Banach spaces containing many complemented subspaces

Manuel González

Departamento de Matemáticas
Universidad de Cantabria, Santander, Spain

Workshop on Banach spaces and Banach lattices
ICMAT Madrid. September 9-13, 2019

Joint work with Javier Pello (URJC, Móstoles)

Subprojective and superprojective Banach spaces

NOTE: Subspaces are always closed.

Definition

A Banach space X is subprojective if every infinite-dim. subspace of X contains an infinite-dim. subspace complemented in X.

The space X is superprojective if every infinite-codim. subspace of X is contained in an infinite-codim. subspace complemented in X.

Remark

X superprojective if and only if every infinite dim. quotient X/M admits an infinite dim. quotient X/N $(M \subset N)$ with N complemented.

References

[W-64] Robert J. Whitley. *Strictly singular operators and their conjugates.* Trans. Amer. Math. Soc. 113 (1964), 252–261.

[OS-15] Timur Oikhberg and Eugeniu Spinu. *Subprojective Banach spaces.* J. Math. Anal. Appl. 424 (2015), 613–635.

[GP-16] Manuel González and Javier Pello. *Superprojective Banach spaces.* J. Math. Anal. Appl. 437 (2016), 1140–1151.

[GGP-17] Elói M. Galego, Manuel González and Javier Pello. *On subprojectivity and superprojectivity of Banach spaces.* Results in Math. 71 (2017), 1191–1205.

[RS-18] César Ruiz and Víctor M. Sánchez. *Subprojective Nakano spaces.* J. Math. Anal. Appl. 458 (2018), 332–344.

[GP-19] Manuel González and Javier Pello. *On subprojectivity of $C(K, X)$.* Proc. Amer. Math. Soc. 147 (2019), 3425–3429.

[GPS-19] Manuel González, Margot Salas-Brown and Javier Pello. *The perturbation classes problem on subprojective and superprojective Banach spaces.* Preprint 2019.

[GP-20] Manuel González and Javier Pello. *Complemented subspaces of J-sums of Banach spaces.* In preparation.

Basic results and examples [W64]

1. Every subspace of a subprojective space is subprojective.
2. Every quotient of a superprojective space is superprojective.
3. Suppose X is reflexive. Then
   X subprojective $\iff X^*$ superprojective, and
   X superprojective $\iff X^*$ subprojective.

1. ℓ_p $(1 < p < \infty)$ is subprojective and superprojective.
2. ℓ_1 and c_0 are subprojective.
3. $L_p(0,1)$ subprojective $\iff 2 \leq p < \infty$.
4. $L_p(0,1)$ superprojective $\iff 1 < p \leq 2$.

Duality for non-reflexive spaces [GP16]

1. X subprojective $\not\Rightarrow$ X^* superprojective: $X = c_0$, $X^* = \ell_1$.
   ℓ_1 has a quotient $\ell_1/M \simeq \ell_2$.
2. X^* subprojective $\not\Rightarrow$ X superprojective: X hereditarily reflexive $\mathcal{L}_\infty$-space with $X^* \simeq \ell_1$ (Bourgain Delbaen).
   X has a quotient $X/M \simeq c_0$.
3. NOTE: X^* isometric to ℓ_1 implies X superprojective.

Question 1 Suppose X non-reflexive.

1. X superprojective $\Rightarrow$ X^* subprojective?
2. X^* superprojective $\Rightarrow$ X subprojective?

Unconditional sums of Banach spaces

1. X, Y subprojective $\iff X \times Y$ subprojective [OS-15].
2. X, Y superprojective $\iff X \times Y$ superprojective [GP-16].

Let E, X_n ($n \in \mathbb{N}$) be Banach spaces. Suppose that E admits an unconditional basis (e_n). We define

$$E(X_n) := \left\{ (x_n) : x_n \in X_n \text{ for each } n \text{ and } \sum_{n=1}^{\infty} \|x_n\| e_n \in E \right\}.$$

1. E, X_n ($n \in \mathbb{N}$) subprojective $\iff E(X_n)$ subprojective [OS-15].
2. E, X_n ($n \in \mathbb{N}$) superprojective $\iff E(X_n)$ superprojective [GP-16].

Some negative criteria [GP-16]

1. If there exists a surjective strictly singular operator $Q : X \to Z$ then X is not superprojective.
2. If there exists an strictly cosingular embedding operator $J : Z \to X$ then X is not subprojective.

Proposition

The classes of superprojective spaces and subprojective spaces fail the three-space property.

Proof. There exists an exact sequence (Z_2 is the Kalton-Peck space)

$$0 \to \ell_2 \xrightarrow{J} Z_2 \xrightarrow{Q} \ell_2 \to 0$$

in which J is strictly cosingular and Q is strictly singular.

Some negative criteria [GP-16]

Proposition

Suppose that X contains a subspace isomorphic to ℓ_1. Then X is not superprojective and X^* is not subprojective.

Proof. If X contains ℓ_1 then there exists a surjective strictly singular $Q : X \to \ell_2$ such that $Q^* : \ell_2 \to X^*$ is strictly cosingular.

Remark

This result suggests that, among the non-reflexive spaces, there are more subprojective spaces than superprojective spaces.

Tensor products

Proposition ([OS-15])

1. $X, Y \in \{c_0, \ell_p\ 1 \leq p < \infty\} \Rightarrow X\hat{\otimes}_\pi Y$ *and* $X\hat{\otimes}_\epsilon Y$ *subprojective.*
2. $2 \leq p, q < \infty \Rightarrow L_p(0,1)\hat{\otimes}_\epsilon L_q(0,1)$ *subprojective.*

Proposition ([GP-16])

1. $X, Y \in \{c_0, \ell_p\ 1 < p < \infty\} \Rightarrow X\hat{\otimes}_\epsilon Y$ *superprojective.*
2. *Let* $1 < p, q < \infty$. *Then*
   $\ell_p\hat{\otimes}_\pi \ell_q$ *superprojective* $\Leftrightarrow p > q/(q-1) \Leftrightarrow \ell_p\hat{\otimes}_\pi \ell_q$ *reflexive.*
3. *For* $1 < p, q \leq 2$, $L_p(0,1)\hat{\otimes}_\pi L_q(0,1)$ *is not superprojective.*

Question 2

1. Is $L_p(0,1)\hat{\otimes}_\pi L_q(0,1)$ subprojective when $2 \leq p, q < \infty$?
2. Is $L_p(0,1)\hat{\otimes}_\epsilon L_q(0,1)$ superprojective when $1 < p, q \leq 2$?

Spaces $C(K, X)$ and $L_p(X)$

Let K be a compact space.
$C(K)$ subprojective $\iff$ $C(K)$ superprojective $\iff$ K scattered.

Theorem (GP-19)

$C(K)$ and X subprojective $\Longrightarrow C(K, X) \equiv C(K)\hat{\otimes}_\epsilon X$ subprojective.

Question 3

$C(K)$ and X superprojective $\Rightarrow C(K, X)$ superprojective?

Proposition (GP-16)

X superprojective $\Longrightarrow C([0, \lambda], X)$ superprojective.

Observation. (F.L. Hernández) [Y. Raynaud 1985]:

For $2 < p < q < \infty$, $L_p(L_q)$ is not subprojective (while L_p and L_q are).

For $1 < s < r < 2$, $L_r(L_s)$ is not superprojective (while L_r and L_s are).

Properties implying superprojectivity [GGP-17]

Definition

We say that X satisfies P_{weak} if each non-weakly compact operator $T : X \to Y$ is an isomorphism on a copy of c_0 and X^* is hereditarily ℓ_1.

We say that X satisfies P_{strong} if each non-compact operator $T : X \to Y$ is an isomorphism on a copy of c_0.

Proposition

1. *X satisfies P_{strong} $\Rightarrow$ X satisfies P_{weak} $\Rightarrow$ X is superprojective.*
2. *If X_n ($n \in \mathbb{N}$) satisfy P_{strong} then $c_0(X_n)$ satisfies P_{strong}.*
3. *If X and Y satisfy P_{strong} then $X \hat{\otimes}_\pi Y$ satisfies P_{strong}.*

Properties implying superprojectivity [GGP-17]

Examples

1. *Isometric preduals of* $\ell_1(\Gamma)$ *satisfy* P_{strong}.
2. $C(K)$ *spaces with* K *scattered satisfy* P_{strong}.
3. *The Hagler space JH satisfies* P_{strong}.
4. *The Schreier space* S *and the predual of the Lorentz space* $d(w,1)$ *satisfy* P_{weak} *but not* P_{strong}.
   In fact $S\hat{\otimes}_\pi S$ *is not superprojective.*

Question 4
Find new examples of sub(super)projective Banach spaces.

J-sums of Banach spaces I [GP-20]

J: James' space. $\dim J^{**}/J = 1$.
J and J^* are subprojective. By duality, J and J^* are superprojective.

[S.F. Bellenot. The J-sum of Banach spaces. J. Funct. Analysis (1982)]

$$(X_1, \|\cdot\|_1) \xrightarrow{i_1} (X_2, \|\cdot\|_2) \xrightarrow{i_2} (X_3, \|\cdot\|_3) \xrightarrow{i_3} \cdots \quad \|i_k\| \leq 1.$$

$$J(X_n)^{lim} = \{(x_i)_{i\in\mathbb{N}} : x_i \in X_i, \|(x_i)\|_J :< \infty\}.$$

where $\|(x_i)\|_J^2 = \sup\{\sum_{i=1}^{k-1} \|x_{p(i+1)} - \phi_{p(i)}^{p(i+1)}(x_{p(i)})\|_{p(i+1)}^2\}$,
and the sup is taken over $k \in \mathbb{N}$ and $p(1) < \cdots < p(k)$. Moreover,

$$J(X_n) = \{(x_i) \in J(X_n)^{lim} : \lim_{i\to\infty} \|(x_i)\|_i :< \infty\}.$$

J-sums of Banach spaces II

Observation [Bellenot].

1. $J(X_n)^{lim}$ is a Banach space and $J(X_n)$ is a subspace of $J(X_n)^{lim}$.
2. If each X_n is reflexive, then $J(X_n)^{**} \equiv J(X_n)^{lim}$.

Theorem

1. *If each X_n is subprojective , then so is $J(X_n)$.*
2. *If each X_n is superprojective, then so is $J(X_n)$.*
3. *If each X_n^* is subprojective, then so is $J(X_n)^*$.*

J-sums of Banach spaces III

Special case:

(X_n) is an increasing sequence of subspaces of a Banach space Y with $\cup_{n=1}^{\infty} X_n$ dense in Y, and $i_k : X_k \to X_{k+1}$ is the inclusion.

Observation [Bellenot].

1. The expression $U(x_k) = \lim_{k\to\infty} x_k$ defines a surjective operator
   $U : J(X_n)^{lim} \to Y$ with kernel $J(X_n)$.

2. If each X_n is reflexive, then $J(X_n)^{**}/J(X_n) \equiv J(X_n)^{lim}/J(X_n) \equiv Y$.

Theorem

If Y is subprojective, then $J(X_n)$ is subprojective.

Thank you
for your attention.