# Ando–Choi–Effros liftings for regular maps between Banach lattices

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# Lifting through a quotient map

## Question

Let $X$ be a Banach space. Suppose that $J \subset X$ is a closed subspace of $X$, and let $q : X \to X/J$ be the quotient map. Let $Y$ be a Banach space and let $T : Y \to X/J$ be a bounded linear map. When does there exist a bounded linear map $L : Y \to X$ such that $q \circ L = T$ ?

$$
\begin{tikzcd}
& X \arrow[d, "q"] \\
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# Ando–Choi–Effros liftings

**Theorem (Ando '75/Choi–Effros '77)**

*Let $X$ be a Banach space. Suppose that $J \subset X$ is an $M$-ideal in $X$, and let $q : X \to X/J$ be the quotient map. Let $Y$ be a separable Banach space with the $\lambda$-BAP, and let $T : Y \to X/J$ be a linear map with $\|T\| = 1$. Then there exists $L : Y \to X$ such that $q \circ L = T$ and $\|L\| \le \lambda$.*

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# What is ACE good for?

**Theorem (Borsuk–Dugundji)**

*Let $K$ be a compact Hausdorff space and let $D \subset K$ be a closed metrizable subset. Then there is a linear extension operator $T : C(D) \to C(K)$ with $\|T\| = 1$, i.e. $(Tx)(t) = x(t)$ for $x \in C(D)$ and $t \in D$.*

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## Theorem (Pełczyński)

*Let $A$ be the disk algebra, and suppose $D$ is a subset of the unit circle with Lebesgue measure 0. Then there is a contractive linear extension operator from $C(D)$ to $A$.*

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### Recent applications

Study of BAP of Lipschitz-free spaces.
[Borel-Mathurin '12], [Godefroy–Ozawa '14], [Godefroy '15], [CD '19], ...

# $M$-ideals

**Definition (Alfsen–Effros '72)**

Let $X$ be a Banach space.

(a) A linear projection $P : X \to X$ is called
- An $M$-projection if for all $x \in X$ we have $\|x\| = \max\{\|Px\|, \|x - Px\|\}$.
- An $L$-projection if for all $x \in X$ we have $\|x\| = \|Px\| + \|x - Px\|$.

(b) A closed subspace $J \subset X$ is called an $M$-ideal if $J^\perp$ is the range of an $L$-projection in $X^*$.

- $P$ is $M$-projection: $X = PX \oplus_\infty (I - P)X$
- $P$ is $L$-projection: $X = PX \oplus_1 (I - P)X$
- $J$ is $M$-ideal: $X^* = J^\perp \oplus_1 \widetilde{J}$

# Why are these called "ideals"?

**Theorem (Effros–Prosser '63)**

*Let $\mathfrak{A}$ be a $C^*$-algebra, and $\mathfrak{I} \subset \mathfrak{A}$ a closed subspace. Then*

$$\mathfrak{I} \text{ is an } M\text{-ideal} \Leftrightarrow \mathfrak{I} \text{ is a 2-sided ideal.}$$

**Theorem (Smith–Ward '78)**

*In a unital Banach algebra, every $M$-ideal is a subalgebra.*

**Theorem (Smith '81)**

*Let $\mathfrak{A}$ be a function algebra on a compact space $K$, and $\mathfrak{I} \subset \mathfrak{A}$ a closed subspace. Then*

$$\mathfrak{I} \text{ is an } M\text{-ideal} \Leftrightarrow \mathfrak{I} \text{ is an ideal containing a bounded approx. unit.}$$

# Examples of $M$-ideals

- $c_0$ is an $M$-ideal in $\ell_\infty$.
- $(E_n)$ sequence of finite-dimensional subspaces of $X$, then

$$c_0(E_n) = \big\{(x_n) : x_n \in E_n \text{ for each } n \in \mathbb{N}, \quad \lim_n x_n = 0\big\}$$

  is an $M$-ideal in

$$c(E_n) = \big\{(x_n) : x_n \in E_n \text{ for each } n \in \mathbb{N}, \quad (x_n) \text{ converges}\big\}$$

- $\mathcal{K}(\ell_p, \ell_q)$ is an $M$-ideal in $\mathcal{L}(\ell_p, \ell_q)$ when $1 < p \leq q < \infty$.
- $\mathcal{K}(Y, c_0)$ is an $M$-ideal in $\mathcal{L}(Y, c_0)$ for any $Y$.

# The Bounded Approximation Property

## Definition

A Banach space $X$ has the $\lambda$-BAP if for each $\varepsilon > 0$ and compact set $K \subset X$, there exists a finite-rank linear map $S : X \to X$ with $\|S\| \leq \lambda$ and $\|S(x) - x\| \leq \varepsilon$ for each $x \in K$.

## Examples

$C(K), L_p(\mu), \ell_p(\Gamma), c_0(\Gamma), \ldots$

# The project

## Goal

Find Ando–Choi–Effros liftings in the context of Banach lattices, where the mappings should additionally behave well with respect to the order structure.

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A Banach lattice is a Banach space $X$ endowed with a partial order satisfying:

- $x \leq y \Rightarrow x + z \leq y + z$ for all $x, y, z \in X$.
- $x \geq 0, a \geq 0 \Rightarrow ax \geq 0$ for $x \in X$, $a \in \mathbb{R}$.
- For all $x, y \in X$ there exist $x \vee y, x \wedge y \in X$.
- $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$, where $|x| = x \vee (-x)$.

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# Quotients of Banach lattices

To make sense out of

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in the category of Banach lattices, we would like for $X/J$ to be a Banach lattice as well.

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in the category of Banach lattices, we would like for $X/J$ to be a Banach lattice as well.

In order to have a natural lattice structure for $X/J$, we need

## Definition

Let $X$ be a Banach lattice, $J \subseteq X$ a closed linear subspace. $J$ is an order ideal in $X$ if

$$\big[x \in J, \quad |y| \leq |x|\big] \Rightarrow y \in J.$$

# Operators between Banach lattices

Let $T : X \to Y$ be a linear operator between Banach lattices.

- $T$ is called positive if $x \geq 0 \Rightarrow Tx \geq 0$.
- $T : X \to Y$ is called regular if it is the difference of two positive operators.

We write $\mathcal{L}_r(X, Y)$ for the Banach space of all regular operators from $X$ to $Y$, endowed with the norm

$$\|T\|_r = \inf \left\{ \|S\| \ : \ S : X \to Y \text{ positive, such that } \pm T \leq S \right\}.$$

In general $\mathcal{L}_r(X, Y)$ need not be a lattice, but it is if either:

- $\dim(X) < \infty$.
- $Y$ is Dedekind complete, that is, every nonempty subset bounded above has a supremum (e.g. when $Y = Z^*$) .

# Order $M$- and $L$-projections, Order $M$-ideals

## Recall

$X$ Banach space, $P : X \to X$ projection, $J \subseteq X$ closed subspace.

- $P$ is $M$-projection: $X = PX \oplus_\infty (I - P)X$
- $P$ is $L$-projection: $X = PX \oplus_1 (I - P)X$
- $J$ is $M$-ideal: $X^* = J^\perp \oplus_1 \widetilde{J}$

## Definition

Let $X$ be a Banach lattice.

- An order $M$- (resp. $L$-) projection for $X$ is an $M$- (resp. $L$-) projection $P : X \to X$ such that $0 \leq P \leq Id_X$.
- An order $M$-ideal is a closed subspace $J \subseteq X$ such that $J^\perp$ is the range of an order $L$-projection.

# Order $M$-ideals

**Remark**

Order $M$-ideals had already been considered in the literature: [Haydon '77], [Ando '73].

Order $M$-ideals are relatively easy to find:

**Proposition (CD '19+)**

*An order ideal which is also an $M$-ideal is automatically an order M-ideal.*

# The Bounded Positive Approximation Property

## Definition

A Banach lattice $Y$ has the $\lambda$-BPAP if for each $\varepsilon > 0$ and compact set $K \subset X$, there exists a positive finite-rank linear map $S : Y \to Y$ with $\|S\| \leq \lambda$ and $\|S(x) - x\| \leq \varepsilon$ for each $x \in K$.

## Examples

$C(K), L_p(\mu), \ell_p(\Gamma), c_0(\Gamma), \ldots$

# What was known in this regard?

- [Ando '73], [Vesterstrøm '73] proved lifting theorems for certain Banach spaces with a partial order, but they require the finite-rank approximations to be positive projections and consider only $T = Id_{X/J} : X/J \to X/J$.

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- [Andersen '74] proved a closely related result, stated in the language of extensions between certain function spaces. It relaxes the approximation requirement to the BPAP, but assumes that the space of functions contains the constants and still considers only $T = Id_{X/J} : X/J \to X/J$.

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- Our results appear to be the first ones to be done for regular operators $T : Y \to X/J$.

# Ando–Choi–Effros liftings for Banach lattices

## Theorem (CD ’19+)

*Suppose that $J$ is an order $M$-ideal in the Banach lattice $X$, and let $q : X \to X/J$ be the canonical quotient map. Let $Y$ be a Banach lattice, and let $T : Y \to X/J$ be a regular map with $\|T\|_r = 1$. If $Y$ is separable and has the $\lambda$-BPAP, and both $X$ and $Y$ are Dedekind complete, then there exists $L : Y \to X$ such that $q \circ L = T$ and $\|L\|_r \le \lambda$.*

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SURPRISE!

# Proving the Ando–Choi–Effros theorem

## Key components of the proof

- A finite-dimensional lemma yielding an actual lift from an almost lift, without increasing the norm.
- The BAP + an inductive procedure work together to give the general result, once we take limits.

# Finite dimensions: from almost-lifts to lifts

## Lemma (CD '19+)

*Suppose that $J$ is an order $M$-ideal in a Banach lattice $X$. Let $F \subseteq E$ be finite-dimensional Banach lattices, and let $T : E \to X/J$ be a regular map with $\|T\|_r \le 1$. Given $\varepsilon > 0$, if there exists $L : E \to X$ such that $\|L\|_r \le 1$ and $\|(q \circ L - T)|_F\|_r \le \varepsilon$, then there exists $\tilde{L} : E \to X$ such that $T = q \circ \tilde{L}$, $\|\tilde{L}\|_r \le 1$ and $\|(\tilde{L} - L)|_F\|_r \le 6\varepsilon$.*

$$
\begin{tikzcd}
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F \arrow[urr, "L|_F"] \arrow[r, hook] & E \arrow[r, "T"'] \arrow[ur, phantom, "\circlearrowleft_{\varepsilon}"] & X/J
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$$

# Finite dimensions: from almost-lifts to lifts

**Lemma (CD '19+)**

*Suppose that $J$ is an order $M$-ideal in a Banach lattice $X$. Let $F \subseteq E$ be finite-dimensional Banach lattices, and let $T : E \to X/J$ be a regular map with $\|T\|_r \leq 1$. Given $\varepsilon > 0$, if there exists $L : E \to X$ such that $\|L\|_r \leq 1$ and $\|(q \circ L - T)|_F\|_r \leq \varepsilon$, then there exists $\tilde{L} : E \to X$ such that $T = q \circ \tilde{L}$, $\left\|\tilde{L}\right\|_r \leq 1$ and $\left\|(\tilde{L} - L)\big|_F\right\|_r \leq 6\varepsilon$.*

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## About the proof

- The arguments follow closely [Choi–Effros '77], using tensor products.
- The duality $E \otimes_\pi X^* \equiv \mathcal{L}(E, X)^*$ is replaced by $E \otimes_{|\pi|} X^* \equiv \mathcal{L}_r(E, X)^*$; this is related to the Principle of Local Reflexivity.

# The inductive argument: Banach space case

- Let $\{y_n\}_{n=1}^{\infty}$ have dense span in $Y$.
- $F_n \subset Y$ finite-dimensional subspace, suppose $L_n : F_n \to X$ is a lift for $T|_{F_n}$.
- By $\lambda$-BAP, find finite-rank $S_n : Y \to Y$ with $\|S_n\| \leq \lambda$ such that $\left\|(S_n - Id)|_{F_n}\right\| \leq 1/2^{n+1}$.
- Enlarge the subspace: $F_{n+1} = F_n + S_n(Y) + \mathbb{R}y_n$.
- Apply the Lemma, rinse and repeat, take limits appropriately.

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- Enlarge the sublattice: $F_{n+1} = F_n + S_n(Y) + \mathbb{R}y_n$. $\leftarrow$ WAIT!
- Apply the Lemma, rinse and repeat, take limits appropriately.

# The big issue!

## Problem

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But when the lattice is Dedekind complete, it is almost true:

## Lemma (attributed to Johnson)

*Let $F$ be a finite-dimensional subspace of a Dedekind complete Banach lattice $X$ and let $\varepsilon > 0$. Then there exist a finite-dimensional sublattice $E$ of $X$ and a linear map $P : F \to E$ such that $\|(P - Id)|_F\| \leq \varepsilon$.*

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- We use a strengthening of a strengthening from [Lissitsin–Oja '14].
- This explains why $Y$ is assumed Dedekind complete.
- The Dedekind completeness assumption for $X$ comes from the last step of taking limits.

# Ando–Choi–Effros liftings, Version 2

## Theorem (Ando '75)

*Let $X$ be a Banach space. Suppose that $J \subset X$ is an $M$-ideal in $X$, and $J$ is an $L^1$-predual. Let $q : X \to X/J$ be the quotient map. Let $Y$ be a separable Banach space, and let $T : Y \to X/J$ be a linear map with $\|T\| = 1$. Then there exists $L : Y \to X$ such that $q \circ L = T$ and $\|L\| \leq 1$.*

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# Ando–Choi–Effros liftings, Version 2

## Theorem (Ando '75)

*Let $X$ be a Banach space. Suppose that $J \subset X$ is an $M$-ideal in $X$, and $J$ is an $L^1$-predual. Let $q : X \to X/J$ be the quotient map. Let $Y$ be a separable Banach space, and let $T : Y \to X/J$ be a linear map with $\|T\| = 1$. Then there exists $L : Y \to X$ such that $q \circ L = T$ and $\|L\| \leq 1$.*

$$\begin{tikzcd} & X \arrow[d, "q"] \\ Y \arrow[ur, dashed, "L"] \arrow[r, "T"'] & X/J \end{tikzcd}$$

## Theorem

*$J$ is an $L^1$-predual $\Leftrightarrow$ $J^{**}$ is an injective Banach space.*

# Ando–Choi–Effros liftings for lattices, Version 2

## Theorem (CD '19+)

*Let $X$ be a Banach lattice. Suppose that $J \subset X$ is an order $M$-ideal in $X$, and and $J$ satisfies Cartwright's property (C). Let $q : X \to X/J$ be the quotient map. Let $Y$ be a separable Banach lattice, and let $T : Y \to X/J$ be a linear map with $\|T\|_r = 1$. Then there exists $L : Y \to X$ such that $q \circ L = T$ and $\|L\|_r \leq 1$.*

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## Theorem (Cartwright '75)

*$J$ satisfies property (C) $\Leftrightarrow$ $J^{**}$ is an injective Banach lattice.*

# Open question

$$\begin{tikzcd} & X \arrow[d, "q"] \\ Y \arrow[ur, dashed, "L"] \arrow[r, "T"'] & X/J \end{tikzcd}$$

- If the original mapping $T$ is positive, can we find a lift $L$ which is also positive?
- The answer is yes in some known special cases (e.g. $X = C(K)$).
- It is possible that the current proof already shows that, but one would need to take a really deep dive into the guts of [Alfsen–Effros '72].

THANKS!