Inversion and extension of the finite Hilbert transform

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September 11, 2019

Workshop on Banach spaces and Banach lattices
ICMAT

Authorship

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The airfoil equation

- *"The study of an ideal flow past a thin airfoil"* lead in aerodynamics to the airfoil equation:

$$(AE) \qquad p.v.\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-t}\,dx = g(t), \quad a.e.\ t \in (-1,1).$$

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  - Birnbaum 1920's; von Kármán 1930's; Söhngen 1940's; Tricomi 1950's.
  - Tricomi *"Integral Equations"* (1957) for the spaces $L^p(-1,1)$.

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  - Nowadays is used in Tomography (image reconstruction).

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- The finite Hilbert transform is defined, for $f \in L^1(-1,1)$, by the principal value integral:

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- The setting of the L^p-spaces is not the most adequate for studying the FHT, because:

$$T\colon X \to X \text{ is injective} \iff L^{2,\infty}(-1,1) \nsubseteq X.$$

$$T\colon X \to X \text{ has non-dense range} \iff X \subseteq L^{2,1}(-1,1).$$

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$$|g| \leqslant |f| \text{ a.e. } \& \; f \in X \quad \Longrightarrow \quad g \in X \; \& \; \|g\|_X \leqslant \|f\|_X.$$

- X is rearrangement invariant:

$$m(\{x : |g(x)| > \lambda\} = m(\{x : |f(x)| > \lambda\}, \quad \text{for all } \lambda > 0$$

and $f \in X \quad \Longrightarrow \quad g \in X$ and $\|g\|_X = \|f\|_X$.

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Examples: L^p spaces, weak-L^p spaces, Orlicz spaces, Lorentz $L^{p,q}$ spaces, Lorentz Λ_ϕ spaces, Marcinkiewicz spaces,.....

Boundedness of T on r.i.s.

- Theorem (M. Riesz): For H the Hilbert transform on $\mathbb{R}$

$$H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \iff 1 < p < \infty.$$

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- Theorem (Boyd): For X r.i.s. on $\mathbb{R}$
$$H\colon X \to X \iff 0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1,$$
where the Boyd indices of X (with $E_{1/t}$ the dilation operator $f \mapsto f(\cdot/t)$):
$$0 \leqslant \underline{\alpha}_X := \lim_{t\to 0^+} \frac{\log \|E_{1/t}\|}{\log t} \leqslant \overline{\alpha}_X := \lim_{t\to\infty} \frac{\log \|E_{1/t}\|}{\log t} \leqslant 1.$$

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- Theorem: For X r.i.s. on $(-1,1)$

$$T\colon X \to X \iff 0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$$

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- We give inversion formulae for r.i.s. X on $(-1,1)$ in two cases :
  - When $1/2 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$.
  - When $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$.
  - Not when $1/2 \in [\underline{\alpha}_X, \overline{\alpha}_X]$.
    For example, X=$L^{2,q}$ for $1 \leqslant q \leqslant \infty$.

The case $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$

Theorem (C., Okada & Ricker, 2018)

Let X be a r.i.s. on $(-1, 1)$ *satisfying* $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$.

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(a) *$T\colon X \to X$ is injective.*

(b) *$\check{T}\colon X \to X$ and satisfies $\check{T}T = I$, for*

$$\check{T}(f)(x) := -\sqrt{1-x^2}\, T\left(\frac{f(t)}{\sqrt{1-t^2}}\right)(x).$$

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(c) *The range of T is* $R(T) = \left\{ f \in X : \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = 0 \right\}$.

(d) *$\check{T}$ is an isomorphism from $R(T)$ onto X.*

(e) $X = \left\{ f \in X : \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = 0 \right\} \oplus \left\langle \mathbf{1} \right\rangle$.

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(a) *$T\colon X \to X$ is surjective and $\mathrm{Ker}(T) = \langle 1/\sqrt{1-x^2}\rangle$.*

(b) *$\widehat{T}\colon X \to X$ and satisfies $T\widehat{T} = I$, for*

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(c) *The range of* $\widehat{T}$ *is* $R(\widehat{T}) = \left\{ f \in X : \int_{-1}^{1} f(x)dx = 0 \right\}$.

(d) $\widehat{T}$ *is an isomorphism onto its range.*

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Solution to the airfoil equation $Tf = g$

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- $1/2 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$: *given $g \in X$, all solutions $f \in X$ of the airfoil equation (AE) are given by*

$$f(x) = \frac{1}{\sqrt{1-x^2}} T\left(\sqrt{1-t^2}\, g(t)\right)(x) + \frac{\lambda}{\sqrt{1-x^2}}, \quad \lambda \in \mathbb{C}.$$

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- $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$: *given $g \in X$ satisfying* $\int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}} dx = 0$, *the unique solution $f \in X$ of the airfoil equation (AE) is*

$$f(x) := -\sqrt{1-x^2}\, T\left(\frac{g(t)}{\sqrt{1-t^2}}\right)(x).$$

Extension of the FHT on L^p

Theorem (Okada, Ricker & Sánchez-Pérez, 2008)

For $1 < p < \infty$ *with* $p \neq 2$*, the finite Hilbert transform*

$$T\colon L^p(-1,1) \to L^p(-1,1)$$

<u>cannot</u> be extended to a larger Banach function space.

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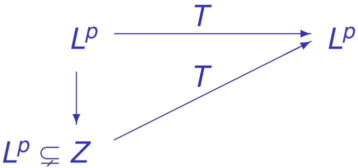

The proof is based on Tricomi's decomposition of L^p in terms of T.

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- There is no close description of functions belonging to $T(L^2)$.

  Okada (1991) gave a (rather complicate) characterization of when $g \in T(L^2)$.
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Question (1991): Is it possible to extend

$$T\colon L^2(-1,1) \to L^2(-1,1)$$

to a larger space: $T\colon Z \to L^2(-1,1)$?

Solution: for all r.i.s.!

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Theorem (C., Okada & Ricker, 2019)

Let X be a r.i.s. on $(-1,1)$ with $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$. The finite Hilbert transform

$$T\colon X \to X$$

cannot be extended to any genuinely larger Banach function space over $(-1,1)$.

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which is the optimal lattice domain for T:

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & X \\ \downarrow & \nearrow_{T} & \\ [T, X] & & \end{array}$$

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- Showing that $[T, X]$ is a Banach function space (with the Fatou property), for the norm

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- Thus, it suffices to prove that

$$[T, X] \subseteq X.$$

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(2) Proving $[T, X] \subseteq X$ is equivalent to showing, for some constant $C > 0$ and all simple functions ϕ, that:

$$(*) \qquad C\|\phi\|_X \leqslant \sup_{|\theta|=1} \|T(\theta\phi)\|_X,$$

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which is showing, for $\phi := \sum_{n=1}^{N} a_n \chi_{A_n}$, that

$$C\Big\| \sum_{n=1}^{N} a_n \chi_{A_n} \Big\|_X \leqslant \sup_{|\theta|=1} \Big\| T\Big(\theta \cdot \sum_{n=1}^{N} a_n \chi_{A_n}\Big) \Big\|_X$$

for some $C > 0$ independent of ϕ.

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The claim follows if we show that

$$(**) \qquad C\|\phi\|_X \leqslant \|F\|_{L^1(\Lambda)} \leqslant \|F\|_{L^\infty(\Lambda)} \leqslant \sup_{|\theta|=1} \|T(\theta\phi)\|_X .$$

Proof of $(**)$: $\|F\|_{L^\infty(\Lambda)} \leqslant \sup_{|\theta|=1} \|T(\theta\phi)\|_X$

(4) Since $\sigma_n = \pm 1$,

$$
\begin{aligned}
\|F\|_{L^\infty(\Lambda)} &= \sup_{\sigma\in\Lambda} \left\| T\left(\sum_{n=1}^{N} \sigma_n a_n \chi_{A_n} \right) \right\|_X \\
&= \sup_{\sigma\in\Lambda} \left\{ \left\| T\left(\theta \sum_{n=1}^{N} a_n \chi_{A_n} \right) \right\|_X : \theta = \sum_{n=1}^{N} \sigma_n \chi_{A_n} \right\} \\
&\leqslant \sup_{|\theta|=1} \left\| T(\theta\phi) \right\|_X .
\end{aligned}
$$

Proof of $(**)$: $C\|\phi\|_X \leqslant \|F\|_{L^1(\Lambda)}$

(5) Fubini's theorem and duality yields

$$\begin{aligned}
\|F\|_{L^1(\Lambda)} &= \int_\Lambda \left\| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right) \right\|_X d\sigma \\
&= \int_\Lambda \left(\sup_{\|g\|_{X'}=1} \int_{-1}^{1} |g(t)| \left| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| dt \right) d\sigma \\
&\geqslant \sup_{\|g\|_{X'}=1} \int_{-1}^{1} |g(t)| \left(\int_\Lambda \left| \sum_{n=1}^{N} \sigma_n a_n T\left(\chi_{A_n}\right)(t) \right| d\sigma \right) dt.
\end{aligned}$$

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For $t \in (-1, 1)$ fixed, via Khintchine inequality, we have

$$\int_\Lambda \left| \sum_{n=1}^{N} \sigma_n \, a_n T(\chi_{A_n})(t) \right| d\sigma \geqslant \frac{1}{\sqrt{2}} \left(\sum_{n=1}^{N} |a_n|^2 \, |T(\chi_{A_n})(t)|^2 \right)^{1/2}.$$

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Consequently, via duality again,

$$\|F\|_{L^1(\Lambda)} \geqslant \frac{1}{\sqrt{2}} \left\| \left(\sum_{n=1}^{N} |a_n|^2 \, |T(\chi_{A_n})|^2 \right)^{1/2} \right\|_X .$$

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(6) From the Stein-Weiss formula for the distribution function of $T(\chi_A)$, it follows that

$$m(\{t \in A : |T(\chi_A)(t)| > \lambda\}) = \frac{2m(A)}{e^{\pi\lambda} + 1}.$$

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We set $\lambda = 1$, and find disjoint sets $A_n^1 \subseteq A_n$ with

$$m(A_n^1) = \delta m(A_n),$$

for some $0 < \delta < 1$, with $|T(\chi_{A_n})| > 1$ on A_n^1.

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So :

$$\left(\sum_{n=1}^{N} |a_n|^2 \, |T\,(\chi_{A_n})|^2 \right)^{1/2} \geqslant \sum_{n=1}^{N} |a_n| \chi_{A_n^1}.$$

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(7) Consequently,

$$\|F\|_{L^1(\Lambda)} \geqslant \frac{1}{\sqrt{2}} \left\| \left(\sum_{n=1}^{N} |a_n|^2 \, |T(\chi_{A_n})|^2 \right)^{1/2} \right\|_X .$$

$$\geqslant \frac{1}{\sqrt{2}} \left\| \sum_{n=1}^{N} |a_n| \chi_{A_n^1} \right\|_X$$

$$\geqslant \frac{1}{\sqrt{2}} \frac{1}{\|E_\delta\|} \left\| \sum_{n=1}^{N} |a_n| \chi_{A_n} \right\|_X = C\|\phi\|_X .$$

Q.E.D.

Consequence

Corollary (C., Okada & Ricker, 2019)

Let X be a r.i.s. on $(-1,1)$ satisfying $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$. Given $f \in L^1(-1,1)$, the following conditions are equivalent.

(a) $f \in X$.

(b) $T(f\chi_A) \in X$ *for every* $A \in \mathcal{B}$.

(c) $T(f\theta) \in X$ *for every* $\theta \in L^\infty$ *with* $|\theta| = 1$ *a.e.*

(d) $T(h) \in X$ *for every* $h \in L^0$ *with* $|h| \leqslant |f|$ *a.e.*

References

By G. P. Curbera, S. Okada and W. J. Ricker:

- *Inversion and extension of the finite Hilbert transform on (-1,1)*,
  Annali di Matematica Pura ed Applicata, to appear.
- *Extension and integral representation of the finite Hilbert transform in rearrangement invariant spaces*,
  Quaestiones Mathematicae, to appear.
  Special issue dedicated to the memory of Joe Diestel.
- *Non-extendability of the finite Hilbert transform*,
  preprint.
- *The fine spectrum of the finite Hilbert transform*,
  in preparation.

Thank you for your attention