Properties of Multisymplectic Manifolds

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Introduction: statement and aim of the talk

Multisymplectic manifolds are the most general and complete tool for describing geometrically (covariant) first and higher-order classical field theories.

Other alternative geometrical models for classical field theories: **polysymplectic,** k-**symplectic** and k-**cosymplectic manifolds**.

All of them are generalizations of **symplectic manifolds** (which are used to describe geometrically mechanical systems).

This talk is devoted to review some of the main properties of multisymplectic geometry:

- Definition of *multisymplectic manifold*.
- Hamiltonian structures.
- Characteristic submanifolds of multisymplectic manifolds.
- Canonical models. Darboux-type coordinates.
- Other kinds of multisymplectic manifolds.
- Other properties: invariance theorems.

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Multisymplectic manifolds

Definition 1

Let M be a differentiable manifold, with $\dim M = n$, and $\Omega \in \Omega^k(M)$ $(k \leq n)$.

- *The form Ω is 1-nondegenerate if, for every $p \in M$ and $X_p \in \mathrm{T}_p M$,*
$$i(X_p)\Omega_p = 0 \iff X_p = 0 \ .$$
- *The form Ω is a multisymplectic form if it is closed and 1-nondegenerate.*
- *A multisymplectic manifold (of degree k) is a couple (M, Ω), where $\Omega \in \Omega^k(M)$ is a multisymplectic form.*

If Ω is only closed then it is called a pre-multisymplectic form.
If Ω is only 1-nondegenerate then it is an almost-multisymplectic form.

Ω is 1-nondegenerate if, and only if, the vector bundle morphism

$$\begin{array}{rccc} \Omega^\flat\colon & \mathrm{T}M & \to & \Lambda^{k-1}\mathrm{T}^*M \\ & X_p & \mapsto & i(X_p)\Omega_p \end{array}$$

and thus the corresponding morphism of $\mathcal{C}^\infty(M)$-modules

$$\begin{array}{rccc} \Omega^\flat\colon & \mathfrak{X}(N) & \to & \Omega^{k-1}(N) \\ & X & \mapsto & i(X)\Omega \end{array}$$

are injective.

Examples:

- Multisymplectic manifolds of degree 2 are just *symplectic manifolds*.
- Multisymplectic manifolds of degree n are *orientable manifolds* and the multisymplectic forms are *volume forms*.
- *Bundles of k-forms (k-multicotangent bundles)* endowed with their *canonical* $(k+1)$*-forms* are multisymplectic manifolds of degree $k+1$.
- *Jet bundles* (over m-dimensional manifolds) endowed with the *Poincaré-Cartan* $(m+1)$*-forms* associated with (singular)*Lagrangian densities* are (pre)multisymplectic manifolds of degree $m+1$.

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Hamiltonian structures

Definition 2

A m-vector field (or a multivector field of degree m) in a manifold M (with $m \leq n = \dim M$) is any section of the bundle $\Lambda^m(\mathrm{T}M) \to M$. (A contravariant, skewsymmetric tensor field of degree m in M). The set of m-vector fields in M is denoted by $\mathfrak{X}^m(M)$.

$\forall p \in M$, $\exists U_p \subset M$ and $\exists X_1, \ldots, X_r \in \mathfrak{X}(U_p)$, $m \leq r \leq \dim M$, such that

$$\mathbf{X}|_{U_p} = \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} X_{i_1} \wedge \ldots \wedge X_{i_m} \ ; \ \text{with } f^{i_1 \ldots i_m} \in \mathrm{C}^\infty(U_p) \ .$$

Definition 3

A multivector field $\mathbf{X} \in \mathfrak{X}^m(M)$ is homogeneous (or decomposable) if there are $X_1, \ldots, X_m \in \mathfrak{X}(M)$ such that $\mathbf{X} = X_1 \wedge \ldots \wedge X_m$. $\mathbf{X} \in \mathfrak{X}^m(M)$ is locally homogeneous (decomposable) if, for every $p \in M$, $\exists U_p \subset M$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$.

Remark: Locally decomposable m-multivector fields $\mathbf{X} \in \mathfrak{X}^m(M)$ are locally associated with m-dimensional distributions $D \subset \mathrm{T}M$.

Every multivector field $\mathbf{X} \in \mathfrak{X}^m(M)$ defines a contraction with differential forms $\Omega \in \mathit{\Omega}^k(M)$, which is the natural contraction between tensor fields:

$$\begin{aligned} i(\mathbf{X})\Omega|_{U_p} &= \sum_{1\leq i_1<\ldots<i_m\leq r} f^{i_1\ldots i_m}\, i(X_1 \wedge \ldots \wedge X_m)\Omega \\ &= \sum_{1\leq i_1<\ldots<i_m\leq r} f^{i_1\ldots i_m}\, i(X_1)\ldots i(X_m)\Omega\ . \end{aligned}$$

Then, for every form Ω we have the morphisms

$$\begin{array}{cccccc} \Omega^\flat & : & \Lambda^m(\mathrm{T}M) & \longrightarrow & \Lambda^{k-m}(\mathrm{T}^*M) & \\ & & \mathbf{X}_p & \mapsto & i(\mathbf{X}_p)\Omega_p & . \\ \Omega^\flat & : & \mathfrak{X}^m(M) & \longrightarrow & \mathit{\Omega}^{k-m}(M) & \\ & & \mathbf{X} & \mapsto & i(\mathbf{X})\Omega & . \end{array}$$

If $\mathbf{X} \in \mathfrak{X}^m(M)$, the Lie derivative of $\Omega \in \mathit{\Omega}^k(M)$ is

$$\mathrm{L}(\mathbf{X})\Omega := [\mathrm{d}, i(\mathbf{X})] = \mathrm{d}\, i(\mathbf{X}) - (-1)^m\, i(\mathbf{X})\mathrm{d}$$

DEFINITION 4

Let (M, Ω) be a multisymplectic manifold of degree k. A diffeomorphism $\varphi \colon M \to M$ is a multisymplectomorphism *if $\varphi^*\Omega = \Omega$.*

DEFINITION 5

1. *$X \in \mathfrak{X}(M)$ whose flow consists of multisymplectic diffeomorphisms is a* locally Hamiltonian vector field. *It is equivalent to demand that $\mathrm{L}(X)\Omega = 0$, or equivalently, $i(X)\Omega \in \Omega^{k-1}(M)$ is a closed form.*
2. *$\mathbf{X} \in \mathfrak{X}^m(M)$ $(m < k)$ is a* locally Hamiltonian multivector field *if $\mathrm{L}(\mathbf{X})\Omega = 0$ or, what is equivalent, $i(\mathbf{X})\Omega \in \Omega^{k-m}(M)$ is a closed form. Then, for every $p \in M$, $\exists U \subset M$ and $\zeta \in \Omega^{k-m-1}(U)$ such that $i(\mathbf{X})\Omega = \mathrm{d}\zeta$ (on U).*
   $\zeta \in \Omega^{k-m-1}(U)$ is a locally Hamiltonian form *for $\mathbf{X}$.*
3. *$\mathbf{X} \in \mathfrak{X}^m(M)$ is a* Hamiltonian multivector field *if $i(\mathbf{X})\Omega \in \Omega^{k-m}(M)$ is an exact form; that is, there exists $\zeta \in \Omega^{k-m-1}(M)$ such that $i(\mathbf{X})\Omega = \mathrm{d}\zeta$.*
   $\zeta \in \Omega^{k-m-1}(M)$ is a Hamiltonian form *for $\mathbf{X}$.*

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Characteristic submanifolds

Definition 6

Let (M,Ω) be a multisymplectic manifold of degree k, and $\mathcal{W}$ a distribution in M. $\forall p\in M$ and $1\leq r\leq k-1$, the r-orthogonal multisymplectic vector space *at p is*

$$\mathcal{W}_p^{\perp,r}=\{v\in \mathrm{T}_pM \mid i(v\wedge w_1\wedge\ldots\wedge w_r)\Omega_p=0,\ \forall w_1,\ldots,w_r\in\mathcal{W}_p\}\ ,$$

the r-orthogonal multisymplectic complement *of $\mathcal{W}$ is the distribution*

$$\mathcal{W}^{\perp,r}:=\cup_{p\in M}\mathcal{W}_p^{\perp,r}\ .$$

1. $\mathcal{W}$ *is an* r-coisotropic distribution *if* $\mathcal{W}^{\perp,r}\subset\mathcal{W}$.
2. $\mathcal{W}$ *is an* r-isotropic distribution *if* $\mathcal{W}\subset\mathcal{W}^{\perp,r}$.
3. $\mathcal{W}$ *is an* r-Lagrangian distribution *if* $\mathcal{W}=\mathcal{W}^{\perp,r}$.
4. $\mathcal{W}$ *is a* multisymplectic distribution *if* $\mathcal{W}\cap\mathcal{W}^{\perp,k-1}=\{0\}$.

Remark: For every distribution $\mathcal{W}$, we have that $\mathcal{W}^{\perp,r}\subset\mathcal{W}^{\perp,r+1}$. As a consequence, every r-isotropic distribution is $(r+1)$-isotropic, and every r-coisotropic distribution is $(r-1)$-coisotropic.

Definition 7

Let (M,Ω) be a multisymplectic manifold of degree k, and N a submanifold of M. If $0 \leq r \leq k-1$, then:

1. *N is an r-coisotropic submanifold of M if $\mathrm{T}N^{\perp,r} \subset \mathrm{T}N$.*
2. *N is an r-isotropic submanifold of M if $\mathrm{T}N \subset \mathrm{T}N^{\perp,r}$.*
3. *N is an r-Lagrangian submanifold of M if $\mathrm{T}N = \mathrm{T}N^{\perp,r}$.*
4. *N is a multisymplectic submanifold of M if $\mathrm{T}N \cap \mathrm{T}N^{\perp,k-1} = \{0\}$.*

Proposition 1

A submanifold N of M is r-Lagrangian $\Longleftrightarrow$ it is r-isotropic and maximal.

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Canonical models. Darboux-type coordinates

Canonical models of multisymplectic manifolds: bundles of forms.

Let Q be a manifold.

- $\rho\colon \Lambda^k(\mathrm{T}^*Q) \to Q$ is the bundle of k-forms in Q.
  The *tautological form* (canonical form) $\Theta_Q \in \mathit{\Omega}^k(\Lambda^k(\mathrm{T}^*Q))$ is defined as follows: if $\alpha \in \Lambda^k(\mathrm{T}^*Q)$, and $V_1,\ldots,V_k \in \mathrm{T}_\alpha(\Lambda^k(\mathrm{T}^*Q))$, then
  $$\Theta_{Q\alpha}(V_1,\ldots,V_k) = i(\rho_* V_k \wedge \ldots \wedge \rho_* V_1)\alpha \ .$$
  Therefore, $\Omega_Q = \mathrm{d}\Theta_Q \in \mathit{\Omega}^{k+1}(\Lambda^k(\mathrm{T}^*Q))$ is a 1-nondegenerate form.
  Then $(\Lambda^k(\mathrm{T}^*Q), \Omega_Q)$ is a multisymplectic manifold of degree $k+1$.
- Let $\rho_r\colon \Lambda^k_r(\mathrm{T}^*Q) \to Q$ be the subbundle of $\Lambda^k(\mathrm{T}^*Q)$ made of the r-horizontal k-forms in Q (with respect to the projection ρ).
  Let $\Theta^r_Q \in \mathit{\Omega}^k(\Lambda^k_r(\mathrm{T}^*Q))$ be the corresponding tautological k-form, and $\Omega^r_Q = \mathrm{d}\Theta^r_Q \in \mathit{\Omega}^{k+1}(\Lambda^k_r(\mathrm{T}^*Q))$.
  $(\Lambda^k_r(\mathrm{T}^*Q), \Omega^r_Q)$ is a multisymplectic manifold of degree $k+1$.

If $(x^i, p_{i_1\dots i_k})$ is a system of *natural coordinates* in $U \subset \Lambda^k(\mathrm{T}^*Q)$

$$\begin{aligned} \Theta_Q^{(r)}\mid_U &= p_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k} \ . \\ \Omega_Q^{(r)}\mid_U &= \mathrm{d}p_{i_1\dots i_k}\wedge\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k} \ . \end{aligned}$$

These are called *Darboux coordinates*.

In general, for a multisymplectic manifold (M, Ω), additional properties are needed in order to have a Darboux theorem which assures the existence of Darboux coordinates.

In particular, in order to have multisymplectic manifolds which locally behave as the canonical models, it is necessary to endow them with additional structures: a 1-isotropic distribution $\mathcal{W}$ satisfying some dimensionality conditions, and a "generalized distribution" ε defined on the space of leaves determined by $\mathcal{W}$.

Definition 8

Let (M,Ω) be a multisymplectic manifold of degree k, and $\mathcal{W}$ a 1-isotropic involutive distribution in (M,Ω).

1. *The triple $(M,\Omega,\mathcal{W})$ is a* multisymplectic manifold of type $(k,0)$ *if, for every $p \in M$, we have that:*
   1. $\dim \mathcal{W}(p) = \dim \Lambda^{k-1}(\mathrm{T}_p M/\mathcal{W}(p))^*$.
   2. $\dim (\mathrm{T}_p M/\mathcal{W}(p)) > k-1$.
2. *A* multisymplectic manifold of type (k,r) *$(1 \leq r \leq k-1)$ is a quadruple $(M,\Omega,\mathcal{W},\mathcal{E})$ such that $\mathcal{E}$ is a "generalized distribution" on M (in the sense that, for every $p \in M$, $\mathcal{E}(p) \subset \mathrm{T}_p M/\mathcal{W}(p)$ is a vector subspace) and, for every $p \in M$, denoting by $\pi_p \colon \mathrm{T}_p M \to \mathrm{T}_p M/\mathcal{W}(p)$ the canonical projection, we have that:*
   1. $i(v_1 \wedge \ldots \wedge v_r)\Omega_p = 0$, *for every $v_i \in \mathrm{T}_p M$ such that $\pi_p(v_i) \in \mathcal{E}(p)$ $(i = 1,\ldots,r)$.*
   2. $\dim \mathcal{W}(p) = \dim \Lambda^{k-1}_r(\mathrm{T}_p M/\mathcal{W}(p))^*$, *where the horizontal forms are considered with respect to the subspace $\mathcal{E}(p)$.*
   3. $\dim (\mathrm{T}_p M/\mathcal{W}(p)) > k-1$.

Proposition 2

Every multisymplectic manifold (M,Ω) of type $(k,0)$ (resp. of type (k,r)) is locally multisymplectomorphic to a bundle of $(k-1)$-forms $\Lambda^{k-1}(\mathrm{T}^*Q)$ (resp. $\Lambda^{k-1}_r(\mathrm{T}^*Q)$), for some manifold Q; that is, to a canonical multisymplectic manifold.
Therefore, there is a local chart of Darboux coordinates around every point $p\in M$.

Definition 9

Multisymplectic manifolds which are locally multisymplectomorphic to bundles of forms are called locally special multisymplectic manifolds.

Definition 10

A special multisymplectic manifold *is a multisymplectic manifold (M,Ω) (of degree k) such that:*

1. $\Omega = \mathrm{d}\Theta$, *for some* $\Theta \in \Omega^{k-1}(M)$.
2. *There is a diffeomorphism* $\phi\colon M \to \Lambda^{k-1}(\mathrm{T}^*Q)$, dim $Q = n \geq k-1$, *(or* $\phi\colon M \to \Lambda^{k-1}_r(\mathrm{T}^*Q)$*), and a fibration* $\pi\colon M \to Q$ *such that* $\rho\circ\phi = \pi$ *(resp.* $\rho_r\circ\phi = \pi$*), and* $\phi^*\Theta_Q = \Theta$ *(resp.* $\phi^*\Theta^r_Q = \Theta$*). (*$(M,\Omega)$ *is* multisymplectomorphic to a bundle of forms*).*

Every special multisymplectic manifold is a locally special multisymplectic manifold and hence has charts of *Darboux coordinates* at every point.

M. de León D. Martín de Diego, A. Santamaría-Merino, "Tulczyjew triples and Lagrangian submanifolds in classical field theories", in *Applied Differential Geometry and Mechanics* (eds. W. Sarlet, F. Cantrijn), Univ. Gent, Gent, Academia Press, (2003), 2147.

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Other kinds of multisymplectic manifolds

In general, locally Hamiltonian vector fields in a multisymplectic manifold (M, Ω) do not span the tangent bundle of this manifold, and the group of multisymplectic diffeomorphisms does not act transitively on M.

Definition 11

Let M be a differentiable manifold, $p \in M$ and a compact set K with $p \in \overset{\circ}{K}$. A local Liouville *or* local Euler-like vector field *at p with respect to K is a vector field $\Delta^p \in \mathfrak{X}(M)$ such that:*

1. $\operatorname{supp} \Delta^p := \overline{\{x \in M \mid \Delta^p(x) \neq 0\}} \subset K$,
2. *there exists a diffeomorphism* $\varphi \colon \overbrace{\operatorname{supp} \Delta^p}^{\circ} \to \mathbb{R}^n$ *such that* $\varphi_* \Delta^p = \Delta$, *where* $\Delta = x^i \dfrac{\partial}{\partial x^i}$ *is the standard Liouville or dilation vector field in* $\mathbb{R}^n$.

Definition 12

A form $\Omega \in \Omega^k(M)$ is said to be locally homogeneous *at $p \in M$ if, for every open set $U \subset M$ containing p, there exists a local Euler-like vector field Δ^p at p with respect to a compact set $K \subset U$ such that*

$$\mathrm{L}(\Delta^p)\Omega = f\Omega \; ; \quad f \in \mathrm{C}^\infty(U) \; .$$

Ω is locally homogeneous *if it is locally homogeneous for all $p \in M$.*
A locally homogeneous manifold *is a couple (M, Ω), where M is a manifold and $\Omega \in \Omega^k(M)$ is locally homogeneous.*

Proposition 3

Let (M, Ω) be a locally homogeneous multisymplectic manifold. Then the family of locally Hamiltonian vector fields span locally the tangent bundle of M; that is, $\forall\, p \in M$, $\mathrm{T}_pM = \mathrm{span}\{X_p \mid X \in \mathfrak{X}(M) \; , \; \mathrm{L}(X)\Omega = 0\}$.

Theorem 1

The group of multisymplectic diffeomorphisms $G(M, \Omega)$ of a locally homogeneous multisymplectic manifold (M, Ω) acts transitively on M.

Remark:

- Locally special multisymplectic manifolds have local Euler-like vector fields; in particular, the local vector fields $x^i \dfrac{\partial}{\partial x^i} + p_{i_1 \dots i_k} \dfrac{\partial}{\partial p_{i_1 \dots i_k}}$. Then, the corresponding multisymplectic forms are locally homogeneous.
- As a consequence, if (M, Ω) is a locally special multisymplectic manifold, then the family of locally Hamiltonian vector fields span locally the tangent bundle of M and the group of multisymplectic diffeomorphisms acts transitivelly on M. In fact, the local vector fields $\left\{ \dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial p_{i_1 \dots i_k}} \right\}$ are locally Hamiltonian.

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INVARIANCE THEOREMS

(Partial) generalization of *Lee Hwa Chung's Theorem* for symplectic manifolds (which characterizes all the differential forms which are invariant under infinitesimal symplectomorphisms):

THEOREM 2

Let (M,Ω) be a locally homogeneous multisymplectic manifold of degree k and $\alpha \in \Omega^p(M)$, with $p = k-1, k$, such that:

- *α is invariant by the set of locally Hamiltonian $(k-1)$-vector fields; that is, $\mathrm{L}(\mathbf{X})\alpha = 0$, for every $\mathbf{X} \in \mathfrak{X}^{k-1}_{lh}(M)$.*
- *α is invariant by the set of locally Hamiltonian vector fields; that is, $\mathrm{L}(Z)\alpha = 0$, for every $Z \in \mathfrak{X}_{lh}(M)$.*

Therefore:

1. *If $p = k$ then $\alpha = c\,\Omega$, with $c \in \mathbb{R}$.*
2. *If $p = k-1$ then $\alpha = 0$.*

Generalization of *Theorems of Banyaga* for symplectic manifolds:

Theorem 3

Let (M_i, Ω_i), $i = 1, 2$, be local homogeneous multisymplectic manifolds of degree k and $G(M_i, \Omega_i)$ their groups of multisymplectic automorphisms. Let $\Phi \colon G(M_1, \Omega_1) \to G(M_2, \Omega_2)$ be a group isomorphism (which is a homeomorphism when $G(M_i, \Omega_i)$ are endowed with the point-open topology). Then, there exists a diffeomorphism $\varphi \colon M_1 \to M_2$, such that :

1. $\Phi(\psi) = \varphi \circ \psi \circ \varphi^{-1}$, *for every* $\psi \in G(M_1, \Omega_1)$.
2. *The map φ_* maps locally Hamiltonian vector fields of (M_1, Ω_1) into locally Hamiltonian vector fields of (M_2, Ω_2).*
3. *In addition, if φ_* maps locally Hamiltonian multivector fields of (M_1, Ω_1) into locally Hamiltonian multivector fields of (M_2, Ω_2), then there is a constant c such that $\varphi^* \Omega_2 = c\, \Omega_1$.*

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Discussion and final remarks

The properties and characteristics of multisymplectic manifolds are, in general, more elaborated and richer than for symplectic manifolds.

Other interesting properties of multisymplectic manifolds are, for instance:

- The *graded Lie algebra structure* of the sets of *Hamiltonian forms* and *Hamiltonian multivector fields*.
- Polarized multisymplectic manifold and its general structure theorem.
- Other properties and relevance of r-coisotropic, r-isotropic and, especially, of r-Lagrangian distributions and submanifols.
- Characterizations of multisymplectic transformations.
- ....

LIVE LONG AND PROSPER, ALBERTO !