \lambda\|T\|$ for every T in the ideal under consideration.