Generalized Geometry and Double Field Theory: a toy Model

Patrizia Vitale

Dipartimento di Fisica Università di Napoli Federico II and INFN

with V. Marotta (Heriot-Watt Edimbourgh), and F. Pezzella (INFN Napoli)

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- Motivation
- Geometric formulation of the rigid rotator on configuration space $SU(2)$
- Dual model and generalization to the Drinfeld double group $SL(2, C)$
- Recognizing geometric structures of generalized and double geometry
- The Principal Chiral Model with target space $SU(2)$
- Dual model and symmetry under duality
- Double field theory formulation
- Conclusions and perspectives

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- The new action contains a number of variables which is doubled with respect to the original one, as in double field theory. Geometric structures can be understood in terms of generalized geometry

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$$y^i = -\frac{i}{2}\,\mathrm{Tr}\,g\sigma_i, \quad y^0 = \frac{1}{2}\,\mathrm{Tr}\,g\sigma_0, \quad i = 1,..,3$$

Since $$g^{-1}\dot{g} = i(y^0\dot{y}^i - y^i\dot{y}^0 + \epsilon^i{}_{jk}y^j\dot{y}^k)\sigma_i = i\dot{Q}^i\sigma_i$$

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- Tangent bundle coordinates: $(Q^i, \dot{Q}^i)$
- Equations of motion $\ddot{Q}^i = 0$ *or*, $\frac{d}{dt}\left(g^{-1}\frac{dg}{dt}\right) = 0$

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- PB's:

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Fiber coordinates I_i are associated to the angular momentum components and the base space coordinates (y^0, y^i) to the orientation of the rotator.
I_i are constants of the motion, g undergoes a uniform precession.

Remarks:

- As a group $T^*SU(2) \simeq SU(2) \ltimes \mathbb{R}^3$ with Lie algebra

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The double group $SL(2,\mathbb{C})$ can be endowed with PB's which generalize both those of $T^*SU(2)$ and of $T^*SB(2\mathbb{C})$ [[Semenov-Tyan-Shanskii '91, Alekseev-Malkin '94]]

$$\{\gamma_1, \gamma_2\} = -\gamma_1\gamma_2 r^* - r\gamma_1\gamma_2$$

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- Consider now r^* as an independent solution of the Yang Baxter equation $\rho = \mu e_k \otimes e^k$ and expand $g \in SU(2)$ as a function of the parameter μ:
  $g = \mathbf{1} + i\mu\tilde{I}e_i + O(\mu^2)$
  By repeating the same analysis as above we get back the canonical Poisson structure on $T^*SB(2,C)$, with position coordinates and momenta now interchanged. In particular we note

$$\{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k \tilde{I}^k$$

Last but not least, it is possible to consider a different Poisson structure on the double [Semenov], given by

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Notice that here C-brackets satisfy Jacobi identity because they stem from a Lie bi-algebra (the generalized tangent bundle is a Lie bi-algebroid)

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On denoting by C_+, C_- the two subspaces spanned by $\{e_i\}$, $\{b_i\}$ respectively, we notice that the splitting $\mathfrak{d} = C_+ \oplus C_-$ defines a positive definite metric on $\mathfrak{d}$ via

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G is a pseudo-orthogonal metric -
the sum $\alpha L + \beta G$ is the generalized metric of DFT

Remark 1: Both scalar products have applications in theoretical physics to build invariant action functionals; two relevant examples

- 2+1 gravity with cosmological term as a CS theory of $SL(2,\mathbb{C})$ [Witten '88]
- Palatini action with Holst term [Holst, Barbero, Immirzi..]

Remark 2: While the first product is nothing but the Cartan-Killing metric of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, the Riemannian structure G can be mathematically formalized in a way which clarifies its role in the context of generalized complex geometry [freidel '17]: it can be related to the structure of *para-Hermitian manifold* of $SL(2,\mathbb{C})$ and therefore generalized to even-dimensional real manifolds which are not Lie groups.

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On $T^*SB(2,\mathbb{C})$ we may define

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However:
The usual rigid rotator can be equivalently formulated on *the whole* $SL(2,\mathbb{C})$ [[Marmo et al '94] and one could try a similar analysis for the $SB(2,\mathbb{C})$ model.

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Scope: Introduce an action functional on $SL(2,\mathbb{C})$ (doubled coordinates) which reduces to previous models when constrained

- The action

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The matrix $(\alpha L_{IJ} + \beta R_{IJ})$ is invertible for $\alpha/\beta \neq \pm 1$ and we repeat exactly the same analysis as for the rigid rotator.
We reduce to the two dual models by gauging the global symmetries.
Preliminary analysis in Sfetsos '99, Reid-Edwards '10

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Happy Birthday Alberto!