Measures, supports, and extremality in Lipschitz-free spaces

Ramón J. Aliaga

Universitat Politècnica de València

(joint work with Eva Pernecká,
Colin Petitjean & Antonín Procházka)

Workshop on Banach Spaces and Banach Lattices
Madrid, 11 Sep 2019

Lipschitz spaces

Let (M,d) be a complete metric space.
Fix a base point $0 \in M$.
The *Lipschitz constant* of $f : M \to \mathbb{R}$ is

$$\|f\|_L := \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y \in M \right\}.$$

The spaces of Lipschitz functions on M are

$$\mathrm{Lip}(M) = \{f : M \to \mathbb{R} : \|f\|_L < \infty\}$$
$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R} : \|f\|_L < \infty, f(0) = 0\}$$

$\mathrm{Lip}_0(M)$ is a Banach space with norm $\|\cdot\|_L$.

Lipschitz-free spaces

For $x \in M$, consider the evaluation operators

$$\delta(x) : f \mapsto f(x).$$

Then $\delta \colon M \to \mathrm{Lip}_0(M)^*$ is a (nonlinear) isometric embedding.

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Lipschitz-free space

$$\mathcal{F}(M) = \overline{\operatorname{span}}\, \delta(M) \subset \mathrm{Lip}_0(M)^*$$

Theorem (Arens, Eells 1956)

$$\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M)$$

Lipschitz-free subspaces

Theorem (Kadets 1985)

If $M_0 \subset M$, then $\mathcal{F}(M_0) \subset \mathcal{F}(M)$ isometrically:

$$\mathcal{F}(M_0) \cong \overline{\operatorname{span}}\, \delta(M_0)$$

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We will assume $0 \in M_0$. Otherwise, we mean

$$\mathcal{F}(M_0) \equiv \mathcal{F}(M_0 \cup \{0\}).$$

The intersection theorem

Theorem (Aliaga, Pernecká 2019)

Let $K_i \subset M$ be closed subsets. Then

$$\bigcap_i \mathcal{F}(K_i) = \mathcal{F}\left(\bigcap_i K_i\right)$$

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Let $m \in \mathcal{F}(M)$. We define the *support of m* as

$$\mathrm{supp}(m) = \bigcap \{S \subset M \text{ closed} : m \in \mathcal{F}(S)\}$$

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Theorem (Aliaga, Pernecká 2019)

$$m \in \mathcal{F}(\operatorname{supp}(m))$$

Supports in $\mathcal{F}(M)$

Proposition

Let $m \in \mathcal{F}(M)$, $K \subset M$ closed. TFAE:

- $\mathrm{supp}(m) \subset K$
- $m \in \mathcal{F}(K)$
- If $f, g \in \mathrm{Lip}_0(M)$ satisfy $f|_K = g|_K$, then $\langle m, f \rangle = \langle m, g \rangle$

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Proposition

Let $m \in \mathcal{F}(M)$, $p \in M$. TFAE:

- $p \in \mathrm{supp}(m)$
- For every neighborhood U of p, there is $f \in \mathrm{Lip}_0(M)$ supported on U such that $\langle m, f\rangle \neq 0$

Weighting in $\mathrm{Lip}_0(M)$ and $\mathcal{F}(M)$

Proposition

Let $h \in \mathrm{Lip}(M)$ with bounded support. If $f \in \mathrm{Lip}_0(M)$ then $f \cdot h \in \mathrm{Lip}_0(M)$ and

$$T_h : f \mapsto f \cdot h$$

is a w^*-w^*-continuous linear operator on $\mathrm{Lip}_0(M)$.

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Thus T_h has a continuous preadjoint $(T_h)_* : \mathcal{F}(M) \to \mathcal{F}(M)$ defined by $(T_h)_*(m) = m \circ T_h$:

$$\langle m \circ T_h, f \rangle = \langle m, T_h(f) \rangle = \langle m, f \cdot h \rangle \quad \text{for } f \in \mathrm{Lip}_0(M)$$

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Moreover

$$\mathrm{supp}(m \circ T_h) \subset \mathrm{supp}(m) \cap \mathrm{supp}(h)$$

Decomposition in $\mathcal{F}(M)$

Proposition

Let $m \in \mathcal{F}(M)$, $\mathrm{supp}(m) \subset S_1 \cup S_2$ where S_1, S_2 are closed and disjoint, $d(S_1, S_2) > 0$, and S_1 is bounded. Then there is a unique decomposition

$$m = m_1 + m_2$$

where $\mathrm{supp}(m_1) \subset S_1$ and $\mathrm{supp}(m_2) \subset S_2$.

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where $\operatorname{supp}(m_1) \subset S_1$ and $\operatorname{supp}(m_2) \subset S_2$.

Proof of uniqueness:
Let $m = m_1 + m_2 = m_1' + m_2'$.
Then $m_1 - m_1' = m_2' - m_2$.
But $\operatorname{supp}(m_1 - m_1') \subset S_1 \cap S_2 = \varnothing$, so $m_1 - m_1' = 0$. $\square$

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where $\mathrm{supp}(m_1) \subset S_1$ and $\mathrm{supp}(m_2) \subset S_2$.

Proof of existence:

Let $m_1 = m \circ T_h$ where $h(x) = \max\left\{1 - \dfrac{d(x, S_1)}{d(S_1, S_2)}, 0\right\}$. $\square$

Note: $\|m_1\| \leq \|m\| \cdot \left(1 + \dfrac{\mathrm{rad}(S_1)}{d(S_1, S_2)}\right)$

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$$m = m_1 + m_2$$

where $\operatorname{supp}(m_1) \subset S_1$ and $\operatorname{supp}(m_2) \subset S_2$.

Both conditions are needed in general.
If m is positive then they can be removed.

Measures on M as functionals on $\mathrm{Lip}_0(M)$

Given $\mu \in \mathcal{M}(M)$, define the functional $\mathcal{L}\mu$ on $\mathrm{Lip}_0(M)$ by

$$\mathcal{L}\mu(f) = \int_M f\, d\mu$$

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We only consider $\mathcal{M}_0(M) = \{\mu \in \mathcal{M}(M) : \mu(\{0\}) = 0\}$.

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Proposition

Let $m \in \mathcal{F}(M)$ and $\mu \in \mathcal{M}_0(M)$. Suppose $m = \mathcal{L}\mu$. Then

- $\mathrm{supp}(m) = \mathrm{supp}(\mu)$
- m is positive iff μ is positive

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In particular, if $m = \mathcal{L}\mu$ then $\mu \in \mathcal{M}_0(M)$ is unique.

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Proposition

Let $\mu \in \mathcal{M}_0(M)$. TFAE:

- $d(\cdot, 0) \in L_1(\mu)$
- $\mathcal{L}\mu = \int_M \delta(x) \, d\mu(x)$ as a Bochner integral
- $\mathcal{L}\mu \in \mathrm{Lip}_0(M)^*$
- $\mathcal{L}\mu \in \mathcal{F}(M)$

If $\mathrm{diam}(M) < \infty$, every measure induces an element of $\mathcal{F}(M)$.

Measures on M as functionals on $\mathrm{Lip}_0(M)$

Given $\mu \in \mathcal{M}_0(\beta M)$, define the functional $\mathcal{L}\mu$ on $\mathrm{Lip}_0(M)$ by

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Let $\mu \in \mathcal{M}_0(\beta M)$. TFAE:

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Measures on βM as functionals on $\mathrm{Lip}_0(M)$

Suppose $\mu \in \mathcal{M}_0(\beta M)$ and $\mathcal{L}\mu \in \mathrm{Lip}_0(M)^*$.
When does $\mathcal{L}\mu \in \mathcal{F}(M)$?

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Suppose $\mu \in \mathcal{M}_0(\beta M)$ and $\mathcal{L}\mu \in \mathrm{Lip}_0(M)^*$.
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It is sufficient that μ is concentrated on M.

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If μ is positive, then $\mathcal{L}\mu \in \mathcal{F}(M)$ iff μ is concentrated on M.

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Proposition

If μ is positive, then $\mathcal{L}\mu \in \mathcal{F}(M)$ iff μ is concentrated on M.

It is not necessary in general:
Let $\xi_1 \neq \xi_2 \in \beta M$ such that $f(\xi_1) = f(\xi_2)$ for every Lipschitz f.
Take $\mu = \delta_{\xi_1} - \delta_{\xi_2}$.
Then $\mathrm{supp}(\mu) = \{\xi_1, \xi_2\} \not\subseteq M$, but $\mathcal{L}\mu = 0 \in \mathcal{F}(M)$.

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Suppose $\mu \in \mathcal{M}_0(\beta M)$ and $\mathcal{L}\mu \in \mathrm{Lip}_0(M)^*$.
When does $\mathcal{L}\mu \in \mathcal{F}(M)$?

It is sufficient that $\mathcal{L}\mu = \mathcal{L}(\mu|_M)$, that is

$$\int_{\beta M} f\,d\mu = \int_M f\,d\mu \quad \text{for } f \in \mathrm{Lip}_0(M)$$

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Proposition

If M is locally compact, then $\mathcal{L}\mu \in \mathcal{F}(M)$ iff $\mathcal{L}\mu = \mathcal{L}(\mu|_M)$.

Elements of $\mathcal{F}(M)$ representable as measures

In general, not all elements of $\mathcal{F}(M)$ are represented by measures.

Theorem

Let M be a bounded complete metric space. TFAE:

- every $m \in \mathcal{F}(M)$ is $m = \mathcal{L}\mu$ for some $\mu \in \mathcal{M}(M)$
- M is uniformly discrete

Elements of $\mathcal{F}(M)$ representable as measures

Theorem

Let $m \in \mathcal{F}(M)$. If $m = \mathcal{L}\mu$ for some $\mu \in \mathcal{M}(M)$, then m is the limit of elements

$$\sum_{k=1}^{n} a_k \delta(p_k) \in \mathcal{F}(M)$$

such that $\sum_{k=1}^{n} |a_k|$ is uniformly bounded.
If m is positive or M is locally compact, the converse also holds.

Elements of $\mathcal{F}(M)$ representable as measures

Theorem

Let $m \in \mathcal{F}(M)$ such that $0 \notin \mathrm{supp}(m)$. TFAE:

- $m = \mathcal{L}\mu$ for some $\mu \in \mathcal{M}(M)$
- $m = m^+ - m^-$ for some positive $m^+, m^- \in \mathcal{F}(M)$

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In particular:

$$m \text{ positive, } 0 \notin \operatorname{supp}(m) \quad \Longrightarrow \quad m = \mathcal{L}\mu$$
$$\text{for some finite positive measure } \mu \text{ on } M$$

If $0 \in \operatorname{supp}(m)$ this is no longer true.

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If $0 \in \operatorname{supp}(m)$ this is no longer true.

Theorem

Every positive element of $\mathcal{F}(M)$ is represented by a σ-finite positive measure on M.

Extremal structure of $\mathcal{F}(M)$

Research program

Let M be a complete pointed metric space.
What are the extreme points of $B_{\mathcal{F}(M)}$?

Elementary molecules

An *elementary molecule* is $u_{pq} = \dfrac{\delta(p) - \delta(q)}{d(p,q)} \in S_{\mathcal{F}(M)}$.

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Properties:

- $\|f\|_L = \sup\left\{\langle u_{pq}, f\rangle : p, q \in M\right\}$ since

$$\langle u_{pq}, f\rangle = \frac{f(p) - f(q)}{d(p,q)}$$

- $B_{\mathcal{F}(M)} = \overline{\operatorname{conv}}\left\{u_{pq} : p, q \in M\right\}$
- For every $\varepsilon > 0$, $m \in \mathcal{F}(M)$ admits an expression

$$m = \sum_{n=1}^{\infty} a_n u_{p_n q_n} \quad \text{where} \quad \sum_{n=1}^{\infty} |a_n| < \|m\| + \varepsilon$$

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Theorem (Weaver 1995)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule.

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Corollary

Every extreme point of $B_{\mathcal{F}(M)}$ with finite support is an elementary molecule.

Extreme molecules

Let $p, q \in M$. The *metric segment* between p and q is

$$[p,q] = \{x \in M : d(p,x) + d(x,q) = d(p,q)\} .$$

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If u_{pq} is an extreme point of $B_{\mathcal{F}(M)}$, then $[p, q] = \{p, q\}$.

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Proposition

If u_{pq} is an extreme point of $B_{\mathcal{F}(M)}$, then $[p,q] = \{p,q\}$.

Proof: If $x \in [p,q]$ then $u_{pq} \in [u_{px}, u_{xq}]$:

$$\begin{aligned} u_{pq} = \frac{\delta(p) - \delta(q)}{d(p,q)} &= \frac{\delta(p) - \delta(x)}{d(p,q)} + \frac{\delta(x) - \delta(q)}{d(p,q)} \\ &= \frac{d(p,x)}{d(p,q)} u_{px} + \frac{d(x,q)}{d(p,q)} u_{xq}. \qquad \square \end{aligned}$$

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Theorem

The intersection of all faces of $B_{\mathcal{F}(M)}$ that contain u_{pq} is contained in $\mathcal{F}([p,q])$.

Extreme molecules

Theorem (Petitjean, Procházka 2018)

u_{pq} is an exposed point of $B_{\mathcal{F}(M)}$ iff $[p,q] = \{p,q\}$.

$x \in S_X$ is *exposed* $\equiv$ there is $f \in S_{X^*}$ that attains its norm at x and nowhere else.

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Theorem

The face of $B_{\mathcal{F}(M)}$ determined by f_{pq} is contained in $\mathcal{F}([p,q])$.

Exposed molecules

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The face of $B_{\mathcal{F}(M)}$ determined by f_{pq} is contained in $\mathcal{F}([p,q])$.

The exposing functional for u_{pq} is

$$f_{pq}(x) = \frac{d(p,q)}{2} \frac{d(x,q) - d(x,p)}{d(x,q) + d(x,p)} - \text{constant}$$

with the property:

$$\langle u_{xy}, f_{pq} \rangle \geq 1 - \varepsilon \quad \text{implies} \quad x, y \in [p,q]_\varepsilon$$

where

$$[p,q]_\varepsilon = \left\{ x \in M : d(p,x) + d(x,q) \leq \frac{d(p,q)}{1-\varepsilon} \right\}$$

Exposed molecules

Theorem (Petitjean, Procházka 2018)

If $m \in B_{\mathcal{F}(M)}$ and $\langle m, f_{pq} \rangle = 1$, then $\mathrm{supp}(m) \subset [p,q]$.

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Proof: Fix $\delta, \varepsilon > 0$ and write

$$m = \sum_{n=1}^{\infty} a_n u_{x_n y_n} \quad \text{where} \quad \sum_{n=1}^{\infty} |a_n| < 1 + \delta\varepsilon\,.$$

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Let $I = \left\{ n : \langle u_{x_n y_n}, f_{pq} \rangle \geq 1 - \varepsilon \right\}$. Then

$$\begin{aligned} 1 = \langle m, f_{pq} \rangle &= \sum_{n=1}^{\infty} a_n \langle u_{x_n y_n}, f_{pq} \rangle = \sum_{n \in I} + \sum_{n \notin I} \\ &\leq \sum_{n \in I} |a_n| + (1 - \varepsilon) \sum_{n \notin I} |a_n| < (1 + \delta\varepsilon) - \varepsilon \sum_{n \notin I} |a_n| \,. \end{aligned}$$

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Theorem (Petitjean, Procházka 2018)

If $m \in B_{\mathcal{F}(M)}$ and $\langle m, f_{pq} \rangle = 1$, then $\operatorname{supp}(m) \subset [p, q]$.

That is, $\sum_{n \notin I} |a_n| < \delta$. So

$$\left\| m - \sum_{n \in I} a_n u_{x_n y_n} \right\| = \left\| \sum_{n \notin I} a_n u_{x_n y_n} \right\| \leq \sum_{n \notin I} |a_n| < \delta.$$

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Thus m is δ-close to $\sum_{n \in I} a_n u_{x_n y_n}$, whose supp is in $[p, q]_\varepsilon$.
Hence $\operatorname{supp}(m) \subset [p, q]_\varepsilon$.

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Thus m is δ-close to $\sum_{n \in I} a_n u_{x_n y_n}$, whose supp is in $[p,q]_\varepsilon$.
Hence $\mathrm{supp}(m) \subset [p,q]_\varepsilon$. So

$$\mathrm{supp}(m) \subset \bigcap_{\varepsilon > 0} [p,q]_\varepsilon = [p,q]. \quad \square$$

Extreme points of the positive ball

Theorem (Aliaga, Pernecká, Petitjean, Procházka 2019)

The extreme points of $B^+_{\mathcal{F}(M)}$ are 0 and $\dfrac{\delta(p)}{\|\delta(p)\|}$, $p \in M$.

Extreme points of the positive ball

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The extreme points of $B^+_{\mathcal{F}(M)}$ are 0 and $\dfrac{\delta(p)}{\|\delta(p)\|}$, $p \in M$.

Proof of necessity: Let $m \in S^+_{\mathcal{F}(M)}$ and suppose $p \neq q \in \operatorname{supp}(m)$.

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Let $h \in \operatorname{Lip}(M)$, $0 \leq h \leq 1$, $h = 1$ near p, $h = 0$ near q.
Then $m \circ T_h$ is positive and $\neq 0$.
So is $m - m \circ T_h = m \circ T_{1-h}$.

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So is $m - m \circ T_h = m \circ T_{1-h}$. So

$$\|m \circ T_h\| + \|m \circ T_{1-h}\| = \|m\| = 1$$

and

$$m = \|m \circ T_h\| \frac{m \circ T_h}{\|m \circ T_h\|} + \|m \circ T_{1-h}\| \frac{m \circ T_{1-h}}{\|m \circ T_{1-h}\|} \qquad \square$$

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If $m, m' \in \mathcal{F}(M)$ and $0 \leq m \leq m'$ then $\mathrm{supp}(m) \subset \mathrm{supp}(m')$.

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Suppose $\frac{\delta(p)}{\|\delta(p)\|} = \frac{1}{2}(m_1 + m_2)$ for positive m_1, m_2.
Since $0 \leq \frac{1}{2}m_i \leq \frac{\delta(p)}{\|\delta(p)\|}$, by the lemma $\operatorname{supp}(m_i) \subset \{p\}$. $\square$

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Lemma

If $m \in \mathcal{F}(M)$, $\phi \in \mathcal{F}(M)^{**}$ and $0 \leq \phi \leq m$ then $\phi \in \mathcal{F}(M)$.

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If $m \in \mathcal{F}(M)$, $\phi \in \mathcal{F}(M)^{**}$ and $0 \leq \phi \leq m$ then $\phi \in \mathcal{F}(M)$.

Suppose $\frac{\delta(p)}{\|\delta(p)\|} = \frac{1}{2}(\phi_1 + \phi_2)$ for positive $\phi_1, \phi_2 \in \mathrm{Lip}_0(M)^*$.
Since $0 \leq \frac{1}{2}\phi_i \leq \frac{\delta(p)}{\|\delta(p)\|}$, by the lemma $\phi_i \in \mathcal{F}(M)$.

Almost positive extreme points

In particular:
If an extreme point of $B_{\mathcal{F}(M)}$ is positive, then it is a molecule.

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Theorem (Aliaga, Pernecká, Petitjean, Procházka 2019)

Let m be an extreme point of $B_{\mathcal{F}(M)}$. Suppose

$$m = \lambda + \mu$$

where λ is positive and μ has finite support.
Then m is a molecule.

Open problem

Main question

Let M be a complete pointed metric space.
Are all extreme points of $B_{\mathcal{F}(M)}$ molecules?

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Known to be true when:

- M is compact and $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$ (Weaver 1999)
  - M is compact Hölder
  - M is compact and countable (Dalet 2015)
  - M is compact and ultrametric (Dalet 2015)

$$\mathrm{lip}_0(M) = \left\{ f \in \mathrm{Lip}_0(M) : \frac{|f(p) - f(q)|}{d(p,q)} \to 0 \text{ unif. as } d(p,q) \to 0 \right\}$$

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- $\mathcal{F}(M)$ has a natural predual that is a subspace of $\mathrm{lip}_0(M)$ (García-Lirola, Petitjean, Procházka, Rueda 2017)

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- $\mathcal{F}(M)$ has a natural predual that is a subspace of $\mathrm{lip}_0(M)$ (García-Lirola, Petitjean, Procházka, Rueda 2017)
- M is a subset of an $\mathbb{R}$-tree (Aliaga, Petitjean, Procházka 2019)

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Equivalently,

(1) Are all extreme points of $B_{\mathcal{F}(M)}$ exposed?
(2) Are all exposed points of $B_{\mathcal{F}(M)}$ molecules?

Thank you for your attention!

References:

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