Transitivity and Ramsey properties of L_p-spaces

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Outline

1. Transitivities of isometry groups
2. Fraïssé theory and the KPT correspondence
3. Fraïssé Banach spaces
4. The Approximate Ramsey Property for ℓ_p^n's

Joint work with J. Lopez-Abad, B. Mbombo, S. Todorcevic
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Notation

X, Y denote an infinite dimensional Banach spaces, E, F, G, H finite dimensional ones.

$S_X =$ unit sphere of X.

$GL(X)$=the group of linear automorphisms of X

$\mathrm{Isom}(X)$= group of linear surjective isometries of X.

$\mathrm{Emb}(F, X)$ = set of linear isometric embeddings of F into X.

p a real number in the separable Banach range: $1 \leq p < +\infty$

L_p denotes the Lebesgue space $L_p([0, 1])$.

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Topologies on $GL(X)$

Three topologies are relevant on $GL(X)$:

- the norm topology $\|T\| = \sup_{x \in S_X} \|Tx\|$.
- the SOT (strong operator topology) i.e. pointwise convergence on X,
- the WOT (weak operator topology) i.e. weak pointwise convergence on X,

Fact

$GL(X), \|.\|$ is a topological group

It is not so for $GL(X)$ with SOT in general, but things get better if one looks at bounded subgroups (in particular $\mathrm{Isom}(X)$).

The group $\mathrm{Isom}(X)$

Fact

$\mathrm{Isom}(X)$ *is a topological group for SOT.*

But $\mathrm{Isom}(X), \|.\|$ is not separable in general, for X separable.

Fact

If X is separable then $(\mathrm{Isom}(X), SOT)$ is separable. Actually it is a Polish group, i.e. SOT is separable completely metrizable

- Regarding WOT it coincides with SOT on $Isom(X)$ as soon as X has PCP (e.g. reflexive) - see Megrelishvilii 2000 - so for L_p, $1 < p < \infty$.
- On $\mathrm{Isom}(L_1)$ WOT and SOT are different (Antunes 2019)
- Things are more involved in $C(K)$-spaces, but those will not be studied here.

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Summing up:

$\mathrm{Isom}(X)$ will always be equipped with the Strong Operator Topology SOT.

Regarding $\mathrm{Emb}(F, X)$, for F finite dimensional, we shall usually equip $\mathrm{Emb}(F, X)$ with the distance induced by the norm on $\mathcal{L}(F, X)$. But note that here SOT and the norm topology are equivalent.

Classical isometry groups

1. If H=Hilbert, then $\mathrm{Isom}(H)$ is the unitary group $\mathcal{U}(H)$. It acts transitively on S_H, meaning there is a single and full orbit for the action $\mathrm{Isom}(H) \curvearrowright S_H$.

2. For $1 \leq p < +\infty$, $p \neq 2$, every isometry on $L_p = L_p(0,1)$ is of the form
$$T(f)(.) = h(.)f(\phi(.)),$$
where ϕ is a measurable transformation of $[0,1]$ onto itself, and h such that $|h|^p = d(\lambda \circ \phi)/d\lambda$, λ the Lebesgue measure (Banach-Lamperti 1932-1958). So

3. $\mathrm{Isom}(L_p)$ acts almost transitively on S_{L_p}, meaning that the action $\mathrm{Isom}(L_p) \curvearrowright S_{L_p}$ admits dense orbits.

4. However $\mathrm{Isom}(L_p)$ does not act almost transitively on the sphere unless $p = 2$.

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Classical isometry groups

5 Every isometry on c_0 and ℓ_p, $p \neq 2$ acts as a "signed permutation", i.e. a combination of signs and permutation of the coordinates on the canonical basis.

6 By the Banach-Stone theorem (1932), every isometry of $C(K)$ is of the form

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Mazur rotation problem

If $G = \mathrm{Isom}(\mathrm{X})$ acts transitively on S_X, must X be isomorphic? isometric? to a Hilbert space.

(a) if $\dim X < +\infty$: YES to both

(b) if $\dim X = +\infty$ and is separable: ???

(c) if $\dim X = +\infty$ and is non-separable: NO to both

Proof

(a) Average a given inner product by using the Haar measure on G and observe that this new inner product turns all $T \in G$ into unitaries and therefore, by transitivity, must induce a multiple of the original norm.

$$[x, y] = \int_{T \in G} < Tx, Ty > d\mu(T),$$

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(c) Use ultrapowers.....

Ultrapowers

A normed space is *transitive (resp. almost transitive)* if the associated isometry group acts transitively (resp. almost transitively) on the associated unit sphere.

It is an easy observation that if X is almost transitive then for any non-principal ultrafilter $\mathcal{U}$, $X_{\mathcal{U}}$ is transitive. Actually the subgroup $\mathrm{Isom}(X)_{\mathcal{U}}$ of isometries T of the form

$$T((x_n)_{n\in\mathbb{N}}) = (T_n(x_n))_{n\in\mathbb{N}}$$

where $T_n \in \mathrm{Isom}(X)$, acts transitively on $X_{\mathcal{U}}$.

Proposition

The space $(L_p(0,1))_{\mathcal{U}}$ is transitive.

Note that in these lines Cabello-Sanchez (1998) studies $\Pi_{n\in\mathbb{N}} L_{p_n}(0,1)$ for $p_n \to +\infty$ and obtains a transitive M-space.

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On renormings of classical spaces

For $p \neq 2$, L_p is not transitive, and ℓ_p not almost transitive. Furthermore

Theorem (Dilworth - Randrianantoanina, 2014)

Let $1 < p < +\infty, p \neq 2$. Then ℓ_p does not admit an equivalent almost transitive norm.

Question

Let $1 \leq p < +\infty, p \neq 2$. Show that the space $L_p([0,1])$ does not admit an equivalent transitive norm.

See also Cabello-Sanchez, Dantas, Kadets, Kim, Lee, Martín (2019) for related notions of "microtransitivity".

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Ultrahomogeneity

Definition

Let X be a Banach space.

- X is called ultrahomogeneous when for every finite dimensional subspace E of X and every isometric embedding $\phi : E \to X$ there is a linear isometry $g \in \mathrm{Isom}(X)$ such that $g \restriction E = \phi$; this means the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}(E, X)$ is transitive.
- X is called approximately ultrahomogeneous (AuH) when for every finite dimensional subspace E of X, every isometric embedding $\phi : E \to X$ and every $\varepsilon > 0$ there is a linear isometry $g \in \mathrm{Isom}(X)$ such that $\|g \restriction E - \phi\| < \varepsilon$; this means the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}(E, X)$ is almost transitive (dense orbits).

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Examples

Note that ultrahomogeneous ⇒ transitive, and (AuH)⇒ almost transitive

Fact

Any Hilbert space is ultrahomogeneous.

Theorem

Are (AuH)*, but not ultrahomogeneous:*

- *The Gurarij space, defined by Gurarij in 1966 (Kubis-Solecki 2013).*
- $L_p[0,1]$ *for* $p \neq 2, 4, 6, 8, \ldots$ *(Lusky 1978).*

One original definition of the Gurarij: *a separable Banach space* $\mathbb{G}$ *universal for f.d. spaces such that any linear isometry between f.d. subspaces extends to a* $1+\epsilon$*-linear isometry on* $\mathbb{G}$. By Lusky 1976, it is isometrically unique.

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- the Gurarij is the unique separable, universal, (AuH) space (Lusky 1976 + Kubis-Solecki 2013).
- Lusky's result abour L_p's is based on the *equimeasurability theorem* by Plotkin / Rudin, 1976. His proof gives (AuH).
- L_p is not (AuH) for $p = 4, 6, 8, \ldots$:
  B. Randrianantoanina (1999) proved that for those $p's$ there are two isometric subspaces of L_p (due to Rosenthal), with an unconditional basis, complemented/ uncomplemented.

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It uses

Proposition (Plotkin and Rudin (1976))

For $p \notin 2\mathbb{N}$, suppose that $(f_1, \dots, f_n) \in L_p(\Omega_0, \Sigma_0, \mu_0)$ and $(g_1, \dots, g_n) \in L_p(\Omega_1, \Sigma_1, \mu_1)$ and

$$\|1 + \sum_{j=1}^{n} a_j f_j\|_{\mu_0} = \|1 + \sum_{j=1}^{n} a_j g_j\|_{\mu_1} \text{ for every } a_1, \dots, a_n.$$

Then $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ are equidistributed

Equidistributed here means that for any Borel $B \in \mathbb{R}^n$,

$$\mu_0((f_1, \dots, f_n)^{-1}(B)) = \mu_1((g_1, \dots, g_n)^{-1}(B)).$$

Lp's for p non even are "like" the Gurarij

Let us cite Lusky:

"We show that a certain homogeneity property holds for $L_p(0,1)$; $p \neq 4, 6, 8, \ldots$, *which is similar to a corresponding property of the Gurarij space..."*

We aim to give a more complete meaning to this similarity.

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Fraïssé theory (abusively) summarized

- Given a (hereditary) class $\mathcal{F}$ of finite (or sometimes finitely generated) structures, Fraïssé theory (Fraïssé 1954) investigates the existence of a countable structure $\mathcal{A}$, universal for $\mathcal{F}$ and ultrahomogeneous (any t isomorphism between finite substructures extends to a global automorphism of $\mathcal{A}$)
- Fraïssé theory shows that this is equivalent to certain amalgamation properties of $\mathcal{F}$.
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if $\mathcal{F}$*=the class of finite sets, then* $\mathcal{A} = \mathbb{N}$

In this case isomorphisms of the structure are just bijections.

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Fraïssé theory is related to Extreme Amenability through the KPT correspondence (Kechris-Pestov-Todorcevic 2005).

Definition

A topological group G is called extremely amenable (EA) when every continuous action $G \curvearrowright K$ on a compact K has a fixed point; that is, there is $p \in K$ such that $g \cdot p = p$ for all $g \in G$.

Examples of extremely amenable groups

1. The group $\mathrm{Aut}(\mathbb{Q}, <)$ of strictly increasing bijections of $\mathbb{Q}$ (with the pointwise convergence topology) (Pestov,1998);
2. but $S_\infty = \mathrm{Aut}(\mathbb{N})$ is not extremely amenable;
3. The group of isometries of the Urysohn space with pointwise convergence topology. (Pestov, 2002);
4. The unitary group $U(H)$ endowed with SOT (Gromov-Milman,1983);
5. The group $\mathrm{Isom}(L_p)$ of linear isometries of the Lebesgue spaces $L_p[0,1]$, $1 \leq p \neq 2 < \infty$, with the SOT (Giordano-Pestov, 2006);
6. The group $\mathrm{Isom}(\mathbb{G})$ of linear isometries of the Gurarij space (Bartosova-LopezAbad-Lupini-Mbombo, 2017)

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The KPT correspondence

For finite structures, when $\mathcal{A}$ is the Fraïssé limit of $\mathcal{F}$, then holds the Kechris-Pestov-Todorcevic correspondence.

Theorem (Kechris-Pestov-Todorcevic, 2005)

The group $(\mathrm{Aut}(\mathcal{A}),$ ptwise cv topology$)$ is extremely amenable if and only if $\mathcal{F}$ is "rigid" and satisfies the Ramsey property.

For example Pestov's result that $\mathrm{Aut}(\mathbb{Q},<)$ is EA is a combination of "$(\mathbb{Q},<) =$ Fraïssé limit of finite ordered sets" and of the classical finite Ramsey theorem on $\mathbb{N}$.

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Outline

1. Transitivities of isometry groups
2. Fraïssé theory and the KPT correspondence
3. Fraïssé Banach spaces
4. The Approximate Ramsey Property for ℓ_p^n's

Fraïssé Banach spaces

Several works exist about extension of the Fraïssé theory to the metric setting (i.e. with epsilons), and settle the case of the Gurarij space,

(i.e. allow to see the Gurarij as the Fraissé limit of the class of finite dimensional spaces)

but they are often at the same time too general and too restrictive for us - and in particular do not apply in a satisfactory way to the L_p's. We focus on the Banach space setting.

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Fraïssé Banach spaces

Given two Banach spaces E and X, and $\delta \geq 0$, let $\mathrm{Emb}_\delta(E, X)$ be the collection of all linear δ-isometric embeddings $T : E \to X$, i.e. such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$ (T^{-1} defined on $T(E)$), equipped with the distance induced by the norm.

We consider the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$

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X is Fraïssé if and only if for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $E \subset X$ of dimension k, the action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is "ε-transitive"
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TFAE for X:

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- L_p is not Fraïssé for $p = 4, 6, 8, \ldots$ since not AUH.

Since ε depends only on δ and not on n, we say that the Hilbert and the Gurarij are "stable" Fraïssé".

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(F.,Lopez-Abad, Mbombo, Todorcevic) The spaces $L_p[0,1]$ for $p \neq 4, 6, 8, \ldots$ are Fraïssé.

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Properties of Fraïssé spaces

Proposition

Assume X and Y are Fraissé, and that X is separable. Then are equivalent:

(1) *X is finitely representable in Y*
(2) *every finite dimensional subspace of X embeds isometrically into Y*
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In particular (by Dvoretsky) ℓ_2 is the minimal separable Fraïssé space; and the Gurarij is the maximal one.

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Let

- $Age(X)$=the set of finite dimensional subspaces of X, and
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Assume X and Y are separable Fraïssé. Then are equivalent

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We also obtained internal characterizations of classes of finite dimensional spaces which are $\equiv$ to the age of some Fraïssé ("amalgamation properties"). For such a class $\mathcal{F}$ we write X=Fraïssé lim $\mathcal{F}$ to mean "X separable and $Age(X) \equiv \mathcal{F}$"

Fraïssé is an ultraproperty

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The following are equivalent.

1) *X is weak Fraïssé.*
2) *For every $E \in \mathrm{Age}(X_{\mathcal{U}})$ the action $(\mathrm{Isom}(X))_{\mathcal{U}} \curvearrowright \mathrm{Emb}(E, X_{\mathcal{U}})$ is (almost) transitive.*

Furthermore, the following are equivalent:

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In particular, it follows that if X is Fraïssé, then its ultrapowers are Fraïssé and ultrahomogeneous.

Corollary

The non-separable L_p-space $(L_p(0,1))_{\mathcal{U}}$ is ultrahomogeneous.

A similar fact was observed for the Gurarij, by Aviles, Cabello, Castillo, Gonzalez, Moreno, 2013.
This is related to the theory of "strong Gurarij" spaces (Kubis).
Note: they must be non-separable.

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Note the result by G. Schechtman 1979 (+ Dor 1975 for $p = 1$) - as observed by D. Alspach 1983.

Theorem (Dor - Schechtman)

For any $1 \leq p < \infty$ any $\varepsilon > 0$, there exists $\delta = \delta_p(\epsilon) > 0$ such that

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for every $n \in \mathbb{N}$, and finite measure μ.

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Note however that Schechtman's result holds for $p = 4, 6, 8, \ldots$, so things have to be more complicated for other subspaces and $p \neq 4, 6, 8, \ldots$.

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Proposition (Plotkin and Rudin (1976))

For $p \notin 2\mathbb{N}$, suppose that $(f_1, \ldots, f_n) \in L_p(\Omega_0, \Sigma_0, \mu_0)$ and $(g_1, \ldots, g_n) \in L_p(\Omega_1, \Sigma_1, \mu_1)$ and

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Then $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are equidistributed (i.e. $\mu_0((f_1(\omega), \ldots, f_n(\omega)) \in B) = \mu_1((g_1(\omega), \ldots, g_n(\omega)) \in B)$ for every $B \subset \mathbb{R}^n$ Borel)

Also recall that we sketched the proof by Lusky that

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Those L_p's are (AuH).

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To prove that those L_p's are Fraïssé, the main step is to prove a "continuous" version of Plotkin-Rudin, in the sense that if

$$(1+\delta)^{-1}\|1+\sum_{j=1}^{n} a_j g_j\|_{\mu_1} \leq \|1+\sum_{j=1}^{n} a_j f_j\|_{\mu_0} \leq (1+\delta)\|1+\sum_{j=1}^{n} a_j g_j\|_{\mu_1}$$

then $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are "ε-equimeasurable" in some sense.

more precisely, we measure proximity of associated measures on $\mathbb{R}^n$ in the Lévy-Prokhorov metric.

$$d_{\mathcal{LP}}(\mu, \nu) := \inf \{\varepsilon > 0 \mid \mu(A) \leq \nu(A_\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A_\varepsilon) + \varepsilon \ \forall A\}.$$

Fraissé limits of non hereditary classes

It is also possible and useful to develop a Fraïssé theory with respect to certain classes of finite dimensional subspaces, which are not $\equiv$ to the Age of any X, because they are not hereditary.

For $L_p(0,1)$ we can use the family of ℓ_p^n's and the perturbation result of Dor -Schechtmann to give meaning to

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Fraïssé Banach lattices

By considering lattice embeddings and appropriate notions of δ-lattice embeddings, we may develop a Fraïssé theory in the lattice setting, defining Fraïssé Banach lattices, i.e. some unique universal object for classes of finite dimensional lattices with an approximate lattice ultrahomogeneity property.

For example for $1 \leq p < +\infty$, $L_p(0,1)$ is a Fraïssé Banach lattice.

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A related construction: the lattice Gurarij

Recall that the Gurarij space is obtained as the Fraissé limit of the class of finite dimensional normed spaces, or equivalently, as the limit of the class of spaces isometric to ℓ_∞^n's. See Bartosova - Lopez-Abad - Mbombo - Todorcevic (2017).

The point here is that isometric embeddings between ℓ_p^n's respect the lattice structure if $p < +\infty$, but not if $p = +\infty$.

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The "lattice Gurarij'

Our construction is strongy inspired by some work of Cabello-Sanchez (using $\Pi_{p \in \mathbb{N}} L_p(0,1)$ as ambient space).

Theorem (F. Cabello-Sanchez, 1998)

There exists a renorming of $C(0,1)$ as an M-space with almost transitive norm.

Theorem (the "lattice Gurarij")

There exists a renorming of $C(0,1)$ as an M-space $\mathbb{G}_{lattice}$ which is the Fraïssé limit of the ℓ_∞^n's with isometric lattice embeddings.
In particular, for any $\epsilon > 0$, for any lattice isometry t between two finite dimensional sublattices of $G_{lattice}$, there is a lattice isometry T on $\mathbb{G}_{lattice}$ such that

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Outline

1. Transitivities of isometry groups
2. Fraïssé theory and the KPT correspondence
3. Fraïssé Banach spaces
4. The Approximate Ramsey Property for ℓ_p^n's

The Approximate Ramsey Property

There is relatively well known form of the KPT correspondence, i.e. combinatorial characterization of the extreme amenability of an isometry group in terms of a Ramsey property of the Age, for metric structures.
This applies without difficulty to $(\mathrm{Isom}(X), SOT)$ for a Fraïssé Banach space X.

Definition

A collection $\mathcal{F}$ of finite dimensional normed spaces has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every bicoloring c of $\mathrm{Emb}(F, H)$ admits an embedding $\varrho \in \mathrm{Emb}(G, H)$ which is ε-monochromatic for c.

Here ε-monochromatic means that for some color i, $\varrho \circ \mathrm{Emb}(F, G) \subset c^{-1}(i)_\varepsilon := \{\tau \in \mathrm{Emb}(F, H) : d(c^{-1}(i), \tau) < \varepsilon\}$.

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Theorem (KPT correspondence for Banach spaces)

For X (AuH) the following are equivalent:

- $\mathrm{Isom}(X)$ *is extremely amenable.*
- $\mathrm{Age}(X)$ *has the approximate Ramsey property.*

An example of coloring

Consider $X = L_p$ and E a finite dimensional subspace of X. Color $\phi \in \mathrm{Emb}(E, X)$ blue if $\phi(E)$ is K-complemented in X and red otherwise.

Fact

If $p = 4, 6, 8, \ldots$ then the collection of finite dimensional subspaces of L_p does not satisfy the ARP.

PROOF.

Pick F a space with a well and a badly complemented copy inside L_p. Pick G some ℓ_p^n (and therefore 1-complemented in L_p) large enough to contain these two kinds of copies of F. This proves that ϕ defines a bad coloring. □

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The Approximate Ramsey Property for ℓ_p^n's

The KPT correspondence extends to the setting of ℓ_p^n-subspaces of L_p. This means we can recover the extreme amenability of $\mathrm{Isom}(L_p)$ through internal properties: i.e. through an approximate Ramsey property of isometric embeddings between ℓ_p^n's.

Theorem (Ramsey theorem for embeddings between ℓ_p^n's)

Given $1 \leq p < \infty$, integers d, m, r, and $\epsilon > 0$ there exists $n = n_p(d, m, r, \epsilon)$ such that whenever c is a coloring of $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ with r colors, there is some isometric embedding $\gamma : \ell_p^m \to \ell_p^n$ which is ϵ-monochromatic.

The case $p = \infty$ is due to Bartosova - Lopez-Abad - Mbombo - Todorcevic (2017). We have a direct proof for $p < \infty$, $p \neq 2$.

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Comment and previous Ramsey results

- Odell-Rosenthal-Schlumprecht (1993) proved that that for every $1 \leq p \leq \infty$, every $m \in \mathbb{N}$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for every finite coloring c on $S_{\ell_p^n}$ there is $Y \subset \ell_p^n$ isometric to ℓ_p^m so that S_Y is ϵ-monochromatic. Their proof uses tools from Banach space theory (like unconditionality) to find many symmetries;
- Note that Odell-Rosenthal-Schlumprecht is the case $d = 1$!
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A multidimensional Borsuk-Ulam antipodal theorem

We can relate our Ramsey result to an equivalent form of Borsuk-Ulam called Lyusternik-Schnirelman theorem (1930):

Theorem (a form of Borsuk-Ulam)

If the unit sphere $\mathcal{S}^{n-1}$ of ℓ_2^n is covered by n open sets, then one of them contains a pair $\{-x, x\}$ of antipodal points.

By the fact that every finite open cover of a finite dimensional sphere is the ϵ-fattening of some smaller open cover, for some $\epsilon > 0$, our result for $d = 1, m = 1$ may be seen as a version of Lyusternik-Schnirelman theorem (for $n \geq n_2(1, 1, r, \epsilon)$), and the result for d, m arbitrary may be seen as a multidimensional Borsuk-Ulam theorem.

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Consequences

We recover the result of Giordano-Pestov through KPT correspondence, but also (through the Fraïssé Banach space notion) some non-separable versions of it.

Theorem

The topological group $(\mathrm{Isom}(L_p), SOT)$ is extremely amenable (Giordano-Pestov).
The topological group $(\mathrm{Isom}((L_p)_{\mathcal{U}}), SOT)$ is also extremely amenable.

Since it is easy to prove the approximate Ramsey property for *lattice* isometric embeddings between ℓ_∞^n's, we also deduce:

Theorem

The group of lattice isometries on $\mathbb{G}_{lattice}$, with SOT, is extremely amenable.

What are the separable Fraïssé spaces?

Question

Find a separable Fraïssé (or even AUH) space different from the Gurarij or some $L_p(0,1)$.

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Are the Hilbert and the Gurarij the only stable separable Fraïssé spaces (Fraïssé property independent of the dimension)?

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Are the $L_p(0,1)$ spaces stable Fraïssé for p non even?

Also:

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Show that $L_p(0,1)$ does not admit an ultrahomogeneous renorming if $p \neq 2$.

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THANK YOU - GRACIAS!

F. Cabello-Sánchez, *Regards sur le problème des rotations de Mazur*, Extracta Math. 12 (1997), 97–116.

V. Ferenczi, J. Lopez-Abad, B. Mbombo, S. Todorcevic, *Amalgamation and Ramsey properties of L_p-spaces*, arXiv 1903.05504.

A. S. Kechris, V. G. Pestov, and S. Todorcevic, *Frass limits, Ramsey theory, and topological dynamics of automorphism groups.* Geom. Funct. Anal. **15** (2005), no. 1, 106–189.

V. Pestov, *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon.* Revised edition of Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005; MR2164572]. University Lecture Series, 40. American Mathematical Society, Providence, RI, 2006.