Encoding symplectic fiberwise actions on complete Lagrangian fibrations

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Alberto as speaker wonderful talks!!!

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Alberto as speaker wonderful talks!!!

Alberto as part of the public

interesting questions and useful comments!!!

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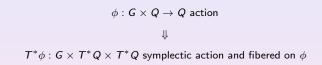
interesting questions and useful comments!!!

enjoy with you for many years!!!!

$\phi: G imes Q o Q$ action \Downarrow $T^*\phi: G imes T^*Q imes T^*Q$ symplectic action and fibered on ϕ

• How are the (symplectic) lifts of the action ϕ to the cotangent bundle T^*Q ?

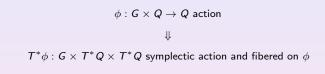
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• How are the (symplectic) lifts of the action ϕ to the cotangent bundle T^*Q ?

 $\pi: \mathcal{T}^* \mathcal{Q}
ightarrow \mathcal{Q}$ is a Lagrangian fibration

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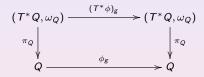


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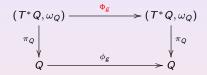
 $\pi: \mathcal{T}^*\mathcal{Q}
ightarrow \mathcal{Q}$ is a Lagrangian fibration

• How are the (symplectic) lifts of the action ϕ to a Lagrangian fibration $\pi: M \to Q$?

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Is it possible to consider other symplectic G-actions on (T^*Q, ω_Q) which fiber on ϕ ?

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$$\widehat{\uparrow}$$
$$\exists f: Q \to Q \text{ diffeomorphism such that } F = T^* f$$

F restricted to the zero section of π_Q is a diffeomorphism onto the zero section of π_Q

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 $F: T^*Q \to T^*Q$ diffeomorphism with $\pi_Q \circ F = \pi_Q$ and $F^*(\omega_Q) = \omega_Q$

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 $\label{eq:field} \begin{array}{c} \updownarrow \\ \exists f: Q \to Q \text{ diffeomorphism such that } F = T^*f \end{array}$

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 $\begin{aligned} & & \\ \exists \alpha : \mathbf{Q} \to \mathcal{T}^* \mathbf{Q} \text{ closed 1-form such that} \\ & F(\gamma_q) = \gamma_q + \alpha(q) \end{aligned}$

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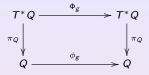
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 $\label{eq:alpha}$ $\label{eq:alpha}$ $\exists lpha: {\it Q}
ightarrow {\it T}^{*}{\it Q}$ closed 1-form such that ${\it F}(\gamma_q) = \gamma_q + lpha(q)$

 $T_{\gamma_q}(F-Id_{T^*Q})$ is null on vertical vectors $\Longrightarrow F(\gamma_q) - \gamma_q$ doesn't depend on the $\gamma_q \in T_q^*Q$

 $\alpha(q) = F(\gamma_q) - \gamma_q$

OTHER FIBERWISE COTANGENT ACTIONS



Symplectic action

$$\Phi_g \circ (\mathcal{T}^* \phi)_g^{-1} = t_{\mathcal{A}_g}, ext{ for all } g \in \mathcal{G}.$$

$$A: G \times Q \rightarrow T^*Q, \quad A: G \rightarrow \Omega^1_c(Q)$$

$$\Phi_g(\gamma_g) = (\mathcal{T}^*\phi)_g(\gamma_q) + A_g(q), ext{ for all } g \in \mathcal{G}.$$

- Φ is an affine action.
- Φ is linear if and only if Φ is the cotangent lift $T^*\phi$ of ϕ .

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The cohomology complex induced by the action ϕ

n-cochains:
$$A: G \times .^{n} . \times G \to \Omega^{1}(Q)$$

 $C^n(G, \Omega^1(Q))$ the set of the *n*-cochains.

 $\delta_\phi: C^n(\mathcal{G}, \Omega^1(\mathcal{Q}))
ightarrow C^{n+1}(\mathcal{G}, \Omega^1(\mathcal{Q}))$ is given by

$$egin{aligned} &(\delta_{\phi} A)(g_1,\ldots,g_{n+1}) &= &(-1)^{n+1} A(g_2,\ldots,g_{n+1}) + \ &+ \sum_{i=1}^n (-1)^{n+i+1} A(g_1,\ldots,g_{i-1},g_i\cdot g_{i+1},\ldots,g_{n+1}) + \ &+ \phi_{g_{n+1}}^* (A(g_1,\ldots,g_n)). \end{aligned}$$

 $(C^{\bullet}(G, \Omega^{1}_{c}(Q)), \delta_{\phi})$ is a subcomplex of $(C^{\bullet}(G, \Omega^{1}(Q)), \delta_{\phi})$

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 $A: G \to \Omega^1_c(Q)$

Φ^A: G × T^{*}Q → T^{*}Q given by Φ^A_g = t_{Ag} ∘ (T^{*}φ)_g is an action if and only if A is a one-cocycle with respect to the cohomology complex (C[•](G, Ω¹(Q)), δ_φ).

$$A_{gh}(q) = A_h(q) + \phi_h^*(A_g(q)), \qquad orall \, g, h \in G \, \, ext{and} \, \, orall \, q \in Q.$$

Φ^A is also symplectic if and only if A is a one-cocycle with respect to the cohomology subcomplex (C•(G, Ω¹_c(Q)), δ_φ).

$$(\Phi_g^A)^*(\omega_Q) = \omega_Q - \pi_Q^*(dA_g).$$

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If $A, B : G \to \Omega^1_c(Q)$ two one-cocycles with respect to $(C^{\bullet}(G, \Omega^1_c(Q)), \delta_{\phi})$,

\Downarrow

 $\exists F: T^*Q \rightarrow T^*Q$ a symplectomorphism such that

•
$$\pi_Q \circ F = \pi_Q$$

• $F \circ \Phi_g^A = \Phi_g^B \circ F$, for all $g \in G$
 $(A] = [B] \in H^1(G, \phi, \Omega_c^1(Q))$

A is a coboundary of $(C^{\bullet}(G, \Omega_c^1(Q)), \delta_{\phi})$, there is a symplectomorphim which is fibered for π_Q and equivariant with respect to the actions Φ^A and $T^*\phi$.

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$$\pi_Q: T^*Q o Q \quad \pi_Q^{-1}(q) = T_q^*Q$$
 Lagrangian space with respect ω_Q

Lagrangian fibration

A fiber bundle $\pi:(M,\omega)\to Q$ with $\pi^{-1}(q)$ Lagrangian submanifold of M, for all $q\in Q$

ker
$$T_x \pi = (\ker T_x \pi)^{\omega}$$
, for all $x \in \pi^{-1}(q)$

$$(\ker T_x\pi)^{\omega} = \{v \in T_xM/\omega(x)(v, u) = 0 \text{ for all } u \in \ker T_x\pi\}$$

• $\pi: (M, \omega) \to Q$ a Lagrangian fibration

•
$$\alpha \in \Omega^1(Q) \Rightarrow X_{\pi^*\alpha} \in \mathfrak{X}(M)$$

$$i_{X_{\pi^*\alpha}}\omega = \pi^*\alpha$$

Complete Lagrangian fibration (M, Q, π, ω)

 $X_{\pi^*\alpha}$ is complete for all $\alpha \in \Omega^1(Q)$

• μ is an action: $\mu(\alpha + \beta, x) = \mu(\alpha, \mu(\beta, x))$ • μ is fibered: $\pi(\mu(\alpha, x)) = \pi(x)$

• μ induces a transitive action of the abelian group T_q^*Q on $\pi^{-1}(q)$

$$\mu_q: T_q^*Q \times \pi^{-1}(q) \to \pi^{-1}(q) \quad (\alpha_q, x) \mapsto \mu(\alpha, x)$$

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$$egin{aligned} \mu_q : \ T^*_q Q imes \pi^{-1}(q) o \pi^{-1}(q) \ & (lpha_q, x) \mapsto \mu(lpha, x) \end{aligned}$$

In general, μ_q is not free

$$\begin{split} \Lambda_q = \{ \alpha_q \in \mathcal{T}_q^* Q \mid \mu_q(\alpha_q, x) = x \quad \forall x \in \pi^{-1}(q) \} \text{ discrete space} \\ & \Downarrow \end{split}$$

 $(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q)$ is a complete Lagrangian fibration

• $\widetilde{\pi}_Q$: $T^*Q/\Lambda \to Q$ fibration

- $\tilde{\omega}_Q$ the induced symplectic structure on T^*Q/Λ
- $\widetilde{\mu}$: $T_q^* Q / \Lambda_q \times M \to M$ is a free action

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$(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q)$ Jacobian Lagrangian fibration

associated to the complete Lagrangian fibration (M, Q, π, ω)

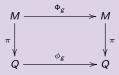
 $\sigma: \mathcal{Q} \to \mathcal{M}$ section of $\pi: \mathcal{M} \to \mathcal{Q}$

 $(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q) \cong (M, Q, \pi, \omega)$

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An G-action on a Lagrangian fibration

A Lagrangian fibration (M, Q, π, ω) , G a Lie group, $g \in G$



Φ is symplectic

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 $(M, Q, \pi, \omega, \phi, \Phi)$ is a G-Lagrangian fibration

Examples

$$(T^*Q, Q, \pi_Q, \omega_Q, \phi, T^*\phi), \quad (T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q, \phi, \widetilde{T^*\phi})$$

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 $(M, Q, \pi, \omega, \phi, \Phi)$ complete G-Lagrangian fibration

How is the rest of symplectic fiberwise actions on π

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 $(M, Q, \pi, \omega, \phi, \Phi)$ complete G-Lagrangian fibration $(T^*Q/\Lambda, Q, \widetilde{\pi}_Q, \widetilde{\omega}_Q)$ Jacobian Lagrangian fibration

 Φ^1 : $G \times M \to M$ is a fiberwise symplectic action

↓

for all $g \in G$, there exists a Lagrangian section $\Sigma_g : Q \to T^*Q/\Lambda$ of $\widetilde{\pi}_Q$

 $\Phi_g^1(x) = \Phi_g(\widetilde{\mu}(\Sigma_g(\pi(x)), x)), \text{ with } x \in M$

 $\widetilde{\mu}: T^*Q/\Lambda \times M \rightarrow M$

Lagrangian section of $\tilde{\pi}_Q$: $(T^*Q/\Lambda, \tilde{\omega}_Q) \to Q =$ section $\Sigma_g : Q \to T^*Q/\Lambda$ of $\tilde{\pi}_Q$ such that

$$\Sigma_g^*(\widetilde{\omega}_Q) = 0$$

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 $(M, Q, \pi, \omega, \phi, \Phi)$ complete G-Lagrangian fibration

a Lagrangian section $\Sigma_g: Q
ightarrow T^*Q/\Lambda$ of $\widetilde{\pi}_Q$

$$\Phi_g^{\Sigma}(x) = \Phi_g(\widetilde{\mu}(\Sigma_g(\pi(x)), x)), \text{ with } x \in M$$

 $\Sigma: G \to \Gamma(T^*Q/\Lambda)$ is a one-cocycle in $(C^{\bullet}(G, \Gamma(T^*Q/\Lambda)), \delta_{\phi})$

• A *n*-cochain is a map $\Sigma : G \times .^n . \times G \to \Gamma(T^*Q/\Lambda)$

• The coboundary operator is $\delta_{\phi} : C^n(G, \Gamma(T^*Q/\Lambda)) \to C^{n+1}(G, \Gamma(T^*Q/\Lambda))$ given by

$$\begin{split} \delta_{\phi} \Sigma(g_1, \dots, g_{n+1}) &= (-1)^{n+1} \Sigma(g_2, \dots, g_{n+1}) + \\ &+ \sum_{i=1}^n (-1)^{n+i+1} \Sigma(g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_{n+1}) + \\ &+ (\widetilde{T^*\phi})_{g_{n+1}^{-1}} \circ \Sigma(g_1, \dots, g_n) \circ \phi_{g_{n+1}} \end{split}$$

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If Σ : $G \to \Gamma(T^*Q/\Lambda)$ a map,

 $\Phi_g^{\Sigma}(x) = \Phi_g(\widetilde{\mu}(\Sigma_g(\pi(x)), x))$, with $x \in M$, defines an action of G on M

$\$

 Σ is a one-cocycle in the cohomology of $(C^{\bullet}(G, \Gamma(T^*Q/\Lambda)), \delta_{\phi})$.

The action Φ_{g}^{Σ} is also symplectic

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 Σ is a one-cocycle of the cohomology in the subcomplex $(C^{\bullet}(G, \Gamma_L(T^*Q/\Lambda)), \delta_{\phi})$

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- $(M, Q, \pi, \omega, \pi, \Phi)$ a complete G-Lagrangian fibration
- $\Sigma^1, \Sigma^2 : G \to \Gamma_L(T^*Q/\Lambda)$ are two one-cocycles with respect to $(C^{\bullet}(G, \Gamma_L(T^*Q/\Lambda)), \delta_{\phi})$, and Φ^{Σ^1} and Φ^{Σ^2} their symplectic actions on M respectively.

There is a symplectomorphism $\widehat{F}: M \to M$ such that

$$\pi \circ \widehat{F} = \pi \tag{1}$$

$$\widehat{F} \circ \Phi_g^{\Sigma^1} = \Phi_g^{\Sigma_2} \circ \widehat{F}, \text{ for all } g \in G$$
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$$[\Sigma^1] = [\Sigma^2] \in H^1(G, \phi, \Gamma_L(T^*Q/\Lambda))$$

In the previous part of the talk, we saw that it was necessary to suppose the additional topological completeness condition on the Lagrangian fibration. Now , we will work with infinitesimal actions in order to obtain results without these completeness condition.

An infinitesimal action of a Lie algebra \mathfrak{g} on a manifold M is a Lie algebra antimorphism $\Psi : \mathfrak{g} \to \mathfrak{X}(M)$ with respect to the Lie brackets on \mathfrak{g} and $\mathfrak{X}(M)$, that is

$$\Psi_{[\xi,\eta]} = -[\psi_{\xi},\psi_{\eta}]$$

for all $\xi, \eta \in \mathfrak{g}$.

If ω is a symplectic structure on M, then Ψ is *symplectic* if for all $\xi \in \mathfrak{g}$, the vector field $\Psi(\xi)$ is symplectic, i.e. $\mathfrak{L}_{\Psi(\xi)}\omega = 0$.

An infinitesimal action $\Psi : \mathfrak{g} \to \mathfrak{X}(M)$ on a fibration $\pi : M \to Q$ is *projectable* if there exists an infinitesimal action $\psi : \mathfrak{g} \to \mathfrak{X}(Q)$ on Q such that

$$T\pi(\Psi_{\xi}) = \psi_{\xi} \circ \pi \tag{3}$$

for all $\xi \in \mathfrak{g}$.

If (M, Q, π, ω) is also a Lagrangian fibration and Ψ is symplectic and projectable , then we said that $(M, Q, \pi, \omega, \Psi, \psi)$ is a g-infinitesimal Lagrangian fibration.

Let (M, Q, π, ω) be a Lagrangian fibration and $\Psi, \widetilde{\Psi} : \mathfrak{g} \to \mathfrak{X}(M)$ be symplectic infinitesimal actions of the Lie algebra \mathfrak{g} on M whose projection onto Q is $\psi : \mathfrak{g} \to \mathfrak{X}(Q)$. Then, for each $\xi \in \mathfrak{g}$, there exists a closed one-form α_{ξ} on Q such that

$$\widetilde{\Psi_{\xi}} = \Psi_{\xi} + X_{\pi^*(\alpha_{\xi})}.$$

 $\alpha:\mathfrak{g}\to\Omega^1_c(Q)$

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$$egin{array}{rcl}
ho_\psi: \mathfrak{g} imes \Omega^1(Q) & o & \Omega^1(Q) \ (\xi,\lambda) & \mapsto &
ho_\psi(\xi)(\lambda) = -\mathfrak{L}_{\psi(\xi)}\lambda \end{array}$$

In this case the cohomogy complex is defined by:

- A n-cochain is a map κ_n : g× .ⁿ. ×g → Ω¹(Q) and Cⁿ(g, Ω¹(Q)) denotes the set of the n-cochains.
- The coboundary operator is $\partial_{\psi}: C^n(\mathfrak{g}, \Omega^1(Q)) \to C^{n+1}(\mathfrak{g}, \Omega^1(Q))$ given by

$$\partial_{\psi} \kappa_{n}(\xi_{1}, \dots, \xi_{n+1}) = \sum_{\substack{i=1 \\ j < k}}^{n+1} (-1)^{i+1} \rho_{\psi}(\xi_{i}) (\kappa_{n}(\xi_{1}, \dots, \widehat{\xi}_{i}, \dots, \xi_{n+1})) \\ + \sum_{\substack{j,k=1 \\ j < k}}^{n+1} (-1)^{j+k} \kappa_{n}([\xi_{j}, \xi_{k}], \xi_{1}, \dots, \widehat{\xi}_{j}, \dots, \widehat{\xi}_{k}, \dots, \xi_{n+1})$$

If we consider the *n*-cochains with values to the closed 1-forms on Q, we have a subcomplex $(C^{\bullet}(\mathfrak{g}, \Omega_c^1(Q)), \partial_{\psi})$ of $(C^{\bullet}(\mathfrak{g}, \Omega^1(Q)), \partial_{\psi})$.

Let $(M, Q, \pi, , \psi, \Psi)$ be a g-infinitesimal Lagrangian fibration and let $\alpha : \mathfrak{g} \to \Omega^1(Q)$ be a linear map. Then,

1 The map $\Psi^{\alpha} : \mathfrak{g} \to \mathfrak{X}(M)$ given by

$$\Psi_{\xi}^{\alpha} = \Psi_{\xi} + X_{\pi^*(\alpha_{\xi})} \tag{4}$$

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is an infinitesimal action of the Lie algebra \mathfrak{g} on M if and only if α is a one-cocyle with respect to $(\mathcal{C}^{\bullet}(\mathfrak{g}, \Omega^{1}(Q)), \partial_{\psi})$.

W is also symplectic if and only if α is a one-cocycle with respect to the subcomplex (C[•](g, Ω¹_c(Q)), ∂_ψ). Let $(M, Q, \pi, \omega, \psi, \Psi)$ be a g-infinitesimal Lagrangian fibration, $\alpha, \beta : \mathfrak{g} \to \Omega^1_c(Q)$ be two 1-cocycles with respect to $(C^n(\mathfrak{g}, \Omega^1_c(Q)), \partial_{\psi})$, and Ψ^{α} and Ψ^{β} be their symplectic infinitesimal actions on (M, ω) respectively, defined by the relation (4).

 $[\alpha] = [\beta] \in H^1(\mathfrak{g}, \psi, \Omega^1_c(Q)) \text{ if and only if there exists a closed 1-form } \lambda \text{ on } Q \text{ such that the map } F : \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$F(X) = X + [X, X_{\pi^*(\lambda)}]$$

satisfies the following property

$${\sf F}(\Psi^lpha_\xi)=\Psi^eta_\xi$$
 for all $\xi\in {\mathfrak g}$

Moreover, F is a symplectic map, that is, for all $X \in \mathfrak{X}(M)$, F(X) is a symplectic vector field.

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 $\phi: \mathbf{G} \times \mathbf{Q} \rightarrow \mathbf{Q}$

- In the case of $\pi_Q : T^*Q \to Q$: The fiberwise symplectic action on T^*Q is determined by $T^*\phi : G \times \Omega^1(Q) \to \Omega^1(Q)$ and a one-cocycle $A : G \to \Omega^1_c(Q)$ for the cohomology associated with $T^*\phi : G \times \Omega^1(Q) \to \Omega^1(Q)$
- In the case of a complete Lagrangian fibration π_Q : M → Q: If Φ is a fiberwise symplectic action on M, the other fiberwise symplectic actions on M are determined by Φ and a one-cocycle Σ : G → Γ_L(T*Q/Λ) for the cohomology associated with T*φ : G × Γ(T*Q/Λ) → Γ(T*Q/Λ)
- In the case of infinitesimal g-Lagrangian fibration $\pi_Q : M \to Q$: If Ψ is a projectable symplectic action on M, the other projectable symplectic actions on M are determined by Ψ and a one-cocycle $\alpha : G \to \Omega_c^1(Q)$ for the cohomology associated with $\rho_{\psi} : \mathfrak{g} \times \Omega^1(Q) \to \Omega^1(Q)$.

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Thanks and congratulation Alberto!!!



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