Quantum Control on the Boundary

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QUITEMAD
Quantum Information Technologies in Madrid

Outline

- Controllability of Quantum Systems
- Unbounded operators and self-adjointness
- Quantum Control on the boundary

- Time dependent Schrödinger Equation

$$i\frac{\partial\Psi}{\partial t} = H(t)\Psi$$

- $H(t)$ is a family of self-adjoint operators
- The solution of the equation is given in terms of a unitary propagator
  - $U : \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathcal{H})$
  - $U(t,t) = \mathbb{I}_{\mathcal{H}}$
  - $U(t,s)U(s,r) = U(t,r)$
- $\Psi(t) = U(t,t_0)\Psi_0$ is a solution of Schrödinger's Equation with initial value Ψ_0

- Finite dimensional quantum System $\mathcal{H} = \mathbb{C}^n$
- Simple situation. Linear controls:

$$i\frac{\partial \Psi}{\partial t} = \left(H_0 + c(t)H_1\right)\Psi$$

- H_0, H_1 self-adjoint operators (Hermitean matrices).
- $c : \mathbb{R} \to \mathcal{C}$ Space of controls
- Use the controls to steer the state of the system from $\Psi_0 \to \Psi_f$.

- Ultimate Objective (not today): Find a curve $c(t) \subset \mathcal{C}$ that drives the system from $\Psi_0 \to \Psi_f$.
- Optimal control: The solution $\Psi(t) = U(t, t_0)\Psi_0$ must minimize some functional.
  - Minimal time
  - Minimal energy
- First of all: Decide wether or not the system is controllable.
  - If there exists $c(t) \subset \mathcal{C}$ such that for some T

$$\Psi_f = \Psi(T) = U(t, t_0)\Psi_0$$

Controllability of finite dimensional Quantum systems

- Study the dynamical Lie algebra:

$$\mathfrak{Lie}\{iH_0, iH_1\}$$

- The reachable set of Ψ_0 is the orbit through Ψ_0 of the exponential map of the dynamical Lie algebra.

- The finite dimensional quantum system is controllable if the dynamical Lie algebra is the Lie algebra of $U(N)$.

Harmonic Oscillator

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{1}{2}x^2\Psi + c(t)x\Psi = \left[\frac{1}{2}(p^2+q^2) + c(t)q\right]\Psi$$

$$p\Psi = -i\frac{\mathrm{d}\Psi}{\mathrm{d}x} \qquad q\Psi = x\Psi(x)$$

- Harmonic Oscillator algebra:

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip) \qquad a = \frac{1}{\sqrt{2}}(q + ip) \qquad N = a^\dagger a$$

$$N|n\rangle = n|n\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle$$

Harmonic Oscillator

$$i\frac{\partial\Psi}{\partial t} = \underbrace{\boxed{-\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{1}{2}x^2\Psi}}_{H_0} + c(t)\underbrace{\boxed{x\Psi}}_{H_1} = \left[\frac{1}{2}(p^2+q^2) + c(t)q\right]\Psi$$

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- Generators of the dynamic:

$$H_0 = N + \frac{1}{2} \qquad H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Finite-dimensional approximation by the first n eigen-states

$$(H_0^n)_{ij} = \langle i|H_0|j\rangle \qquad (H_1^n)_{ij} = \langle i|H_0|j\rangle$$

Example: Truncation of the Harmonic Oscillator

- Finite-dimensional approximation by the first n eigen-states

$$(H_0^n)_{ij} = \langle i|H_0|j\rangle \qquad (H_1^n)_{ij} = \langle i|H_0|j\rangle$$

- The finite dimensional approximation is controllable for all n

$$\dim \mathfrak{Lie}\{iH_0, iH_1\} = n^2$$

Controllability of the Harmonic Oscillator

- Generators of the dynamic:

$$H_0 = N + \frac{1}{2}$$

$$H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Dynamical Lie Algebra of the Harmonic Oscillator

$$[a, a^\dagger] = \mathbb{I} \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger$$

$$[iH_0, iH_1] = -\frac{1}{\sqrt{2}}[N, a^\dagger + a] = -\frac{1}{\sqrt{2}}(a^\dagger - a) = ip = iH_2$$

Controllability of the Harmonic Oscillator

- Generators of the dynamic:

$$H_0 = N + \frac{1}{2} \qquad\qquad H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Dynamical Lie Algebra of the Harmonic Oscillator

$$[a, a^\dagger] = \mathbb{I} \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger$$

$$[iH_0, iH_1] = iH_2 \quad [iH_0, iH_2] = iH_1 \quad [iH_1, iH_2] = i\mathbb{I} = iH_3$$

Controllability of the Harmonic Oscillator

- Generators of the dynamic:

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- Four dimensional Lie algebra!
- The infinite dimensional Harmonic Oscillator is not controllable.

Approximate Controllability: A linear control system is approximately controllable if for every Ψ_0, $\Psi_1 \in \mathcal{S}$ and every $\epsilon > 0$ there exist $T > 0$ and $c(t) \subset \mathcal{C}$ such that

$$\|\Psi_1 - U(T, t_0)\Psi_0\| < \epsilon$$

- Reasonable for infinite dimensions
- Space is defined as equivalence classes of convergent sequences
- Is natural to expect this if one has exact controllability of every finite dimensional subsystem

Outline

- Controllability of Quantum Systems
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- Unbounded operators are not continuous
- One needs to define a suitable domain
- There are two major probems
  - The domain of the adjoint operator

$$\langle \Phi, T\Psi \rangle = \langle T^\dagger \Phi, \Psi \rangle$$

  - The sum of operators is not well defined in general

$$T = A + B \qquad \mathcal{D}(T) = \mathcal{D}(A) \cap \mathcal{D}(B)$$

- Quadratic Form defined by the operator

$$Q(\Phi, \Psi) = \langle \Phi, T\Psi \rangle$$

  - If the operator is lower semibounded, there exists a unique self-adjoint operator that represents it.

- Q.F. Laplace operator

$$\begin{aligned} Q(\Phi, \Psi) &= \langle \mathsf{d}\Phi, \mathsf{d}\Psi \rangle - \langle \varphi, \dot{\varphi} \rangle \\ &= \langle \mathsf{d}\Phi, \mathsf{d}\Psi \rangle - \langle \varphi, A_U\varphi \rangle \end{aligned}$$

$$\dot{\varphi} = -i \left(\frac{U - \mathbb{I}}{U + \mathbb{I}} \right) \varphi = A_U \varphi$$

- Splitting of the boundary conditions

$$U = -1P + P^{\perp}U$$

$$\begin{aligned} P\varphi &= 0 \\ P^{\perp}\dot{\varphi} &= A_U P^{\perp}\varphi \end{aligned}$$

Approximate Controllability

- Consider the Linear Control System:

$$i\frac{\partial \Psi}{\partial t} = (H_0 + c(t)H_1)\,\Psi$$

  - H_0, H_1 are self-adjoint.
  - $\{\Phi_n\}_{n\in\mathbb{N}}$ O.N.B of eigenvectors of H_0
  - $\Phi_n \in \mathcal{D}(H_1)$ for every $n \in \mathbb{N}$

- The linear control system is approximately controllable with piecewise constant controls if [Chambrion, Mason, Sigalotti, Boscain 2009]:
  - $(\lambda_{n+1} - \lambda_n)_{n\in\mathbb{N}}$ are $\mathbb{Q}$-linearly independent.
  - $\langle B\Phi_n, \Phi_{n+1}\rangle \neq 0$ for any $n \in \mathbb{N}$

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Quantum Control at the boundary

- Time dependent Schrödinger Equation

$$i\frac{\partial \Psi}{\partial t} = H(t)\Psi$$

- $H(t)$ is a family of different self-adjoint extension of the same operator

$$\big(H, \mathcal{D}\left(c(t)\right)\big)$$

- Advantage. There is no need to apply an external field
- Problem. Even the existence of solutions of the dynamics is compromised.

- Assumption:
  - The spectrum of $\left(H, \mathcal{D}(c)\right)$ only contains eigenvalues with finite degeneracy.
  - Then $\{\Phi_n^c\}_{n\in\mathbb{N}}$ forms a complete orthonormal base.
- Fix a reference extension $\left(H, \mathcal{D}(c_0)\right)$
  - Define the unitary operator

$$\begin{aligned} V_c \colon \quad & \mathcal{H} \to \mathcal{H} \\ & \Phi_n^c \to \Phi_n^0 \end{aligned}$$

  - One needs to require additionally that $V_c \colon \mathcal{D}(c) \to \mathcal{D}(c_0)$

Quantum Control at the boundary

- Write the Schrödinger Equation on the reference domain:

$$\Psi \in \mathcal{D}(c) \qquad \Psi_0 := V_c \Psi \in \mathcal{D}(c_0)$$

- $V_c H V_c^\dagger$ is a self-adjoint operator with domain $\mathcal{D}(c_0)$
- The Schrödinger Equation becomes

$$i\frac{\partial}{\partial t}\left(V_c^\dagger \Psi_0\right) = H V_c^\dagger \Psi_0$$

$$i\frac{\partial}{\partial t}\Psi_0 = V_c H V_c^\dagger \Psi_0 - i V_c \dot{V}_c^\dagger \Psi_0$$

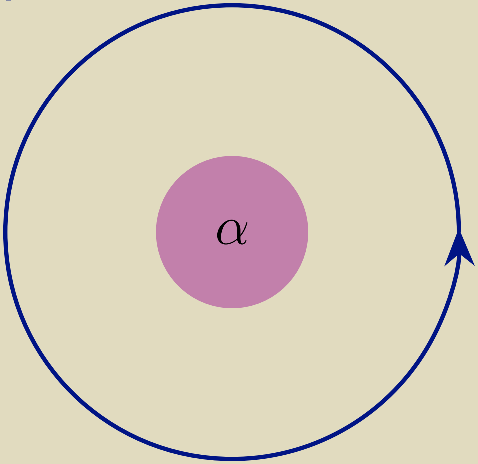
- Theorem (J. Kisynski '63):

If the family $\left(H, \mathcal{D}\left(c(t)\right)\right)$ is uniformly semibounded in time and under suitable differentiability conditions of $V_{c(t)}$ there exists a unitary propagator solving the time dependent Schrödinger equation.

Example: Varying Quasiperiodic Boundary Conditions

$$i\frac{\mathrm{d}}{\mathrm{d}t}\Psi = [i\frac{\mathrm{d}}{\mathrm{d}\theta} - \alpha]^2\Psi + \theta\dot{\alpha}\Psi$$

- By the theorem [Chambrion, Mason, Sigalotti, Boscain 2009] this system is approximately controllable by piecewise constant controls.



Particle Moving in a circular wire

Magnetic flux of intensity α

Quantum Faraday Law

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