From Classical Trajetories to Quantum Commutation Relations

"We would like to avoid the dry kind of approach:

Let M be a manifold and X a vector field on M

Where this manifold comes from,
how a vector field happens to exist on it
what the relation of this vector field is
to the observations of an experimentalist

A centuries old observation :

"Now the path of investigation must lie from what is more immediately cognizable and clear to us, to what is cleorer and more intimately cognizable in its own nature ...

So we must advance from the coucrete data when we have analysed them ...

So we must advance from the coucrete whole to the several constituents which it embraces ..."

Aristotle , Physics, 184a
Translation of Ph Wicksteed

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Geometry from Dynamics, Classical and Quantum

Springer

Structure of Classical Mechanics

- Experimental Data : Trajectories (Parametrized)
- Differential Equations , Implicit versus Explicit
- Description in terms of a "potential function" : Lagrangian description
- Phase space , a universal model for regular Lagrangians
- The Inverse Problem : Lagrangian and Hamiltonian

From Classical to Quantum

Symplectic Abelian Vector Group : Weyl

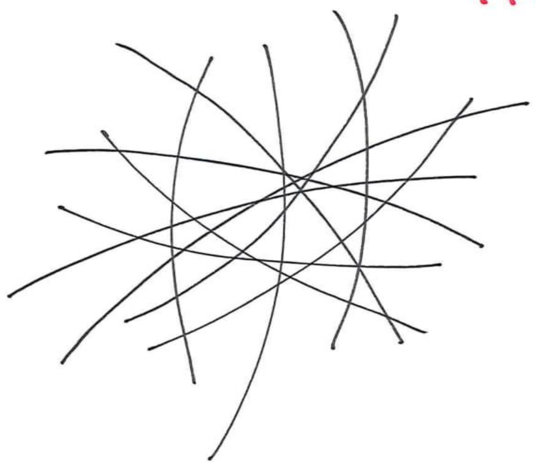
Hilbert spaces and Unitary trausformations

Weyl algebra : Group algebra

Alternative descriptions

Second Quantization

Trajectories and "configuration spaces"



$S \equiv \{ \gamma : I \to Q \}$ $\quad I \subset \mathbb{R}$ interval

Build

$tS \equiv \{ t\gamma : I \to TQ \}$

$t\gamma \equiv (\gamma(t), \dot{\gamma}(t))$

$t(tS) \equiv \{ t(t\gamma) : I \to T(TQ) \}$

$t(t\gamma) : I \longrightarrow T(TQ)$

$t(t\gamma) \equiv (\gamma(t), \dot{\gamma}(t), \dot{\gamma}(t), \ddot{\gamma}(t))$

tS defines $M \subset TQ$

$t(tS)$ defines $\Sigma \subset T(TQ)$

Regularity conditions

Σ image of $\Gamma : TQ \to T(TQ)$

second order vector field

Implicit equations

$$\dim \Sigma > \dim TQ$$

$\dim M < \dim TQ$ $\quad M \subset TQ$

Nonholonomic constraints

(degenerate Lagrangians)

"Potential Functions"

$$\frac{dq_j}{dt} = v_j \qquad \frac{d}{dt} v_j = F_j$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Euler-Lagrange equations

Inverse problem

$$\frac{\partial^2 \mathcal{L}}{\partial v_j \partial v_k} F_k = \frac{\partial \mathcal{L}}{\partial q_j} - \frac{\partial^2 \mathcal{L}}{\partial v_j \partial q_r} v_r$$

A PDE for $\mathcal{L}$

"Potential functions": Hamiltonian

$$\frac{d}{dt} q_j = v_j = \{H, q_j\}$$

$$\frac{d}{dt} v_j = \{H, \{H, q_j\}\} = F_j$$

Poisson structure and H on TQ

Collective coordinates

$$\frac{d}{dt} x_j = \Lambda_{jk} \frac{\partial H}{\partial x_k}$$

Requiring $\{q_j, q_k\} = 0$

The inverse problem for Hamiltonians and Lagrangians are equivalent (locally)

Uniqueness and existence

Math Helmholtz, Kasner, Douglas

Phys Wigner, Feynman, Bergmann, Haves, Currie-Saletan

Bihamiltonian systems and complete integrability

(F. Magri)

Basic conceptual aspects : linear systems

Consider the inverse problem for the Hamiltonian description of linear systems

Replace vector fields and tensor fields with matrices

$$A = \Gamma = \Lambda \cdot H$$

"factorization probeem" for a matrix A into the product of a skew-symmetric matrix Λ and a symmetric H

Proposition :
For a generic A , necessary and sufficient conditions

$$\mathrm{Tr}(A)^{2k+1} = 0$$

Uniqueness

if T is an invertible matrix, $[T, A] = 0$,

$$A = (T \wedge T^t) \cdot ((T^t)^{-1} H \, T^{-1})$$

is an alternative decomposition if T is not a canonical transformation

Prop.

$$T_\lambda = e^{\lambda A^2}$$

are noncanonical transformations

A remarkabele result

A stable linear Hamiltonian vector field preserves a Kählerian structure

The flow is unitary

it represents a quantum-like system

Few comments

$A = | A_j^{\ k} |$ $\qquad \Lambda = | \Lambda_{jk} |$ $\qquad H = | H^{jk} |$

we may define

$A_j^{\ k}\, dx_k \otimes \frac{\partial}{\partial x_j}$ $\qquad \Lambda = \Lambda_{jk} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$ $\qquad H = dx_j \otimes dx_k\, H^{jk}$

or, by contracting with $\Delta = x_j \frac{\partial}{\partial x_j}$.

$A_j^{\ k}\, x_k \frac{\partial}{\partial x_j}$, $\quad \Lambda = \Lambda_{jk} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$ $\qquad H = H^{jk} x_j x_k$

When H positive definite, the polar decomposition of $A = |A| J$ defines a complex structure

(polar decomposition with respect to H)

Nonuniqueness : consequences

- On the relation between symmetries and conservation laws

- Alternative linear structures

$\omega_{\mathcal{L}} = d\left(\frac{\partial \mathcal{L}}{\partial v_j} dq_j\right)$ Define

$i_{X_j}\omega_{\mathcal{L}} = dq_j$ $\quad i_{Y_j}\omega_{\mathcal{L}} = d\left(\frac{\partial \mathcal{L}}{\partial v_j}\right)$

$\Rightarrow \quad [X_j, Y_k] = 0 \;, \quad [X_j, X_k] = 0 \;, \quad [Y_j, Y_k] = 0$

They integrate (if complete) to an action of an Abelian Vector Group

$\Rightarrow$ alternative linear structures on the same carrier space

Poisson Brackets and Jordan Brackets

Linear structure $\Delta = x_j \frac{\partial}{\partial x_j}$

$\Lambda = \Lambda_{jk} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$ $\quad R = R_{jk} \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_k}$ $\quad R_{jk} = R_{kj}$

Λ and R define a complex structure J, $J^2 = -\mathbb{1}$.

Observables:
Functions whose Hamiltonian vector fields preserve R
They define a Lie algebra under Poisson Brackets
Those which satisfy $L_\Delta f = 2f$ close on a Lie-Jordan algebra by means of Λ and R
By a complex extension we get a C^*-algebra

Alternative (Λ, R) for the same dynamics define alternative C^*-algebras

As a cousequeuce of the non-uniqueness of the factorization $\Gamma = \Lambda \cdot H$

we get : Alternative Hermitiau products

Alternative C^*-Algebras

wheuever H is positive definite

The flow associated with Γ is

Unitary for alternative Hermitian products

Automorphism for alternative C^*-algebras

Standard approach : Weyl Systems

Abelian, symplectic vector group (V, ω)

$W : V \to \mathcal{U}(\mathcal{H})$

$$W(v_1)\, W(v_2)\, W^{\dagger}(v_1)\, W^{\dagger}(v_2) = e^{i\omega(v_1, v_2)}\, \mathbb{1}$$

Weyl algebra : Group algebra of Weyl operators

Alternative Linear structures $\Rightarrow$ alternative
Weyl Systems
Weyl algebras

Simple example : (q, p) in $\mathbb{R}^2$ $\qquad Q = q(1+q^2+p^2)\ , \ P = p(1+q^2+p^2)$

(q, p) linear structure $\qquad \Delta = q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p}$

(Q, P) $\qquad \Delta = Q\frac{\partial}{\partial Q} + P\frac{\partial}{\partial P}$

Weyl systems over a Hilbert space : Second Quantization

$\mathcal{H} \equiv \mathcal{L}^2(Q)$ $\quad$ Q a Lagrangian subspace of the Abelian Vector group

$$W(\psi_1)\,W(\psi_2)\,W^+(\psi_1)\,W^+(\psi_2) = e^{im\langle\psi_1|\psi_2\rangle}\,\mathbb{1}$$

Alternative ambiguities with respect to the usual ones

Recent times

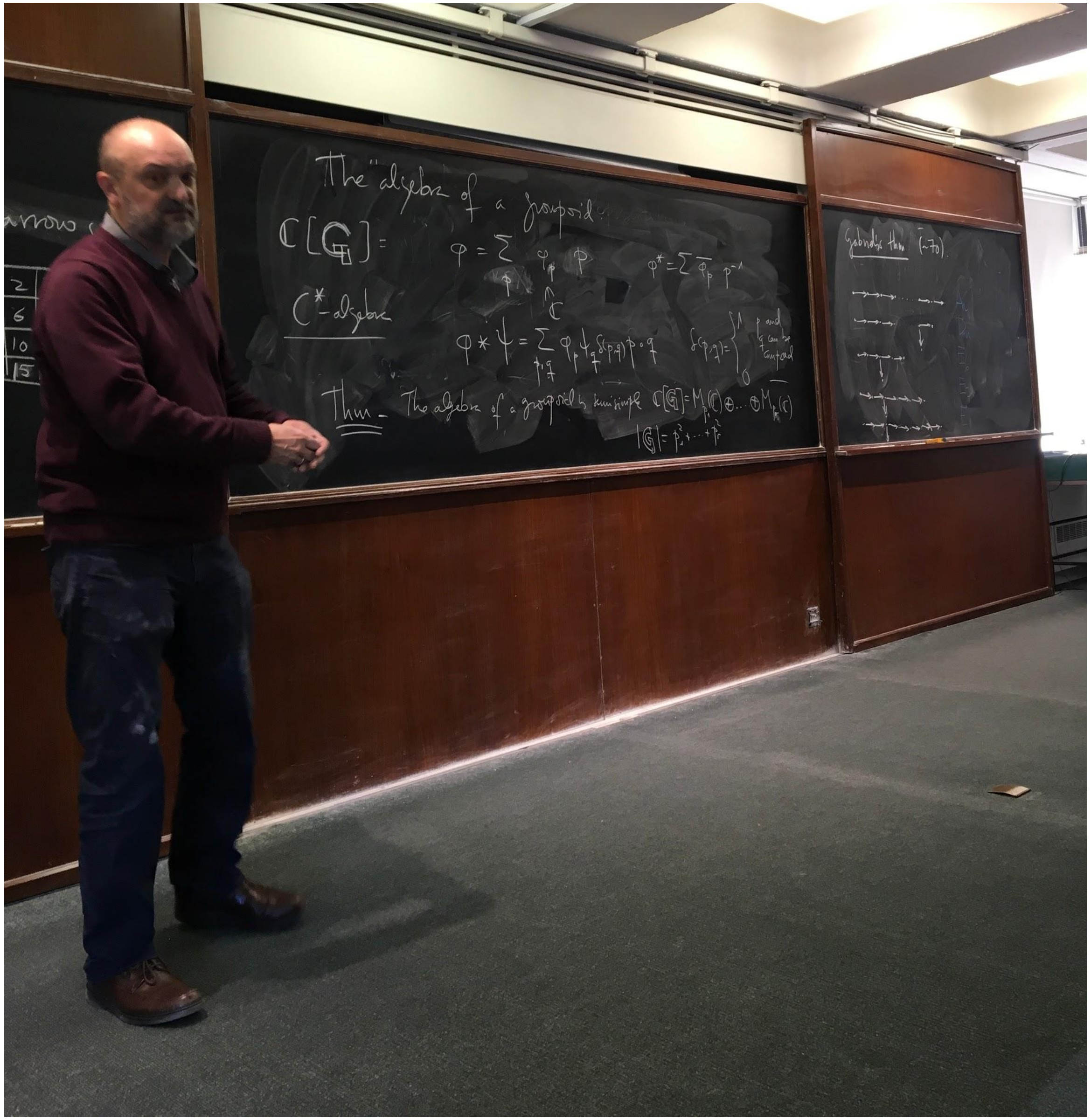
Tomography for Field Theories

Peierls Brackets, Jacobi Brackets for Field Theories

Schwinger Picture of Quantum Theories

Non-Markovian dynamics on quantum states

Quantum Information Geometry


The "algebra" of a groupoid
$\mathbb{C}[\mathbb{G}] =$
$\varphi = \sum \varphi_p \, p$
$\varphi^* = \sum \overline{\varphi_p} \, p^{-1}$
C*-algebra
$\varphi * \psi = \sum_{p,q} \varphi_p \psi_q \delta(p,q) \, p \circ q$
$\delta(p,q) = \begin{cases} 1 & p \text{ and } q \text{ can be composed} \\ 0 \end{cases}$
Thm. The algebra of a groupoid is semisimple $\mathbb{C}[\mathbb{G}] = M_{p_1}(\mathbb{C}) \oplus \dots \oplus M_{p_r}(\mathbb{C})$
$|\mathbb{G}| = p_1^2 + \dots + p_r^2$