Spin chains: Thermodynamics and criticality

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Outline

Introduction

Spin chains of Haldane–Shastry type have been extensively studied as the prototypical examples of one-dimensional lattice models with long-range interactions, due to their remarkable physical and mathematical properties.

Applications:

- conformal field theory
- fractional statistics and anyons,
- quantum chaos vs. integrability
- quantum information theory
- quantum simulation of long-range magnetism.

HS spin chain

- Connection with the (dynamical) spin Sutherland model
- Polychronakos's freezing trick $\to$ chain's partition function
- Other models:
  - Calogero $\to$ Polychronakos–Frahm (PF) spin chain
  - Inozemtsev $\to$ Frahm–Inozemtsev (FI) spin chain

$$\text{spin} = \mathrm{su}(m)\ \text{spin}$$

Supersymmetric models:

$$\mathrm{su}(m|n),\ \text{sites are either } \mathrm{su}(m) \text{ bosons or } \mathrm{su}(n) \text{ fermions}$$

Thermodynamics of spin chains of HS type

- Haldane
- Transfer matrix method, used by Frahm and Inozemtsev (magnetization in an external constant magnetic field)
- Spin 1/2 chains of HS type in a constant magnetic field (Enciso, Finkel, González-López)
- Supersymmetric case, $\mathrm{su}(1|1)$ HS chain (with a chemical potential term): equivalence to a free, translationally invariant fermion system (Carrasco, Finkel, González-López, Rodríguez, Tempesta)
- It cannot be applied to the $\mathrm{su}(1|1)$ PF and FI chains nor to chains of HS type with $m > 1$ or $n > 1$

Conformal field theory

- Connection between $\mathrm{su}(2)$ HS chain and the level-1 $\mathrm{su}(2)$ Wess–Zumino–Novikov–Witten conformal field theory (CFT)
- $\mathrm{su}(n)$ HS chain (with no magnetic field or chemical potential term) is critical (gapless), with central charge $c = n - 1$.
- Extended to the $\mathrm{su}(m|n)$, $m \geqslant 1$, PF chain with central charge $c = m - 1 + n/2$
- $\mathrm{su}(1|1)$ HS chain with a chemical potential: critical with central charge $c = 1$ (for a certain range of values of the chemical potential

Thermodynamics and critical behavior of $\mathrm{su}(m|n)$ spin chains of HS type with a general chemical potential term

- chains' partition functions (connection with vertex models)
- transfer matrix
- free energy per site in the thermodynamic limit
- thermodynamics and criticality of supersymmetric chains of HS type with $1 \leqslant m, n \leqslant 2$
    - Low-temperature behavior of the free energy per site
    - Values of the chemical potentials for which these chains are critical, central charge.
    - Phase transitions at zero temperature

THE HAMILTONIAN

$$\mathcal{H}_0 = \sum_{1\leqslant i<j\leqslant N} J_{ij}(\mathbb{1} - P_{ij}^{(m|n)})$$

Haldane, Shastry, Polychronakos, Frahm, Inozemtsiev, ...

Spin states, $\mathrm{su}(M)$:

$$|s_1,\ldots,s_N\rangle, \quad s_i \in \{1,\ldots,M\}, \quad V = \otimes_{i=1}^N \mathbf{R}^M, \quad \dim V = M^N$$

Coupling constants: $J_{ij} > 0$

Exchange operators: $P_{ij}^{(m|n)}$

- Polychronakos-Frahm (PF):

$$J_{ij} = \frac{J}{(\xi_i - \xi_j)^2}, \quad \xi_i \equiv \text{zeros of Hermite polynomials}$$

- Haldane-Shastry (HS):

$$J_{ij} = \frac{J}{2\sin^2(\xi_i - \xi_j)}, \quad \xi_i = \frac{i\pi}{N}$$

- Frahm-Inozemtsiev (FI):

$$J_{ij} = \frac{J}{2\sinh^2(\xi_i - \xi_j)}, \quad \mathrm{e}^{2\xi_i} \equiv \text{zeros of Laguerre polynomials}$$

Exchange operators

Bosonic model ($s_i \in \{1, \dots, m\}$)

Polychronakos

$$P_{ij}|s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle = |s_1, \dots, s_j, \dots, s_i, \dots, s_N\rangle$$

Supersymmetric model ($s_i \in \{1, \dots, m+n\}$)

Basu-Mallick, Bondyopadhaya, Hikami, Sen, González-López, Finkel, Enciso, Barba, ...

$$\begin{aligned} s_i &\in B = \{1, \dots, m\} \equiv \text{bosons} \\ s_i &\in F = \{m+1, \dots, m+n\} \equiv \text{fermions} \end{aligned}$$

$$P_{ij}|s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle = \epsilon_{i,i+1,\dots,j}|s_1, \dots, s_j, \dots, s_i, \dots, s_N\rangle$$

$$\epsilon_{i,i+1,\dots,j} = \begin{cases} 1, & s_i, s_j \text{ bosons} \\ (-1)^p, & \{s_i, s_j\} \equiv \{\text{fermion}, \text{boson}\}, \\ & p = \text{number of fermions in positions } i+1, \dots, j-1 \\ -1, & s_i, s_j \text{ fermions} \end{cases}$$

Supersymmetric su(1|1), $N = 3$

$$\mathcal{H}_{\mathrm{HS}} = \frac{2}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$E = \{0_2, 2_4, 4_2\}, \quad \mathcal{Z}_3^{(1|1)} = 2 + 4q^2 + 2q^4, \quad q = \mathrm{e}^{-\beta}, \quad \beta = \frac{1}{k_{\mathrm{B}} T}$$

Rational case, scalar model

$$H^{\mathrm{sc}} = -\sum_i \partial_{x_i}^2 + a^2 \sum_i x_i^2 + 2a \sum_{i<j} \frac{a-1}{(x_i - x_j)^2}$$

$$E = E_0 + 2a \sum_{i=1}^{N} n_i, \quad E_0 = aN(a(N-1)+1)$$

$$\mathbf{n} = (n_1, \dots, n_N) \in \mathbf{Z}_+^N$$

Rational case, spin dynamical model

$$H_0 = -\sum_i \partial_{x_i}^2 + a^2 \sum_i x_i^2 + 2a \sum_{i<j} \frac{a - P^{(m|n)}}{(x_i - x_j)^2}$$

$$E = E_0 + 2a \sum_{i=1}^{N} n_i$$

$$\psi_{\mathbf{n}}^{\mathbf{s}}(\mathbf{x}) = \rho(\mathbf{x}) \Lambda^{(m|n)} \left(\prod_i x_i^{n_i} |s_1, \ldots, s_N\rangle \right)$$

$$\rho(\mathbf{x}) = \mathrm{e}^{-\frac{a}{2}\sum_j x_j^2} \prod_{i<j} |x_i - x_j|^a, \quad K_{ij} P_{ij}^{(m|n)} \Lambda^{(m|n)} = \Lambda^{(m|n)}$$

Chemical potential

In the non-supersymmetric case we can add a magnetic field (a chemical potencial in the supersymmetric case) to study the behavior of the system regarding the number of particles of different types:

$$\mathcal{H}_\mu = -\sum_{\alpha=1}^{m+n-1} \mu_\alpha \mathcal{N}_\alpha$$

$\mathcal{N}_\alpha$ number operator of $\alpha \in \{1, \dots, n+m\}$ type particles.

$$\mathcal{N}_\alpha |s_1 \cdots s_N\rangle = N_\alpha(\mathbf{s}) |s_1 \cdots s_N\rangle,$$

$$N_\alpha(\mathbf{s}) \equiv \sum_{i=1}^{N} \delta_{s_i,\alpha}$$

is the number of spins of type α in the state $|s_1 \cdots s_N\rangle$.

The Hamiltonian is:

$$H = H_0 + \frac{2a}{J}\mathcal{H}_\mu$$

The operators $\mathcal{N}_\alpha$ commute with the exchange operators and the energy spectrum of the total Hamiltonian H is:

$$E_{\mathbf{n}}^{\mathbf{s}} = E_0 + 2a\sum_i n_i - \frac{2a}{J}\sum_i \mu_{s_i}$$

$$\mathcal{H}^{(m|n)} = \sum_{i<j} J_{ij}(1 - P_{ij}^{(m|n)}) - \sum_{\alpha=1}^{m+n-1} \mu_\alpha \mathcal{N}_\alpha \equiv \mathcal{H}_0 + \mathcal{H}_1,$$

- $\mathcal{H}^{(m|n)}$, is related to $\mathcal{H}^{(n|m)}$ by a duality relation.

$$U : \Sigma^{(m|n)} \to \Sigma^{(n|m)}, \quad U|s_1 \cdots s_N\rangle = (-1)^{\sum_i i\pi(s_i)}|s_1' \cdots s_N'\rangle$$

$$\pi(s_i) = 0,\ s_i \in B, \quad \pi(s_i) = 1,\ s_i \in F, \quad s_i' = m + n + 1 - s_i$$

$$U^{-1} P_{ij}^{(n|m)} U = -P_{ij}^{(m|n)}\,, \qquad U^{-1}\mathcal{N}_\alpha U = \mathcal{N}_{m+n+1-\alpha}\,,$$

$$U^{-1}\mathcal{H}^{(n|m)} U = E_0 - \mathcal{H}^{(m|n)}\Big|_{\mu_\alpha \to -\mu_{m+n+1-\alpha}}, \qquad E_0 \equiv 2\sum_{i<j} J_{ij}\,.$$

Thus the spectra of $\mathcal{H}^{(n|m)}$ and $\mathcal{H}^{(m|n)}$ are related by

$$E_k^{(n|m)}(\mu_1, \ldots, \mu_{m+n}) = E_0 - E_k^{(m|n)}(-\mu_{m+n}, \ldots, -\mu_1)\,.$$

- Changes in the labeling of the bosonic or fermionic degrees of freedom:

$$T_{\alpha\beta} : \Sigma^{(m|n)} \to \Sigma^{(m|n)}, \quad \alpha \neq \beta \in \{1, \ldots, m+n\}$$

replacing all the s_k's equal to α by β, and vice versa.
If $\pi(\alpha) = \pi(\beta)$, $T_{\alpha\beta}$ commutes with $P_{ij}^{(m|n)}$, and with $\mathcal{H}_0$.

$$T_{\alpha\beta}^{-1}\,\mathcal{N}_\alpha\, T_{\alpha\beta} = \mathcal{N}_\beta\,, \quad T_{\alpha\beta}^{-1}\,\mathcal{N}_\beta\, T_{\alpha\beta} = \mathcal{N}_\alpha$$

$$T_{\alpha\beta}^{-1}\,\mathcal{N}_\gamma\, T_{\alpha\beta} = \mathcal{N}_\gamma \quad (\gamma \neq \alpha, \beta),$$

$$T_{\alpha\beta}^{-1}\,\mathcal{H}\, T_{\alpha\beta} = \mathcal{H}_0 - \mu_\alpha \mathcal{N}_\beta - \mu_\beta \mathcal{N}_\alpha - \sum_{\substack{\gamma=1 \\ \gamma\neq\alpha,\beta}}^{m+n} \mu_\gamma \mathcal{N}_\gamma\,.$$

$$E_k^{(m|n)}(\dots,\mu_\alpha,\dots,\mu_\beta,\dots) = E_k^{(m|n)}(\dots,\mu_\beta,\dots,\mu_\alpha,\dots)$$
$$(\pi(\alpha)=\pi(\beta))$$

the spectrum of $\mathcal{H}$ is invariant under permutations of the bosonic or fermionic chemical potentials among themselves.

- $E_k^{(n|m)}(\mu_1,\dots,\mu_{m+n}) = E_0 - E_k^{(m|n)}(-\mu_{\alpha_1},\dots,-\mu_{\alpha_{m+n}})\,,$

$(\alpha_1,\dots,\alpha_{m+n})$ = permutation of $(1,\dots,m+n)$ with
$\{\alpha_1,\dots,\alpha_m\}=\{n+1,\dots,n+m\}$, $\{\alpha_{m+1},\dots,\alpha_{m+n}\}=\{1,\dots,n\}$.

Hamiltonians

$$H^{\mathrm{sc}} = -\sum_i \partial_{x_i}^2 + a^2 \sum_i x_i^2 + 2a \sum_{i<j} \frac{a-1}{(x_i - x_j)^2}$$

$$H_0 = -\sum_i \partial_{x_i}^2 + a^2 \sum_i x_i^2 + 2a \sum_{i<j} \frac{a - P^{(m|n)}}{(x_i - x_j)^2}$$

$$\mathcal{H}_0 = \sum_{i<j} \frac{J}{(\xi_i - \xi_j)^2} (\mathbb{1} - P_{ij}^{(m|n)}), \quad \mathcal{H}_\mu = -\sum_{\alpha=1}^{m+n-1} \mu_\alpha \mathcal{N}_\alpha$$

$$H = H_0 + \frac{2a}{J} \mathcal{H}_\mu, \quad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_\mu$$

$$H = H^{\mathrm{sc}} + \frac{2a}{J} \mathcal{H}\big|_{\xi_i \to x_i}$$

Polychronakos' freezing trick

The points ξ_i (zeros of the Hermite polynomials) are the minimum of the potential:

$$U = \sum_i x_i^2 + \sum_{i<j} \frac{2}{(x_i - x_j)^2}$$

$$H = H^{\mathrm{sc}} + \frac{2a}{J} \mathcal{H}\Big|_{\xi_i \to x_i}$$

In the limit $a \to \infty$:

$$E_{ij} \simeq E_i^{\mathrm{sc}} + \frac{2a}{J} E_j$$

$$Z(T) = \sum_{i,j} \mathrm{e}^{-E_{ij}/(k_B T)} \simeq \sum_{i,j} \mathrm{e}^{-E_i^{\mathrm{sc}}/(k_B T) - \frac{2a}{J} E_j/(k_B T)}$$
$$= \left(\sum_i \mathrm{e}^{-E_i/(k_B T)} \right) \left(\sum_j \mathrm{e}^{-\frac{2a}{J} E_j/(k_B T)} \right)$$

PARTITION FUNCTION OF THE SPIN CHAIN

$$\mathcal{Z}(T) = \lim_{a \to \infty} \frac{Z(2aT/J)}{Z^{\mathrm{sc}}(2aT/J)}$$

$$Z(T) = \sum_{i,j} \mathrm{e}^{-E_{ij}/(k_B T)} \simeq \sum_{i,j} \mathrm{e}^{-E_i^{\mathrm{sc}}/(k_B T) - \frac{2a}{J} E_j/(k_B T)}$$

$$= \left(\sum_i \mathrm{e}^{-E_i/(k_B T)} \right) \left(\sum_j \mathrm{e}^{-\frac{2a}{J} E_j/(k_B T)} \right)$$

Partition function of the spin chain

$$\mathcal{Z}(T) = \lim_{a \to \infty} \frac{Z(2aT/J)}{Z^{\mathrm{sc}}(2aT/J)}$$

Partition function of the dynamical models

- Partition function of the scalar model

$$Z^{\mathrm{sc}}\left(\frac{2aT}{J}\right) = q^{JE_0/(2a)} \prod_{i=1}^{N} \frac{1}{1-q^{Ji}}, \quad q = \mathrm{e}^{-1/T}$$

$$E_0 = aN + a^2 N(N-1)$$

- Partition function of the spin dynamical model

$$Z\left(\frac{2aT}{J}\right) = q^{JE_0/(2a)} \sum_{\mathbf{k}\in\mathcal{P}_N} \Sigma(\mathbf{k}) q^{J\sum_{i=1}^{r-1} K_i} \prod_{i=1}^{r} \frac{1}{1-q^{JK_i}}$$

$$\Sigma(\mathbf{k}) = \sum_{\mathbf{s}\in\mathbf{n}} q^{-\sum_{j=1}^{N} \mu_{s_j}}$$

Polychronakos' freezing trick

$$\frac{Z(2aT/J)}{Z^{\mathrm{sc}}(2aT/J)}$$

- Partition function of the spin chain:

$$\mathcal{Z}(T) = \left(\sum_{\mathbf{k} \in \mathcal{P}_N} \Sigma(\mathbf{k}) q^{\sum_{i=1}^{r-1} JK_i} \prod_{i=1}^{r} \frac{1}{1 - q^{JK_i}} \right) \prod_{j=1}^{N} \left(1 - q^{Jj}\right)$$

Partition function of the spin chain

$$\mathcal{Z} = \sum_{\mathbf{k}\in\mathcal{P}_N} \Sigma(\mathbf{k}) q^{J\sum_{i=1}^{r-1}\mathcal{E}(K_i)} \prod_{i=1}^{N-r}\left(1 - q^{J\mathcal{E}(K_i')}\right)$$

$$\mathcal{E}(i) = \begin{cases} i, & \text{(PF)} \\ i(N-i), & \text{(HS)} \\ i(c+i-1), & \text{(FI)} \end{cases}$$

$$K_i = \sum_{j=1}^{i} k_j, \quad \{K_1', \dots, K_{N-r}'\} = \{1, \dots, N\} \setminus \{K_1, \dots, K_r\}$$

Λ is the total symmetrizer with respect to simultaneous permutations of coordinates and spin variables,

$$K_{ij}P_{ij}^{(m|n)}\Lambda = \Lambda K_{ij}P_{ij}^{(m|n)} = \Lambda, \qquad 1 \leqslant i < j \leqslant N\,,$$

H is represented by an upper triangular matrix in an appropriate basis

$$|\mathbf{n},\mathbf{s}\rangle = \Lambda\Big(\rho(\mathbf{x})\prod_i x_i^{n_i}\cdot|\mathbf{s}\rangle\Big),$$

$$|\mathbf{s}\rangle \equiv |s_1\cdots s_N\rangle, \quad \rho(\mathbf{x}) = \mathrm{e}^{-ar^2/2}\prod_{i<j}|x_i - x_j|^a.$$

The states are a (non-orthonormal) basis of $\Lambda(L^2(\mathbb{R}^N) \otimes \Sigma^{(m|n)})$ if $\mathbf{n}$ and $\mathbf{s}$ satisfy

I) $n_i \geqslant n_{i+1}$ for all $i = 1, \dots, N-1$.

II) If $n_i = n_{i+1}$ then $s_i \leqslant s_{i+1}$ for $s_i \in B$, or $s_i < s_{i+1}$ for $s_i \in F$.

The action of H_0 on this basis is

$$H_0|\mathbf{n}, \mathbf{s}\rangle = E^0_{\mathbf{n},\mathbf{s}}|\mathbf{n}, \mathbf{s}\rangle + \sum_{|\mathbf{n}'|<|\mathbf{n}|,\mathbf{s}'} c_{\mathbf{n}'\mathbf{s}',\mathbf{ns}}|\mathbf{n}', \mathbf{s}'\rangle$$

$$c_{\mathbf{n}'\mathbf{s}',\mathbf{ns}} \in \mathbb{C}, \quad E^0_{\mathbf{n},\mathbf{s}} = 2a|\mathbf{n}| + E_0 \,.$$

$\mathcal{H}_1$ commutes with the symmetrizer Λ

$$\mathcal{H}_1|\mathbf{n}, \mathbf{s}\rangle = -\Big(\sum_i \mu_{s_i}\Big)|\mathbf{n}, \mathbf{s}\rangle \,.$$

and the spectrum of H is given by

$$E_{\mathbf{n},\mathbf{s}} = 2a|\mathbf{n}| - \frac{2a}{J}\sum_i \mu_{s_i} + E_0$$

Some details on the partition function

Parametrize $\mathbf{n}$

$$\mathbf{n} = (\underbrace{\nu_1, \dots, \nu_1}_{k_1}, \dots, \underbrace{\nu_r, \dots, \nu_r}_{k_r})$$

$$\nu_1 > \cdots > \nu_r \geqslant 0, \quad k_1 + \cdots + k_r = N, \quad k_i > 0,\ \forall i$$

$\mathbf{k} = (k_1, \dots, k_r)$: ordered partitions of the integer N, $\mathcal{P}_N$ $(\nu_i, \dots, \nu_i)$: *sector*. Partition function

$$Z(2aT/J) = q^{\frac{JE_{\mathrm{GS}}}{2a}} \sum_{\mathbf{k}\in\mathcal{P}_N} \sum_{\nu_1>\cdots>\nu_r\geqslant 0} q^{\sum_{i=1}^{r} Jk_i\nu_i} \sum_{\mathbf{s}\in\mathbf{n}} q^{-\sum_j \mu_{s_j}},$$

$\mathbf{s} \in \mathbf{n} \equiv$ all possible multiindices $\mathbf{s} \in \{1, \dots, m+n\}^N$ satisfying condition ii) (given $\mathbf{n}$)

$$\Sigma(\mathbf{k}) \equiv \sum_{\mathbf{s}\in\mathbf{n}} q^{-\sum_j \mu_{s_j}} = \prod_{i=1}^{r} \sigma(k_i), \quad \sigma(k) = \sum_{i+j=k} \sum_{1\leqslant s_1\leqslant\cdots\leqslant s_i\leqslant m} q^{-\sum_{l=1}^{i}\mu_{s_l}} \sum_{1\leqslant l_1<\cdots<l_j\leqslant n} q^{-\sum_{p=1}^{j}\mu_{m+l_p}}$$

- Complete symmetric polynomials

$$h_i(x_1, \dots, x_m) \equiv \sum_{p_1+\cdots+p_m=i} x_1^{p_1} \cdots x_m^{p_m} \,,$$

- Elementary symmetric polynomials

$$e_j(x_1, \dots, x_n) \equiv \sum_{1\leqslant l_1<\cdots<l_j\leqslant n} x_{l_1} \cdots x_{l_j} \,,$$

- Supersymmetric elementary polynomial

$$e_k(x_1, \dots, x_m|y_1, \dots, y_n) = \sum_{i+j=k} h_i(x_1, \dots, x_m) e_j(y_1, \dots, y_n).$$

$$\sigma(k) = \sum_{i+j=k} h_i(q^{-\mu_1},\dots,q^{-\mu_m})e_j(q^{-\mu_{m+1}},\dots,q^{-\mu_{m+n}})$$
$$= e_k(q^{-\mu_1},\dots,q^{-\mu_m}|q^{-\mu_{m+1}},\dots,q^{-\mu_{m+n}}),$$

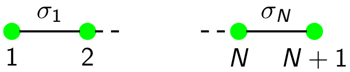
$$\Sigma(\mathbf{k}) = \prod_{i=1}^{r} e_{k_i}(q^{-\mu_1},\dots,q^{-\mu_m}|q^{-\mu_{m+1}},\dots,q^{-\mu_{m+n}})$$
$$\equiv E_{\mathbf{k}}(q^{-\mu_1},\dots,q^{-\mu_m}|q^{-\mu_{m+1}},\dots,q^{-\mu_{m+n}}).$$

Partition function of the spin chain

$$\mathcal{Z} = \sum_{\mathbf{k}\in\mathcal{P}_N} \Sigma(\mathbf{k}) q^{\sum_{i=1}^{r-1} J\mathcal{E}(K_i)} \prod_{i=1}^{N-r}\left(1 - q^{J\mathcal{E}(K_i')}\right)$$

Associated vertex models

$N+1$ vertices, N bonds, $\sigma_i \in \{1,\ldots,m+n\}$ state of the bond i:



Spectrum:

$$E^{(m|n)}(\boldsymbol{\sigma}) = J\sum_{i=1}^{N-1}\delta(\sigma_i,\sigma_{i+1})\mathcal{E}(i)$$

$$\delta(j,k) = \begin{cases} 1, & j>k, \quad \text{or } j=k\in F \\ 0, & j<k, \quad \text{or } j=k\in B \end{cases}$$

Generalized partition function

$$\mathcal{Z}^V(q;\mathbf{x}|\mathbf{y}) \equiv \sum_{\sigma_1,\dots,\sigma_N=1}^{m+n} \prod_{\alpha=1}^{m} x_\alpha^{N_\alpha(\boldsymbol{\sigma})} \cdot \prod_{\beta=1}^{n} y_\beta^{N_{m+\beta}(\boldsymbol{\sigma})} \cdot q^{E^{(m|n)}(\boldsymbol{\sigma})}\,,$$

Partition function of the vertex model

$$\mathcal{Z}^V(q) = \mathcal{Z}^V(q;1^m|1^n).$$

Super Schur polynomials

$S_{\mathbf{k}}(\mathbf{x}|\mathbf{y}) \equiv$ super Schur polynomial associated to the border strip $\langle k_1, \ldots, k_r \rangle$

$$\sum_{\mathbf{k}\in\mathcal{P}_N} S_{\mathbf{k}}(\mathbf{x}|\mathbf{y}) q^{\sum_{i=1}^{r-1} J\mathcal{E}(K_i)} = \sum_{\mathbf{k}\in\mathcal{P}_N} E_{\mathbf{k}}(\mathbf{x}|\mathbf{y}) q^{\sum_{i=1}^{r-1} J\mathcal{E}(K_i)} \prod_{i=1}^{N-r} (1 - q^{J\mathcal{E}(K_i')}) .$$

$$\mathcal{Z}^V(q;\mathbf{x}|\mathbf{y}) = \sum_{\mathbf{k}\in\mathcal{P}_N} S_{\mathbf{k}}(\mathbf{x}|\mathbf{y}) q^{\sum_{i=1}^{r-1} J\mathcal{E}(K_i)}$$

Combining these equations

$$\mathcal{Z}^V(q;\mathbf{x}|\mathbf{y}) = \sum_{\mathbf{k}\in\mathcal{P}_N} E_{\mathbf{k}}(\mathbf{x}|\mathbf{y}) q^{\sum_{i=1}^{r-1} J\mathcal{E}(K_i)} \prod_{i=1}^{N-r} (1 - q^{J\mathcal{E}(K_i')}),$$

$\mathbf{x} \in \mathbf{R}^m$, $\mathbf{y} \in \mathbf{R}^n$.

The partition function of the chain can be written in a simple way using the vertex model description:

$$\mathcal{Z}(q) = \mathcal{Z}^V(q; q^{-\mu_1}, \ldots, q^{-\mu_m} | q^{-\mu_{m+1}}, \ldots, q^{-\mu_{m+n}})$$
$$= \sum_{\sigma_1,\ldots,\sigma_N=1}^{m+n} q^{E^{(m|n)}(\boldsymbol{\sigma}) - \sum\limits_{\alpha=1}^{m+n} \mu_\alpha N_\alpha(\boldsymbol{\sigma})} = \sum_{\sigma_1,\ldots,\sigma_N=1}^{m+n} q^{E^{(m|n)}(\boldsymbol{\sigma}) - \sum_i \mu_{\sigma_i}}$$

Spectrum of the HS-type chains

$$E(\boldsymbol{\sigma}) = E^{(m|n)}(\boldsymbol{\sigma}) - \sum_i \mu_{\sigma_i} = J \sum_{i=1}^{N-1} \delta(\sigma_i, \sigma_{i+1}) \mathcal{E}(i) - \sum_i \mu_{\sigma_i}$$

The vectors $\boldsymbol{\delta}(\boldsymbol{\sigma}) \in \{0,1\}^{N-1}$ with components $\delta_k(\boldsymbol{\sigma}) = \delta(\sigma_k, \sigma_{k+1}) \equiv$ supersymmetric version of the *motifs* (Haldane et al.)

Normalizing the Hamiltonians: the mean energy per site should tend to a finite limit as $N \to \infty$. Since

$$\operatorname{tr} P_{ij}^{(m|n)} = (m+n)^{N-2}(m-n)\,, \qquad \operatorname{tr} \mathcal{N}_\alpha = N(m+n)^{N-1}\,,$$

the mean energy is

$$\mu = \frac{\operatorname{tr}\mathcal{H}}{(m+n)^N} = \left(1 - \frac{m-n}{(m+n)^2}\right)\sum_{i<j} J_{ij} - \frac{N}{m+n}\sum_{\alpha=1}^{m+n}\mu_\alpha\,.$$

$$\sum_{i<j} J_{ij} = \frac{J}{2}\sum_{i=1}^{N-1}\mathcal{E}(i), \quad \mu = \frac{J}{2}\left(1 - \frac{m-n}{(m+n)^2}\right)\sum_{i=1}^{N-1}\mathcal{E}(i) - \frac{N}{m+n}\sum_{\alpha=1}^{m+n}\mu_\alpha\,.$$

$$\sum_{i=1}^{N-1}\mathcal{E}(i) = \begin{cases} \frac{N}{6}(N^2-1), & \text{HS} \\ \frac{N}{2}(N-1), & \text{PF} \\ \frac{N}{6}(N-1)(2N+3c-4), & \text{FI} \end{cases}$$

The mean energy per site will tend to a constant in the thermodynamic limit $N \to \infty$ if J scales as

$$J = \begin{cases} \frac{K}{N^2}, & \text{for the HS and FI chains} \\ \frac{K}{N}, & \text{for the PF chain,} \end{cases}$$

$K \in \mathbf{R}$ independent of N and $\lim_{N\to\infty} c/N \equiv \gamma \geqslant 0$ finite. Then

$$J\mathcal{E}(i) = K\varepsilon(x_i)\,, \qquad x_i \equiv \frac{i}{N}, \quad \gamma_N = (c-1)/N$$

$$\varepsilon(x) = \begin{cases} x(1-x), & \mathrm{HS} \\ x, & \mathrm{PF} \\ x(\gamma_N + x), & \mathrm{FI} \end{cases}$$

The transfer matrices

$$E(\boldsymbol{\sigma}) = \sum_{i=1}^{N-1} \left[K\delta(\sigma_i, \sigma_{i+1})\varepsilon(x_i) - \frac{1}{2}(\mu_{\sigma_i} + \mu_{\sigma_{i+1}}) \right] - \frac{1}{2}(\mu_{\sigma_1} + \mu_{\sigma_N}),$$

$$\mathcal{Z}(q) = \operatorname{tr}\left[A(x_0)A(x_1)\cdots A(x_{N-1}) \right], \quad A_{\alpha\beta}(x) = q^{K\varepsilon(x)\delta(\alpha,\beta) - \frac{1}{2}(\mu_\alpha + \mu_\beta)}.$$

$$A(x) = P(x)J(x)P(x)^{-1}, \quad A_i \equiv A(x_i), \quad J_i \equiv J(x_i), \quad P_i \equiv P(x_i)$$

The partition function is:

$$\mathcal{Z}(q) = \operatorname{tr}\left[P_0 J_0 (P_0^{-1} P_1) J_1 \cdots (P_{N-2}^{-1} P_{N-1}) J_{N-1} P_{N-1}^{-1} \right].$$

$$P_{i+1} = P_i + \frac{1}{N} P'(x_i) + \mathrm{o}(N^{-1}) = P_i + \mathrm{O}(N^{-1}).,\quad P_i^{-1} P_{i+1} = \mathbb{1} + \mathrm{O}(N^{-1}),$$

The dominant contribution to the free energy per spin $f(T) \equiv -(T/N)\log \mathcal{Z}(q)$ in the thermodynamic limit is:

$$f(T) \simeq -\frac{T}{N} \log \operatorname{tr}(UJ_0 \cdots J_{N-1}), \quad U \equiv \lim_{N\to\infty} P_{N-1}^{-1} P_0 = P(1)^{-1} P(0).$$

Assume that $J_0 \cdots J_{N-1}$ is *diagonal*. If $\lambda_\alpha(x)$ are the eigenvalues of $A(x)$

$$\operatorname{tr}(UJ_0 \cdots J_{N-1}) = \sum_{\alpha=1}^{m+n} U_{\alpha\alpha} \Lambda_\alpha\,, \quad \Lambda_\alpha = \prod_{i=0}^{N-1} \lambda_\alpha(x_i)$$

Perron–Frobenius theorem (all the entries of $A(x)$ are strictly positive):
The matrix $A(x)$ has a simple positive eigenvalue, $\lambda_1(x)$, satisfying

$$\lambda_1(x) > |\lambda_\alpha(x)|$$

and it follows that

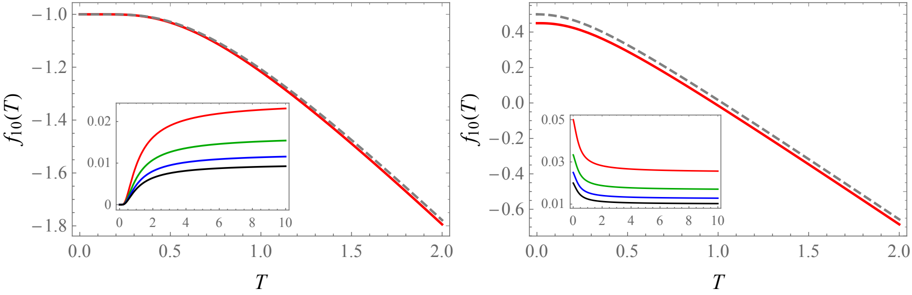
$$\lim_{N\to\infty} \frac{|\Lambda_\alpha|}{\Lambda_1} = 0\,,\ \forall\alpha > 1\,.$$

Then

$$\operatorname{tr}(UJ_0 \cdots J_{N-1}) \simeq U_{11}\Lambda_1 \equiv U_{11} \prod_{i=0}^{N-1} \lambda_1(x_i),$$

if $U_{11} \neq 0$. In this case, the free energy per site in the thermodynamic limit is

$$f(T) = -T \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log\lambda_1(x_i) = -T \int_0^1 \log\lambda_1(x)\,\mathrm{d}x\,.$$



Left: free energy per site of the $\mathrm{su}(1|1)$ PF chain with $\mu_1 \equiv \mu = K > 0$ for $N = 10$ spins, $f_{10}(T)$, as a function of T (solid red line) compared to its thermodynamic limit computed withy the obtained function (dashed gray line). Right: same plot for $\mu = -K$. Insets: difference $f(T) - f_N(T)$ for $N = 10$ (red), 15 (green), 20 (blue) and 25 (black) spins in the range $0 \leqslant T \leqslant 10$. Note: in all plots, f_N, f and T are measured in units of K.

Symmetries of the free energy

$$
\begin{aligned}
f^{(n|m)}(\mu_1, \dots, \mu_{m+n-1}; K) &= K\varepsilon_0 - \mu_{\alpha_1} \\
&+ f^{(m|n)}(-\mu_{\alpha_1}, \mu_{\alpha_{m+n-1}} - \mu_{\alpha_1}, \dots, \mu_{\alpha_2} - \mu_{\alpha_1}; -K),
\end{aligned}
$$

$(\alpha_1, \dots, \alpha_{m+n-1})$ is a permutation of $(1, \dots, m+n-1)$ such that $\{\alpha_1, \dots, \alpha_n\} = \{1, \dots, n\}$.

Density of spins of type α

$$n_\alpha = -\frac{\partial f}{\partial \mu_\alpha}.$$

The variance (per site) of the number of spins of type α

$$\nu_\alpha \equiv \frac{1}{N}\Big(\langle \mathcal{N}_\alpha^2\rangle - \langle \mathcal{N}_\alpha\rangle^2\Big) = -\beta^{-1}\frac{\partial^2 f}{\partial \mu_\alpha^2}\,.$$

The internal energy, heat capacity (at constant volume) and entropy per site are

$$u = \frac{\partial}{\partial\beta}(\beta f), \qquad c_V = -\beta^2\frac{\partial u}{\partial\beta}\,, \qquad s = \beta^2\,\frac{\partial f}{\partial\beta} = \beta(u-f)\,.$$

Transfer matrix

$$A(x) = \begin{pmatrix} q^{-\mu} & q^{-\frac{\mu}{2}} \\ q^{K\varepsilon(x)-\frac{\mu}{2}} & q^{K\varepsilon(x)} \end{pmatrix}, \qquad \mu \equiv \mu_1 ,$$

Eigenvalues

$$\lambda_1(x) = q^{K\varepsilon(x)} + q^{-\mu} \qquad \lambda_2 = 0$$

$A(x)$ is diagonalizable for all $x \in [0, 1]$

$$P(x) = \begin{pmatrix} q^{-(K\varepsilon(x)+\frac{\mu}{2})} & -q^{\frac{\mu}{2}} \\ 1 & 1 \end{pmatrix}$$

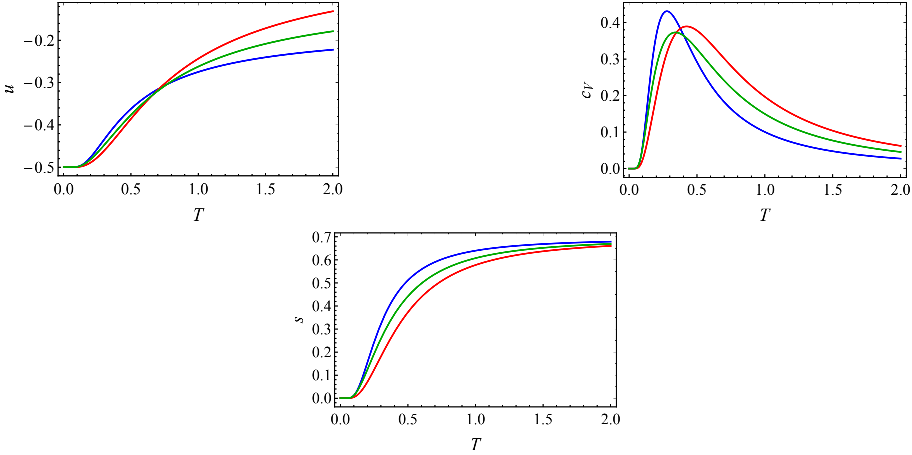
Free energy per site

$$f(T, \mu) = -T \int_0^1 \log\left(q^{K\varepsilon(x)} + q^{-\mu}\right) \mathrm{d}x = -\mu - \frac{1}{\beta} \int_0^1 \log\left(1 + \mathrm{e}^{-\beta(K\varepsilon(x)+\mu)}\right) \mathrm{d}x \, .$$

valid for the three chains of HS type.

Thermodynamic functions of the $\mathrm{su}(1|1)$ chains of HS type:

$$
\begin{aligned}
n_1 &= \int_0^1 \frac{\mathrm{d}x}{1+\mathrm{e}^{-\beta(K\varepsilon(x)+\mu)}}\\
\nu_1 &= \frac{1}{4}\int_0^1 \operatorname{sech}^2\big[\tfrac{\beta}{2}(K\varepsilon(x)+\mu)\big]\,\mathrm{d}x\,,\\
u &= -\mu + \int_0^1 \frac{K\varepsilon(x)+\mu}{1+\mathrm{e}^{\beta(K\varepsilon(x)+\mu)}}\,\mathrm{d}x\\
c_V &= \frac{\beta^2}{4}\int_0^1 (K\varepsilon(x)+\mu)^2 \operatorname{sech}^2\big[\tfrac{\beta}{2}(K\varepsilon(x)+\mu)\big]\,\mathrm{d}x\,,\\
s &= \int_0^1 \bigg\{ \log\Big[2\cosh\big(\tfrac{\beta}{2}(K\varepsilon(x)+\mu)\big)\Big]\\
&\quad - \frac{\beta}{2}(K\varepsilon(x)+\mu)\tanh\big(\tfrac{\beta}{2}(K\varepsilon(x)+\mu)\big)\bigg\}\,\mathrm{d}x\,.
\end{aligned}
$$



Internal energy, specific heat and entropy (right) per site versus the temperature for the HS (blue), PF (red) and FI (with $\gamma = 0$, green) $\mathrm{su}(1|1)$ chains with $\mu/K = 1/2$. The specific heat exhibits the Schottky peak, characteristic of two-level systems like the Ising model at zero magnetic field or paramagnetic spin $1/2$ anyons.

Using the dilogarithm function

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,\mathrm{d}t,$$

the free energy is given by

$$f(T,\mu) = -\mu + \frac{1}{K\beta^2}\Big[\mathrm{Li}_2(-\mathrm{e}^{-\beta\mu}) - \mathrm{Li}_2(-\mathrm{e}^{-\beta(K+\mu)})\Big].$$

Thermodynamic functions

$$n_1 = 1 - \frac{1}{K\beta}\log\left(\frac{1+\mathrm{e}^{-\beta\mu}}{1+\mathrm{e}^{-\beta(K+\mu)}}\right)$$

$$u = \frac{\mu}{K\beta}\log(1+\mathrm{e}^{-\beta\mu}) - \frac{K+\mu}{K\beta}\log(1+\mathrm{e}^{-\beta(K+\mu)}) - f - 2\mu$$

$$s = \beta(u-f)$$

- When $T \to 0$ the free energy per unit length of a $(1+1)$-dimensional CFT behaves as

$$f(T) \simeq f(0) - \frac{\pi c T^2}{6v},$$

  c is the central charge and v is the effective speed of light.
- The value of f at small temperatures is determined by the low energy excitations, then the validity of this relation is taken as a strong indication of the conformal invariance of a quantum system.
- The equation is one of the standard methods to identify the central charge of the Virasoro algebra of a quantum critical system.

$\mu > 0$

$$K\varepsilon(x) + \mu > 0, \quad \forall x \in [0,1]$$

Then

$$|f(T,\mu) - f(0,\mu)| < T\int_0^1 \mathrm{e}^{-\beta(K\varepsilon(x)+\mu)} < T\mathrm{e}^{-\beta\mu}\,,$$

the system is not critical. A similar result holds for $\mu < -K\varepsilon_{\max}$, where

$$\varepsilon_{\max} = \max_{0\leqslant x\leqslant 1}\varepsilon(x) = \begin{cases} \frac{1}{4}\,, & \text{for the HS chain} \\ 1 & \text{for the PF chain} \\ 1+\gamma, & \text{for the FI chain.} \end{cases}$$

$-K\varepsilon_{\max} < \mu < 0$

$$f(T,\mu) + \mu = -\eta T \int_0^{1/\eta} \log\Big(1 + \mathrm{e}^{-\beta(K\varepsilon(x)+\mu)}\Big)\mathrm{d}x\,,$$

$$\eta = \begin{cases} 2\,, & \text{for the HS chain} \\ 1, & \text{for the PF and FI chains.} \end{cases}$$

$$x_0 = \begin{cases} \frac{1}{2}\,(1 - \sqrt{1 + \frac{4\mu}{K}}\,), & \text{for the HS chain} \\ -\frac{\mu}{K}, & \text{for the PF chain} \\ \frac{1}{2}\,(-\gamma + \sqrt{\gamma^2 - \frac{4\mu}{K}}\,), & \text{for the FI chain} \end{cases}$$

unique root of the equation $K\varepsilon(x) + \mu = 0$ in the interval $(0, 1/\eta)$

$K\varepsilon(x)+\mu$ is negative for $0 \leqslant x < x_0$ and positive for $x_0 < x \leqslant 1/\eta$,

$$f(T,\mu) - f(0,\mu) = -\eta T \int_0^{1/\eta} \log\Big(1 + \mathrm{e}^{-\beta|K\varepsilon(x)+\mu|}\Big)\mathrm{d}x.$$

Fix $\Delta < \min(x_0, 1/\eta - x_0)$ independent of T and $A \equiv [0, x_0 - \Delta] \cup [x_0 + \Delta, 1/\eta]$. The integral can be approximated by

$$I(T) \equiv \int_{x_0-\Delta}^{x_0+\Delta} \log\Big(1 + \mathrm{e}^{-\beta|K\varepsilon(x)+\mu|}\Big)\mathrm{d}x$$

Change of variables $y = \beta|K\varepsilon(x)+\mu|$ in each of the intervals $[x_0 - \Delta, x_0]$ and $[x_0, x_0 + \Delta]$:

$$\begin{aligned} I(T) = &\frac{T}{K}\Bigg(\int_0^{\beta|K\varepsilon(x_0-\Delta)+\mu|} \frac{\log(1+\mathrm{e}^{-y})}{\varepsilon'(x)}\,\mathrm{d}y \\ &+ \int_0^{\beta|K\varepsilon(x_0+\Delta)+\mu|} \frac{\log(1+\mathrm{e}^{-y})}{\varepsilon'(x)}\,\mathrm{d}y\Bigg). \end{aligned}$$

$$\frac{1}{\varepsilon'(x)} = \frac{1}{\varepsilon'(x_0)} + \mathrm{O}(x - x_0) = \frac{1}{\varepsilon'(x_0)} + \mathrm{O}(Ty),$$

$$I(T) = \frac{T}{K\varepsilon'(x_0)} \left(\int_0^{\beta|K\varepsilon(x_0-\Delta)+\mu|} + \int_0^{\beta|K\varepsilon(x_0+\Delta)+\mu|} \right) \log(1 + \mathrm{e}^{-y})\,\mathrm{d}y$$
$$+ \mathrm{O}(T^2)\,.$$

$$I(T) = \frac{2T}{K\varepsilon'(x_0)} \int_0^{\infty} \log(1 + \mathrm{e}^{-y})\,\mathrm{d}y + \mathrm{O}(T^2) = \frac{\pi^2 T}{6K\varepsilon'(x_0)} + \mathrm{O}(T^2)\,,$$

$$f(T,\mu) = f(0,\mu) - \frac{\eta\pi^2 T^2}{6K\varepsilon'(x_0)} + \mathrm{O}(T^3)\,.$$

Effective speed of light

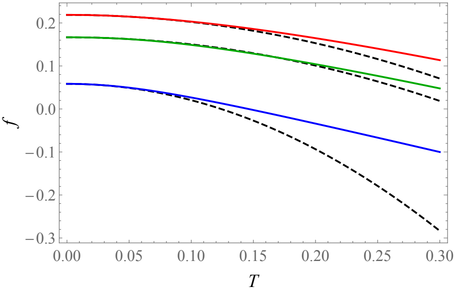
$$v = \frac{\mathrm{d}E}{\mathrm{d}p}\bigg|_{p=2\pi x_0} = \frac{K\varepsilon'(x_0)}{2\pi} \qquad \text{(HS chain)}\,.$$

$$v = \frac{\mathrm{d}E}{\mathrm{d}p}\bigg|_{p=\pi x_0} = \frac{K\varepsilon'(x_0)}{\pi} \qquad \text{(PF and FI chains)}\,.$$

Asymptotic equation for the free energy per site

$$f(T,\mu) = f(0,\mu) - \frac{\pi T^2}{6v} + O(T^3)\,,$$

For $-K\varepsilon_{\max} < \mu < 0$ the chains are critical, with $c = 1$: the free energy per site behaves as that of a CFT with central charge $c = 1$



Free energy per site versus temperature (both in units of K) for the $\mathrm{su}(1|1)$ HS (blue), PF (red) and FI chains (with $\gamma = 0$, green) for $\mu/K = -\varepsilon_{\max}/4$. In all three cases, the dashed black line represents the low-temperature approximation.

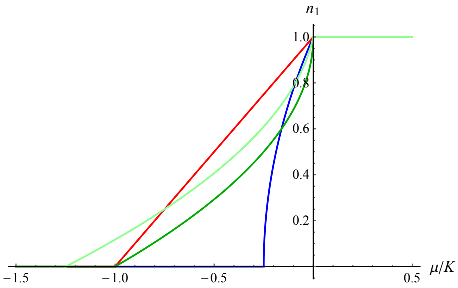
- $\mu = 0$. The HS, PF and FI (with $\gamma \neq 0$) $\mathrm{su}(1|1)$ chains are critical, with central charge $c = 1/2$
- $\gamma = \mu = 0$. The FI chain is not critical:

$$f(T,0) = -\frac{1}{2}\sqrt{\frac{\pi}{K}}\left(1-\frac{1}{\sqrt{2}}\right)\zeta(3/2)\,T^{3/2} + O(T^2),$$

- $\mu = -K\varepsilon_{\max}$. The PF and FI chains are critical with $c = 1/2$. The HS chain behaves as

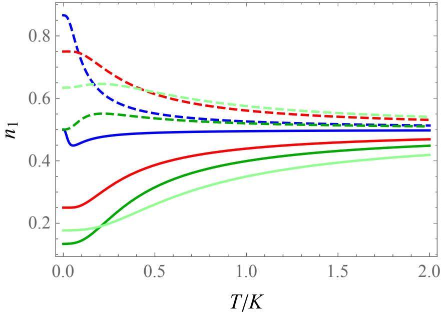
$$f(T,-K/4) = \frac{K}{6} - \sqrt{\frac{\pi}{K}}\left(1-\frac{1}{\sqrt{2}}\right)\zeta(3/2)\,T^{3/2} + O(T^2).$$

Phase transitions



Zero temperature boson density n_1 as a function of μ/K for the HS (blue), PF (red) and FI (green for $\gamma = 0$, light green for $\gamma = 1/4$) chains.

The $\mathrm{su}(1|1)$ boson density presents a second-order (continuous) phase transition at zero temperature



Boson density for $\mu/(K\varepsilon_{\max}) = -3/4$ (solid lines) and $\mu/(K\varepsilon_{\max}) = -1/4$ (dashed lines), HS (blue), PF (red) and FI chains, with $\gamma = 0$ (green) and $\gamma = 1$ (light green)

Its dual su(1|2) version with HS interaction can be mapped to the spin 1/2 Kuramoto–Yokoyama *t*-*J* model in an external magnetic field

Transfer matrix $A(x)$:

$$A(x) = \begin{pmatrix} q^{-\mu_1} & q^{-\frac{1}{2}(\mu_1+\mu_2)} & -q^{-\frac{\mu_1}{2}} \\ q^{K\varepsilon(x)-\frac{1}{2}(\mu_1+\mu_2)} & q^{-\mu_2} & q^{-\frac{\mu_2}{2}} \\ q^{K\varepsilon(x)-\frac{\mu_1}{2}} & q^{K\varepsilon(x)-\frac{\mu_2}{2}} & q^{K\varepsilon(x)} \end{pmatrix},$$

Eigenvalues:

$$\lambda_\pm(x) = a(x) \pm \sqrt{a(x)^2 + q^{-(\mu_1+\mu_2)}(q^{K\varepsilon(x)} - 1)}, \quad 0$$

$$a(x) = \frac{1}{2}\left(q^{-\mu_1} + q^{-\mu_2} + q^{K\varepsilon(x)}\right).$$

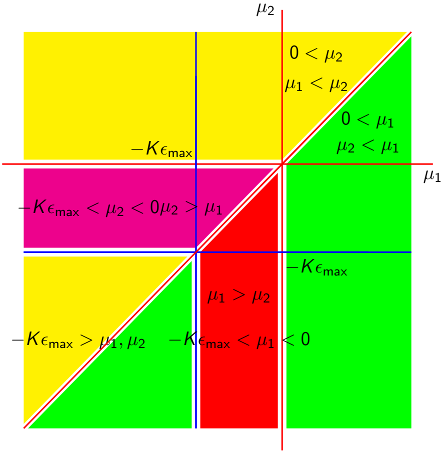
- Perron–Frobenius eigenvalue is $\lambda_1(x) = \lambda_+(x)$.
- $A(x)$ is diagonalizable for $0 < x < 1$M
- Matrix $P(x)$

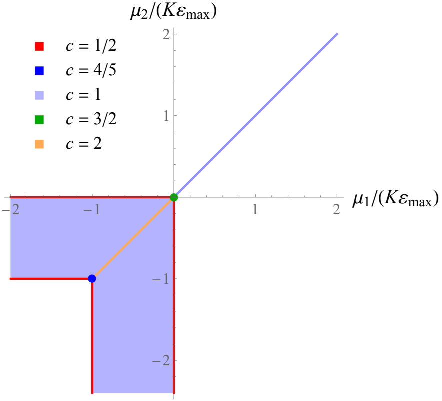
$$P(x) = \begin{pmatrix} q^{\frac{1}{2}(\mu_2-\mu_1)} & 0 & q^{\frac{1}{2}(\mu_2-\mu_1)} \\ 1 + \frac{q^{-\mu_1}}{\lambda_+(x)}(q^{K\varepsilon(x)} - 1) & -q^{\frac{\mu_2}{2}} & 1 + \frac{q^{-\mu_1}}{\lambda_-(x)}(q^{K\varepsilon(x)} - 1) \\ q^{K\varepsilon(x)+\frac{\mu_2}{2}} & 1 & q^{K\varepsilon(x)+\frac{\mu_2}{2}} \end{pmatrix}.$$

$$f(T,\mu_1,\mu_2) = -\frac{1}{2}(\mu_1+\mu_2) - T\int_0^1 \log\Big(b(x) + \sqrt{b(x)^2 + \mathrm{e}^{-K\beta\varepsilon(x)} - 1}\Big)\mathrm{d}x,$$

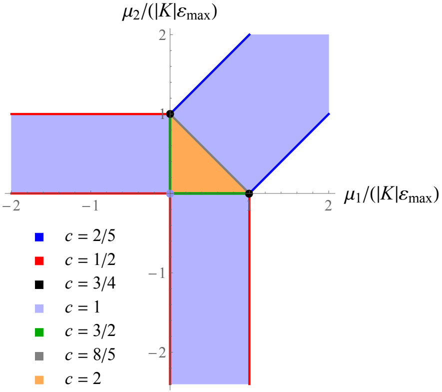
$$b(x) = \frac{1}{2}\,\mathrm{e}^{-\beta[K\varepsilon(x)+\frac{1}{2}(\mu_1+\mu_2)]} + \cosh\Big(\frac{\beta}{2}(\mu_1-\mu_2)\Big).$$

Critical behavior





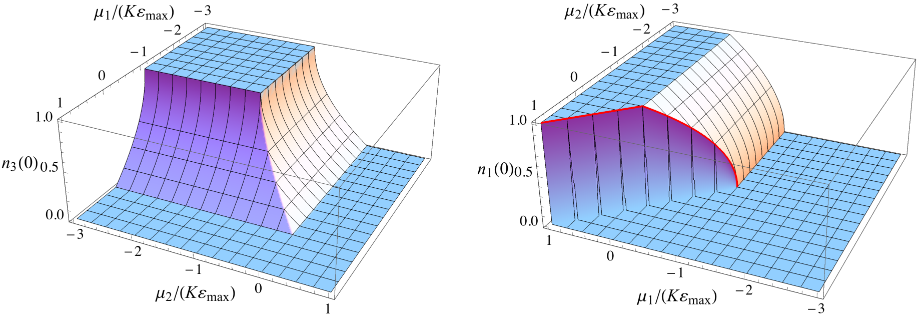
Phase diagram of the $\mathrm{su}(2|1)$ chains of HS type, $K > 0$. The origin and the half-lines $\mu_1 = 0 > \mu_2$, $\mu_2 = 0 > \mu_1$, $\mu_1 = \mu_2 > 0$ are not critical for the FI chain with $\gamma = 0$, while the point $(-K\varepsilon_{\max}, -K\varepsilon_{\max})$ and the half-lines $\mu_1 = -K\varepsilon_{\max} > \mu_2$, $\mu_2 = -K\varepsilon_{\max} > \mu_1$ are not critical for the HS chain.



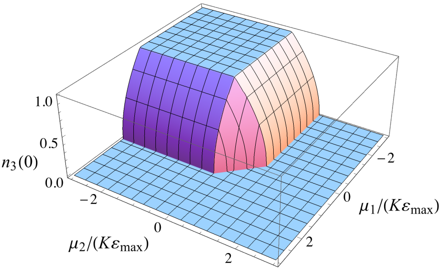
Phase diagram of the $\mathrm{su}(2|1)$ chains of HS type, $K < 0$. The origin and the half-lines $\mu_1 = 0 > \mu_2$, $\mu_2 = 0 > \mu_1$, $\mu_1 = \mu_2 > 0$ are not critical for the FI chain with $\gamma = 0$, while the points $(|K|\varepsilon_{\max}, 0)$, $(0, |K|\varepsilon_{\max})$, the segment $\{\mu_1 + \mu_2 = |K|\varepsilon_{\max},\ 0 < \mu_1 < |K|\varepsilon_{\max}\}$ and the half-lines $\{\mu_1 = |K|\varepsilon_{\max},\ \mu_2 < 0\}$, $\{\mu_2 = |K|\varepsilon_{\max},\ \mu_1 < 0\}$, $\mu_1 = \mu_2 - |K|\varepsilon_{\max} > 0$, $\mu_2 = \mu_1 - |K|\varepsilon_{\max} > 0$ are not critical for the HS chain.

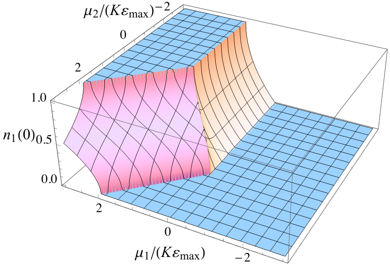
Phase transitions

- The bosonic density n_1 is discontinuous along the half-line $\mu_1 = \mu_2 \geqslant -K\varepsilon_{\max}$, and has a discontinuous gradient along the half-lines $\mu_1 = -K\varepsilon_{\max} \geqslant \mu_2$ and $\mu_1 = 0 \geqslant \mu_2$. The bosonic density n_1 (and hence n_2) presents both first- and second-order phase transitions for appropriate values of the chemical potentials μ_1 and μ_2.
- If $K < 0$ The bosonic density (and hence the remaining one $n_2(0)$) are continuous, although their gradient is discontinuous along several segments and half-lines. Thus when $K < 0$ the chains exhibit only second-order phase transitions at zero temperature.



Left: fermion density at zero temperature for the $\mathrm{su}(2|1)$ HS chain with $K > 0$. Right: same plot for the bosonic density n_1, with a red line drawn to illustrate the discontinuity along the half-line $\mu_1 = \mu_2 \geqslant -K\varepsilon_{\max}$.





Left: fermion density at zero temperature for the $\mathrm{su}(2|1)$ HS chain with $K < 0$. Right: same plot for the bosonic density n_1.

Transfer matrix

$$A(x) = \begin{pmatrix} q^{-\mu_1} & q^{-\frac{1}{2}(\mu_1+\mu_2)} & q^{-\frac{1}{2}(\mu_1+\mu_3)} & q^{-\frac{\mu_1}{2}} \\ q^{K\varepsilon(x)-\frac{1}{2}(\mu_1+\mu_2)} & q^{-\mu_2} & q^{-\frac{1}{2}(\mu_2+\mu_3)} & q^{-\frac{\mu_2}{2}} \\ q^{K\varepsilon(x)-\frac{1}{2}(\mu_1+\mu_3)} & q^{K\varepsilon(x)-\frac{1}{2}(\mu_2+\mu_3)} & q^{K\varepsilon(x)-\mu_3} & q^{-\frac{\mu_3}{2}} \\ q^{K\varepsilon(x)-\frac{\mu_1}{2}} & q^{K\varepsilon(x)-\frac{\mu_2}{2}} & q^{K\varepsilon(x)-\frac{\mu_3}{2}} & q^{K\varepsilon(x)} \end{pmatrix}$$

Eigenvalues: 0 (double) and

$$\lambda_\pm(x) = a(x) \pm \sqrt{a(x)^2 + (q^{K\varepsilon(x)} - 1)(q^{-(\mu_1+\mu_2)} - q^{K\varepsilon(x)-\mu_3})},$$

$$a(x) = \frac{1}{2}\left(q^{-\mu_1} + q^{-\mu_2} + q^{K\varepsilon(x)-\mu_3} + q^{K\varepsilon(x)}\right).$$

Perron–Frobenius eigenvalue: $\lambda_1(x) = \lambda_+(x)$.
$A(x)$ is *not* diagonalizable when $x \in (0,1)$, for $0 < x < 1$ its Jordan canonical form is

$$J(x) = \begin{pmatrix} \lambda_+(x) & 0 & 0 & 0 \\ 0 & \lambda_-(x) & \delta_{0,\lambda_-(x)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Free energy per spin

$$\begin{aligned} f(T,\mu_1,\mu_2,\mu_3) &= -\frac{1}{2}(\mu_1+\mu_2) \\ &- T\int_0^1 \log\Big[b(x) \\ &+ \sqrt{b(x)^2 - (1-\mathrm{e}^{-K\beta\varepsilon(x)})(1-\mathrm{e}^{-\beta(K\varepsilon(x)+\mu_1+\mu_2-\mu_3)})}\Big]\mathrm{d}x, \end{aligned}$$

$$b(x) = \mathrm{e}^{-\beta[K\varepsilon(x)+\frac{1}{2}(\mu_1+\mu_2-\mu_3)]}\cosh\Big(\frac{\beta}{2}\mu_3\Big) + \cosh\Big(\frac{\beta}{2}(\mu_1-\mu_2)\Big).$$

From numerical calculations,

- $K > 0$ the fermionic densities $n_{3,4}$ exhibit only second-order phase transitions at $T = 0$, while the bosonic ones $n_{1,2}$ undergo also a first-order phase transition across (a subset of) the plane $\mu_1 = \mu_2$.
- $K < 0$ the fermionic densities feature only second-order phase transitions at zero temperature while the bosonic ones present also a first-order phase transition across (a subset of) the plane $\mu_3 = 0$.

Conclusions

- The thermodynamics and critical behavior of the three families of $\mathrm{su}(m|n)$ supersymmetric spin chains of Haldane–Shastry type with an additional chemical potential term. The analysis is based on
  - the computation in closed form of the partition function for an arbitrary (finite) number of spins
  - the derivation of a simple description of the spectrum in terms of supersymmetric motifs.
- Using the transfer matrix method, we obtain an analytic expression for the free energy per site,
- For the $\mathrm{su}(1|1)$, $\mathrm{su}(2|1)$ (or $\mathrm{su}(1|2)$) and $\mathrm{su}(2|2)$ chains, we identify the values of the chemical potentials for which the models are critical (gapless) (low-temperature behavior of the free energy per site)
- We show that the central charge can take rational values that are not integers or half-integers, thus excluding the equivalence to a CFT with free bosons and/or fermions.
- We analyze the existence of zero-temperature phase transitions in the spin densities.

¡Feliz cumpleaños!

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