Encoding symplectic fiberwise actions on complete Lagrangian fibrations

Edith Padrón Fernández

mepadron@ull.edu.es

Universidad de La Laguna

Ibort's Fest 2018

The great Alberto!!!!

The great Alberto!!!!

Alberto as speaker

wonderful talks!!!

The great Alberto!!!!

Alberto as speaker

wonderful talks!!!

Alberto as part of the public

interesting questions and useful comments!!!

Alberto as speaker

wonderful talks!!!

Alberto as part of the public

interesting questions and useful comments!!!

enjoy with you for many years!!!!

$$\phi : G \times Q \to Q \text{ action}$$

$$\Downarrow$$

$$T^*\phi : G \times T^*Q \times T^*Q \text{ symplectic action and fibered on } \phi$$

- How are the (symplectic) lifts of the action ϕ to the cotangent bundle T^*Q?

$$\phi : G \times Q \to Q \text{ action}$$

$$\Downarrow$$

$$T^*\phi : G \times T^*Q \times T^*Q \text{ symplectic action and fibered on } \phi$$

- How are the (symplectic) lifts of the action ϕ to the cotangent bundle T^*Q?

$$\pi : T^*Q \to Q \text{ is a Lagrangian fibration}$$

$$\phi : G \times Q \to Q \text{ action}$$

$$\Downarrow$$

$$T^*\phi : G \times T^*Q \times T^*Q \text{ symplectic action and fibered on } \phi$$

- How are the (symplectic) lifts of the action ϕ to the cotangent bundle T^*Q?

$$\pi : T^*Q \to Q \text{ is a Lagrangian fibration}$$

- How are the (symplectic) lifts of the action ϕ to a Lagrangian fibration $\pi : M \to Q$?

THE COTANGENT ACTION

$$
\begin{array}{ccc}
(T^*Q, \omega_Q) & \xrightarrow{(T^*\phi)_g} & (T^*Q, \omega_Q) \\
\Big\downarrow \pi_Q & & \Big\downarrow \pi_Q \\
Q & \xrightarrow{\phi_g} & Q
\end{array}
$$

Other fiberwise cotangent actions

$$\begin{array}{ccc} (T^*Q, \omega_Q) & \xrightarrow{\color{red}{\Phi_g}} & (T^*Q, \omega_Q) \\ \downarrow{\scriptstyle \pi_Q} & & \downarrow{\scriptstyle \pi_Q} \\ Q & \xrightarrow{\phi_g} & Q \end{array}$$

Is it possible to consider other symplectic G-actions on (T^*Q, ω_Q) which fiber on ϕ?

$F : T^*Q \to T^*Q$ diffeomorphism with $\pi_Q \circ F = \pi_Q$ and $F^*(\theta_Q) = \theta_Q$

$$\Updownarrow$$

$\exists f : Q \to Q$ diffeomorphism such that $F = T^*f$

F restricted to the zero section of π_Q is a diffeomorphism onto the zero section of π_Q

Other fiberwise cotangent actions

$$F : T^*Q \to T^*Q \text{ diffeomorphism with } \pi_Q \circ F = \pi_Q \text{ and } F^*(\theta_Q) = \theta_Q$$

$$\Updownarrow$$

$$\exists f : Q \to Q \text{ diffeomorphism such that } F = T^*f$$

F restricted to the zero section of π_Q is a diffeomorphism onto the zero section of π_Q

$$F : T^*Q \to T^*Q \text{ diffeomorphism with } \pi_Q \circ F = \pi_Q \text{ and } F^*(\omega_Q) = \omega_Q$$

$$F : T^*Q \to T^*Q \text{ diffeomorphism with } \pi_Q \circ F = \pi_Q \text{ and } F^*(\theta_Q) = \theta_Q$$

$$\Updownarrow$$

$$\exists f : Q \to Q \text{ diffeomorphism such that } F = T^*f$$

F restricted to the zero section of π_Q is a diffeomorphism onto the zero section of π_Q

$$F : T^*Q \to T^*Q \text{ diffeomorphism with } \pi_Q \circ F = \pi_Q \text{ and } F^*(\omega_Q) = \omega_Q$$

$$\Updownarrow$$

$$\exists \alpha : Q \to T^*Q \text{ closed 1-form such that}$$

$$F(\gamma_q) = \gamma_q + \alpha(q)$$

$$F : T^*Q \to T^*Q \text{ diffeomorphism with } \pi_Q \circ F = \pi_Q \text{ and } F^*(\theta_Q) = \theta_Q$$

$$\Updownarrow$$

$$\exists f : Q \to Q \text{ diffeomorphism such that } F = T^*f$$

F restricted to the zero section of π_Q is a diffeomorphism onto the zero section of π_Q

$$F : T^*Q \to T^*Q \text{ diffeomorphism with } \pi_Q \circ F = \pi_Q \text{ and } F^*(\omega_Q) = \omega_Q$$

$$\Updownarrow$$

$$\exists \alpha : Q \to T^*Q \text{ closed 1-form such that}$$

$$F(\gamma_q) = \gamma_q + \alpha(q)$$

$T_{\gamma_q}(F - Id_{T^*Q})$ is null on vertical vectors $\Longrightarrow F(\gamma_q) - \gamma_q$ doesn't depend on the $\gamma_q \in T^*_q Q$

$$\alpha(q) = F(\gamma_q) - \gamma_q$$

Other fiberwise cotangent actions

$$\begin{array}{ccc} T^*Q & \xrightarrow{\Phi_g} & T^*Q \\ \downarrow{\scriptstyle \pi_Q} & & \downarrow{\scriptstyle \pi_Q} \\ Q & \xrightarrow{\phi_g} & Q \end{array}$$

Symplectic action

$$\Phi_g \circ (T^*\phi)_g^{-1} = t_{A_g}, \text{ for all } g \in G.$$

$$A : G \times Q \to T^*Q, \qquad A : G \to \Omega^1_c(Q)$$

$$\Phi_g(\gamma_g) = (T^*\phi)_g(\gamma_q) + A_g(q), \text{ for all } g \in G.$$

$$\Downarrow$$

- Φ is an affine action.
- Φ is linear if and only if Φ is the cotangent lift $T^*\phi$ of ϕ.

The cohomology complex induced by the action ϕ

n-cochains: $A : G\times \overset{n}{\ldots} \times G \to \Omega^1(Q)$

$C^n(G,\Omega^1(Q))$ the set of the n-cochains.

$\delta_\phi : C^n(G,\Omega^1(Q)) \to C^{n+1}(G,\Omega^1(Q))$ is given by

$$\begin{aligned}(\delta_\phi A)(g_1,\ldots,g_{n+1}) &= (-1)^{n+1}A(g_2,\ldots,g_{n+1}) + \\ &+\sum_{i=1}^{n}(-1)^{n+i+1}A(g_1,\ldots,g_{i-1},g_i\cdot g_{i+1},\ldots,g_{n+1}) + \\ &+\phi^*_{g_{n+1}}(A(g_1,\ldots,g_n)).\end{aligned}$$

$(C^\bullet(G,\Omega^1_c(Q)),\delta_\phi)$ is a subcomplex of $(C^\bullet(G,\Omega^1(Q)),\delta_\phi)$

$A : G \to \Omega^1_c(Q)$

- $\Phi^A : G \times T^*Q \to T^*Q$ given by $\Phi^A_g = t_{A_g} \circ (T^*\phi)_g$ is an action if and only if A is a one-cocycle with respect to the cohomology complex $(C^\bullet(G, \Omega^1(Q)), \delta_\phi)$.

$$A_{gh}(q) = A_h(q) + \phi_h^*(A_g(q)), \qquad \forall\, g, h \in G \text{ and } \forall\, q \in Q.$$

- Φ^A is also symplectic if and only if A is a one-cocycle with respect to the cohomology subcomplex $(C^\bullet(G, \Omega^1_c(Q)), \delta_\phi)$.

$$(\Phi^A_g)^*(\omega_Q) = \omega_Q - \pi_Q^*(dA_g).$$

If $A, B : G \to \Omega^1_c(Q)$ two one-cocycles with respect to $(C^\bullet(G, \Omega^1_c(Q)), \delta_\phi)$,

$$\Downarrow$$

$\exists F : T^*Q \to T^*Q$ a symplectomorphism such that

- $\pi_Q \circ F = \pi_Q$
- $F \circ \Phi^A_g = \Phi^B_g \circ F$, for all $g \in G$

$$\Updownarrow$$

$$[A] = [B] \in H^1(G, \phi, \Omega^1_c(Q))$$

A is a coboundary of $(C^\bullet(G, \Omega^1_c(Q)), \delta_\phi)$, there is a symplectomorphim which is fibered for π_Q and equivariant with respect to the actions Φ^A and $T^*\phi$.

$$\pi_Q : T^*Q \to Q \quad \pi_Q^{-1}(q) = T_q^*Q \text{ Lagrangian space with respect } \omega_Q$$

Lagrangian fibration

A fiber bundle $\pi : (M, \omega) \to Q$ with $\pi^{-1}(q)$ Lagrangian submanifold of M, for all $q \in Q$

$$\ker T_x\pi = (\ker T_x\pi)^\omega, \quad \text{for all } x \in \pi^{-1}(q)$$

$$(\ker T_x\pi)^\omega = \{v \in T_xM / \omega(x)(v, u) = 0 \text{ for all } u \in \ker T_x\pi\}$$

- $\pi : (M, \omega) \to Q$ a Lagrangian fibration
- $\alpha \in \Omega^1(Q) \Rightarrow X_{\pi^*\alpha} \in \mathfrak{X}(M)$

$$i_{X_{\pi^*\alpha}}\omega = \pi^*\alpha$$
$$\Downarrow$$

Complete Lagrangian fibration (M, Q, π, ω)

$$X_{\pi^*\alpha} \text{ is complete for all } \alpha \in \Omega^1(Q)$$

$$F^{X_{\pi^*\alpha}}(t) : M \to M \text{ is the flow of } X_{\pi^*\alpha} \text{ at the value } t \in \mathbb{R}$$
$$(F^{X_{\pi^*\alpha}}(t))^*\omega = \omega + t\pi^*(d\alpha)$$
$$\Downarrow$$
$$\begin{array}{rcl} \mu : \Omega^1(Q) \times M & \to & M \\ (\alpha, x) & \mapsto & F^{X_{\pi^*\alpha}}(1)(x) \end{array}$$

- μ is an action: $\mu(\alpha + \beta, x) = \mu(\alpha, \mu(\beta, x))$
- μ is fibered: $\pi(\mu(\alpha, x)) = \pi(x)$
- μ induces a transitive action of the abelian group T_q^*Q on $\pi^{-1}(q)$

$$\mu_q : T_q^*Q \times \pi^{-1}(q) \to \pi^{-1}(q) \quad (\alpha_q, x) \mapsto \mu(\alpha, x)$$

$$\mu_q : T_q^*Q \times \pi^{-1}(q) \to \pi^{-1}(q)$$
$$(\alpha_q, x) \mapsto \mu(\alpha, x)$$

In general, μ_q is not free

$$\Lambda_q = \{\alpha_q \in T_q^*Q \mid \mu_q(\alpha_q, x) = x \quad \forall x \in \pi^{-1}(q)\} \text{ discrete space}$$

$$\Downarrow$$

$(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q)$ is a complete Lagrangian fibration

- $\widetilde{\pi}_Q : T^*Q/\Lambda \to Q$ fibration
- $\widetilde{\omega}_Q$ the induced symplectic structure on T^*Q/Λ
- $\widetilde{\mu} : T_q^*Q/\Lambda_q \times M \to M$ is a free action

$(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q)$ *Jacobian Lagrangian fibration*

associated to the complete Lagrangian fibration (M, Q, π, ω)

$\sigma : Q \to M$ section of $\pi : M \to Q$

$$(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q) \cong (M, Q, \pi, \omega)$$

An G-action on a Lagrangian fibration

A Lagrangian fibration (M, Q, π, ω), G a Lie group, $g \in G$

$$\begin{array}{ccc} M & \xrightarrow{\Phi_g} & M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Q & \xrightarrow{\phi_g} & Q \end{array}$$

Φ is symplectic

$$\Downarrow$$

$(M, Q, \pi, \omega, \phi, \Phi)$ is a *G-Lagrangian fibration*

Examples

$$(T^*Q, Q, \pi_Q, \omega_Q, \phi, T^*\phi), \qquad (T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q, \phi, \widetilde{T^*\phi})$$

$(M, Q, \pi, \omega, \phi, \Phi)$ complete G-Lagrangian fibration

HOW IS THE REST OF SYMPLECTIC FIBERWISE ACTIONS ON π

$(M, Q, \pi, \omega, \phi, \Phi)$ complete G-Lagrangian fibration

$(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q)$ *Jacobian Lagrangian fibration*

$\Phi^1 : G \times M \to M$ is a fiberwise symplectic action

$$\Downarrow$$

for all $g \in G$, there exists a Lagrangian section $\Sigma_g : Q \to T^*Q/\Lambda$ of $\widetilde{\pi}_Q$

$$\Phi^1_g(x) = \Phi_g(\widetilde{\mu}(\Sigma_g(\pi(x)), x)), \text{with } x \in M$$

$$\widetilde{\mu} : T^*Q/\Lambda \times M \quad \to \quad M$$

Lagrangian section of $\widetilde{\pi}_Q : (T^*Q/\Lambda, \widetilde{\omega}_Q) \to Q$= *section* $\Sigma_g : Q \to T^*Q/\Lambda$ *of* $\widetilde{\pi}_Q$ *such that*

$$\Sigma^*_g(\widetilde{\omega}_Q) = 0$$

$(M, Q, \pi, \omega, \phi, \Phi)$ complete G-Lagrangian fibration

a Lagrangian section $\Sigma_g : Q \to T^*Q/\Lambda$ of $\widetilde{\pi}_Q$

$$\Phi_g^\Sigma(x) = \Phi_g(\widetilde{\mu}(\Sigma_g(\pi(x)), x)), \text{with } x \in M$$

$\Sigma : G \to \Gamma(T^*Q/\Lambda)$ is a one-cocycle in $(C^\bullet(G, \Gamma(T^*Q/\Lambda)), \delta_\phi)$

- A n-cochain is a map $\Sigma : G \times \overset{n}{\ldots} \times G \to \Gamma(T^*Q/\Lambda)$
- The coboundary operator is $\delta_\phi : C^n(G, \Gamma(T^*Q/\Lambda)) \to C^{n+1}(G, \Gamma(T^*Q/\Lambda))$ given by

$$\begin{aligned} \delta_\phi\Sigma(g_1, \ldots, g_{n+1}) &= (-1)^{n+1}\Sigma(g_2, \ldots, g_{n+1}) + \\ &+ \sum_{i=1}^{n}(-1)^{n+i+1}\Sigma(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + \\ &+ (\widetilde{T^*\phi})_{g_{n+1}^{-1}} \circ \Sigma(g_1, \ldots, g_n) \circ \phi_{g_{n+1}} \end{aligned}$$

If $\Sigma : G \to \Gamma(T^*Q/\Lambda)$ a map,

$\Phi_g^\Sigma(x) = \Phi_g(\widetilde{\mu}(\Sigma_g(\pi(x)), x))$, with $x \in M$, defines an action of G on M

$\Updownarrow$

Σ is a one-cocycle in the cohomology of $(C^\bullet(G, \Gamma(T^*Q/\Lambda)), \delta_\phi)$.

The action Φ_g^Σ is also symplectic

$\Updownarrow$

Σ is a one-cocycle of the cohomology in the subcomplex $(C^\bullet(G, \Gamma_L(T^*Q/\Lambda)), \delta_\phi)$

- $(M, Q, \pi, \omega, \pi, \Phi)$ a complete G-Lagrangian fibration
- $\Sigma^1, \Sigma^2 : G \to \Gamma_L(T^*Q/\Lambda)$ are two one-cocycles with respect to $(C^\bullet(G, \Gamma_L(T^*Q/\Lambda)), \delta_\phi)$, and Φ^{Σ^1} and Φ^{Σ^2} their symplectic actions on M respectively.

There is a symplectomorphism $\widehat{F} : M \to M$ such that

$$\pi \circ \widehat{F} = \pi \quad (1)$$

$$\widehat{F} \circ \Phi_g^{\Sigma^1} = \Phi_g^{\Sigma_2} \circ \widehat{F}, \text{ for all } g \in G \quad (2)$$

$$\Updownarrow$$

$$[\Sigma^1] = [\Sigma^2] \in H^1(G, \phi, \Gamma_L(T^*Q/\Lambda))$$

In the previous part of the talk, we saw that it was necessary to suppose the additional topological completeness condition on the Lagrangian fibration. Now , we will work with infinitesimal actions in order to obtain results without these completeness condition.

An infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M is a Lie algebra antimorphism $\Psi : \mathfrak{g} \to \mathfrak{X}(M)$ with respect to the Lie brackets on $\mathfrak{g}$ and $\mathfrak{X}(M)$, that is

$$\Psi_{[\xi,\eta]} = -[\psi_\xi, \psi_\eta]$$

for all $\xi, \eta \in \mathfrak{g}$.

If ω is a symplectic structure on M, then Ψ is *symplectic* if for all $\xi \in \mathfrak{g}$, the vector field $\Psi(\xi)$ is symplectic, i.e. $\mathfrak{L}_{\Psi(\xi)}\omega = 0$.

An infinitesimal action $\Psi : \mathfrak{g} \to \mathfrak{X}(M)$ on a fibration $\pi : M \to Q$ is *projectable* if there exists an infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(Q)$ on Q such that

$$T\pi(\Psi_\xi) = \psi_\xi \circ \pi \tag{3}$$

for all $\xi \in \mathfrak{g}$.

If (M, Q, π, ω) is also a Lagrangian fibration and Ψ is symplectic and projectable , then we said that $(M, Q, \pi, \omega, \Psi, \psi)$ is a *$\mathfrak{g}$-infinitesimal Lagrangian fibration.*

Let (M, Q, π, ω) be a Lagrangian fibration and $\Psi, \widetilde{\Psi} : \mathfrak{g} \to \mathfrak{X}(M)$ be symplectic infinitesimal actions of the Lie algebra $\mathfrak{g}$ on M whose projection onto Q is $\psi : \mathfrak{g} \to \mathfrak{X}(Q)$. Then, for each $\xi \in \mathfrak{g}$, there exists a closed one-form α_ξ on Q such that

$$\widetilde{\Psi_\xi} = \Psi_\xi + X_{\pi^*(\alpha_\xi)}.$$

$$\alpha : \mathfrak{g} \to \Omega^1_c(Q)$$

$$\begin{array}{rcl} \rho_\psi : \mathfrak{g} \times \Omega^1(Q) & \to & \Omega^1(Q) \\ (\xi, \lambda) & \mapsto & \rho_\psi(\xi)(\lambda) = -\mathfrak{L}_{\psi(\xi)}\lambda \end{array}$$

In this case the cohomogy complex is defined by:

- A n-cochain is a map $\kappa_n : \mathfrak{g} \times \overset{n}{\dots} \times \mathfrak{g} \to \Omega^1(Q)$ and $C^n(\mathfrak{g}, \Omega^1(Q))$ denotes the set of the n-cochains.
- The coboundary operator is $\partial_\psi : C^n(\mathfrak{g}, \Omega^1(Q)) \to C^{n+1}(\mathfrak{g}, \Omega^1(Q))$ given by

$$\begin{array}{rcl} \partial_\psi \kappa_n(\xi_1, \dots, \xi_{n+1}) & = & \displaystyle\sum_{i=1}^{n+1} (-1)^{i+1} \rho_\psi(\xi_i)(\kappa_n(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{n+1})) \\ & + & \displaystyle\sum_{\substack{j,k=1 \\ j<k}}^{n+1} (-1)^{j+k} \kappa_n([\xi_j, \xi_k], \xi_1, \dots, \widehat{\xi_j}, \dots, \widehat{\xi_k}, \dots, \xi_{n+1}) \end{array}$$

If we consider the n-cochains with values to the closed 1-forms on Q, we have a subcomplex $(C^\bullet(\mathfrak{g}, \Omega^1_c(Q)), \partial_\psi)$ of $(C^\bullet(\mathfrak{g}, \Omega^1(Q)), \partial_\psi)$.

Let $(M, Q, \pi, , \psi, \Psi)$ be a $\mathfrak{g}$-infinitesimal Lagrangian fibration and let $\alpha : \mathfrak{g} \to \Omega^1(Q)$ be a linear map. Then,

1. The map $\Psi^\alpha : \mathfrak{g} \to \mathfrak{X}(M)$ given by

$$\Psi^\alpha_\xi = \Psi_\xi + X_{\pi^*(\alpha_\xi)} \tag{4}$$

   is an infinitesimal action of the Lie algebra $\mathfrak{g}$ on M if and only if α is a one-cocyle with respect to $(C^\bullet(\mathfrak{g}, \Omega^1(Q)), \partial_\psi)$.
2. Ψ is also symplectic if and only if α is a one-cocycle with respect to the subcomplex $(C^\bullet(\mathfrak{g}, \Omega^1_c(Q)), \partial_\psi)$.

Let $(M, Q, \pi, \omega, \psi, \Psi)$ be a $\mathfrak{g}$-infinitesimal Lagrangian fibration, $\alpha, \beta : \mathfrak{g} \to \Omega^1_c(Q)$ be two 1-cocycles with respect to $(C^n(\mathfrak{g}, \Omega^1_c(Q)), \partial_\psi)$, and Ψ^α and Ψ^β be their symplectic infinitesimal actions on (M, ω) respectively, defined by the relation (4).

$[\alpha] = [\beta] \in H^1(\mathfrak{g}, \psi, \Omega^1_c(Q))$ if and only if there exists a closed 1-form λ on Q such that the map $F : \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$F(X) = X + [X, X_{\pi^*(\lambda)}]$$

satisfies the following property

$$F(\Psi^\alpha_\xi) = \Psi^\beta_\xi \text{ for all } \xi \in \mathfrak{g}$$

Moreover, F is a symplectic map, that is, for all $X \in \mathfrak{X}(M)$, F(X) is a symplectic vector field.

$\phi : G \times Q \to Q$

- In the case of $\pi_Q : T^*Q \to Q$: The fiberwise symplectic action on T^*Q is determined by $T^*\phi : G \times \Omega^1(Q) \to \Omega^1(Q)$ and a one-cocycle $A : G \to \Omega^1_c(Q)$ for the cohomology associated with $T^*\phi : G \times \Omega^1(Q) \to \Omega^1(Q)$
- In the case of a complete Lagrangian fibration $\pi_Q : M \to Q$: If Φ is a fiberwise symplectic action on M, the other fiberwise symplectic actions on M are determined by Φ and a one-cocycle $\Sigma : G \to \Gamma_L(T^*Q/\Lambda)$ for the cohomology associated with $\widetilde{T^*\phi} : G \times \Gamma(T^*Q/\Lambda) \to \Gamma(T^*Q/\Lambda)$
- In the case of infinitesimal $\mathfrak{g}$-Lagrangian fibration $\pi_Q : M \to Q$: If Ψ is a projectable symplectic action on M, the other projectable symplectic actions on M are determined by Ψ and a one-cocycle $\alpha : G \to \Omega^1_c(Q)$ for the cohomology associated with $\rho_\psi : \mathfrak{g} \times \Omega^1(Q) \to \Omega^1(Q)$.

Thanks .... and congratulation Alberto!!!