Quantum Suprematism with Triadas of Malevich's Squares identifying spin (qubit) states; new entropic inequalities

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60 Years Alberto Ibort Fest Classical and Quantum Physics: Geometry, Dynamics and Control

Madrid, 5-9 March, 2018

Abstract

For arbitrary N-level atom states, the density matrix elements are expressed in terms of a set of probability distributions describing the set of "classical coin" positions. For one qubit, the states are identified with three probability distributions [1] and illustrated by three squares on the plane called "Triada of Malevich's Squares" [2].

Using this approach, called the quantum suprematism picture, new entropic inequalities are obtained for density matrix elements of qudits and N-level atom states. Arbitrary quantum observables are bijectively mapped on the set of classical random variables, and formulas for quantum statistics of the observables are expressed in terms of classical-like statistics of random variables.

[1] V.I. Man'ko, G. Marmo, F. Ventriglia, and Vitale, *J. Phys A: Math Theor.*, **50**, 335302 (2017)

[2] V.N.Chernega, O.V.Man'ko, and V.I.Man'ko, *J. Russ. Laser Res.* **38**, 141-149, 234-333, 416-425 (2017)

The aim of the talk is to discuss the possibility to describe the quantum states by the fair probability distributions and quantum observables by classiacal-like random variables. This aim is coherent with tomographic description of quantum states [3] [4].

[3] A. Ibort, V.I. Man'ko, G. Marmo, A. Simoni, and F. Ventriglia, "An introduction to the tomographic picture of quantum mechanics." *Phys. Scr.* **79**, 065013 (2009).

[4] M. Asorey, A. Ibort, G. Marmo, and F Ventriglia, "Quantum Tomography twenty years later." *Phys. Scr.* ***90**, 074031 (2015).*

Quantum suprematism and Malevich squares[5]

$$\sum_{j=1}^{3}(p_j - 1/2)^2 \leq 1/4$$

$$w(m, \mathbf{n}) = (u\rho u^\dagger)_{mm}$$

[5] A. Shatskikh, *Black Square: Malevich and the Origin of Suprematism*. Yale University Press, New Haven (2012).

Probability distribution for three classical coins

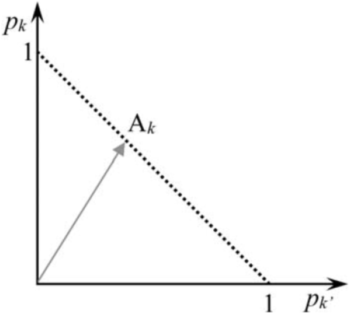


Fig. 1. The probability vector $\mathbf{p}_k$ with the end at point A_k on the simplex line determined by the equality $p_k + p_{k'} = 1$.

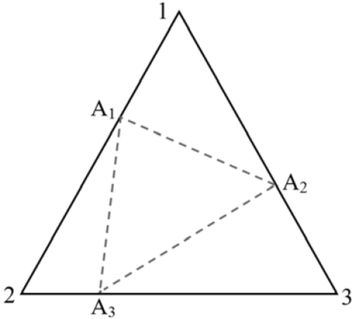
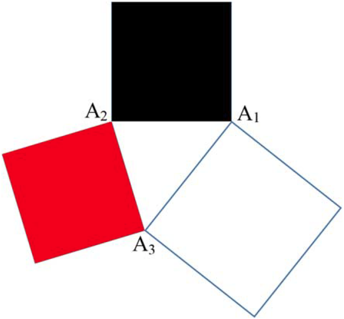


Fig. 2. The equilateral triangle with vertices 1, 2, and 3 and side length $\sqrt{2}$ and vertices A_1, A_2, and A_3 determining the qubit state.

Triada of Malevich's squares determined by the triangle $A_1A_2A_3$ [6]



$$
\begin{aligned}
y_1 &= (2 + 2p_1^2 - 4p_1 - 2p_2 + 2p_2^2 + 2p_1p_2)^{1/2} \\
y_2 &= (2 + 2p_2^2 - 4p_2 - 2p_3 + 2p_3^2 + 2p_2p_3)^{1/2} \\
y_3 &= (2 + 2p_3^1 - 4p_3 - 2p_1 + 2p_1^2 + 2p_3p_1)^{1/2}
\end{aligned}
$$

[6] V. N. Chernega, O. V. Man'ko, and V. I. Man'ko *J. Russ. Laser Res.* **38**, 141 (2017)

Density matrix for spin-1/2 state

$$\rho = \begin{pmatrix} p_3 & p_1 - ip_2 - (1/2) + (i/2) \\ p_1 + ip_2 - (1/2) - (i/2) & 1 - p_3 \end{pmatrix}$$

$$S = 2\left[3\,(1 - p_1 - p_2 - p_3) + 2p_1^2 + 2p_2^2 + 2p_3^2 + p_1p_2 + p_2p_3 + p_3p_1\right]$$

$$S_{min} \leq S \leq S_{max}$$

$S = 3/2$ — for maximally mixed state

Triangle area in terms of probabilities p_1, p_2, p_3

$$y_k = (2 + 2p_k^2 - 4p_k - 2p_{k+1} + 2p_{k+1}^2 + 2p_k p_{k+1})^{1/2}$$

$$S_{\mathrm{tr}} = \frac{1}{4}\left[(y_1 + y_2 + y_3)(y_1 + y_2 - y_3)(y_2 + y_3 - y_1)(y_3 + y_1 - y_2)\right]^{1/2}$$

$S_{tr} = \sqrt{3}/2$ for classical case

$S_{tr} = \sqrt{3}/8$ for maximally mixed case

Qutrit as two ququarts

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$$

$$\rho(1) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & 0 \\ \rho_{21} & \rho_{22} & \rho_{23} & 0 \\ \rho_{31} & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{11} & \rho_{12} & \rho_{13} \\ 0 & \rho_{21} & \rho_{22} & \rho_{23} \\ 0 & \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$$

$$R(1) = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix}, \quad R(2) = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix},$$

$$R(3) = \begin{pmatrix} \rho_{11} & \rho_{13} \\ \rho_{31} & \rho_{33} + \rho_{22} \end{pmatrix}, \quad R(4) = \begin{pmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{11} + \rho_{33} \end{pmatrix}.$$

$$R(k) = \begin{pmatrix} p_3^{(k)} & (p_1^{(k)} - \frac{1}{2}) - i(p_2^{(k)} - \frac{1}{2}) \\ (p_1^{(k)} - \frac{1}{2}) + i(p_2^{(k)} - \frac{1}{2}) & 1 - p_3^{(k)} \end{pmatrix},$$

$k = 1, 2, 3, 4$

$$\rho_{33} = 1 - p_3^{(1)},$$
$$\rho_{22} = 1 - p_3^{(2)} = p_3^{(4)},$$
$$\rho_{11} = p_3^{(1)} + p_3^{(2)} - 1,$$
$$\rho_{21} = (p_1^{(2)} - \frac{1}{2}) + i(p_2^{(2)} - \frac{1}{2}),$$
$$\rho_{31} = (p_1^{(1)} - \frac{1}{2}) + i(p_2^{(1)} - \frac{1}{2}),$$
$$\rho_{32} = (p_1^{(4)} - \frac{1}{2}) + i(p_2^{(4)} - \frac{1}{2}).$$

Coin probabilities for qutrit

$$p_1^{(1)} = p_1^{(3)},\ p_2^{(1)} = p_2^{(3)},\ p_3^{(4)} = 1 - p_3^{(2)},\ p_3^{(1)} = p_3^{(3)} + p_3^{(4)}$$

$$\rho = \begin{pmatrix} p_3^{(1)} + p_3^{(2)} - 1 & (p_1^{(2)} - \frac{1}{2}) - i(p_2^{(2)} - \frac{1}{2}) & (p_1^{(1)} - \frac{1}{2}) - i(p_2^{(1)} - \frac{1}{2}) \\ (p_1^{(2)} - \frac{1}{2}) + i(p_2^{(2)} - \frac{1}{2}) & 1 - p_3^{(2)} & (p_1^{(4)} - \frac{1}{2}) - i(p_2^{(4)} - \frac{1}{2}) \\ (p_1^{(1)} - \frac{1}{2}) + i(p_2^{(1)} - \frac{1}{2}) & (p_1^{(4)} - \frac{1}{2}) + i(p_2^{(4)} - \frac{1}{2}) & 1 - p_3^{(1)} \end{pmatrix}$$

Probabilities $\rho_j^{(k)}$, $j = 1, 2, 3,\ k = 1, 2, 3, 4$ satisfy the inequalities

$\sum_{j=1}^{3} \left(p_j^{(k)} - \frac{1}{2} \right)^2 \leq \frac{1}{4}, \quad k = 1, 2, 3, 4.$

$$p_3^{(1)} = p_3^{(33)}, \quad p_3^{(2)} = p_3^{(22)}, \quad p_1^{(1)} = p_1^{(31)}, \quad p_2^{(1)} = p_2^{(31)},$$

$$p_1^{(2)} = p_1^{(21)}, \quad p_2^{(2)} = p_2^{(21)}, \quad p_1^{(4)} = p_1^{(32)}, \quad p_2^{(4)} = p_2^{(32)}.$$

$$\rho_{jk} = (p_1^{(jk)} - \frac{1}{2}) + i(p_2^{(jk)} - \frac{1}{2}), \quad j > k$$

$$\rho_{jj} = 1 - p_3^{(jj)}, \quad j \geq 2,$$

$$\rho_{11} = 1 - \sum_{j=2}^{3} \rho_{jj}.$$

Density matrix of qutrit in terms of coin probabilities

$$\rho = \begin{pmatrix} p_3^{(33)} + p_3^{(22)} - 1 & (p_1^{(21)} - \frac{1}{2}) - i(p_2^{(21)} - \frac{1}{2}) & (p_1^{(31)} - \frac{1}{2}) - i(p_2^{(31)} - \frac{1}{2}) \\ (p_1^{(21)} - \frac{1}{2}) + i(p_2^{(21)} - \frac{1}{2}) & 1 - p_3^{(22)} & (p_1^{(32)} - \frac{1}{2}) - i(p_2^{(32)} - \frac{1}{2}) \\ (p_1^{(31)} - \frac{1}{2}) + i(p_2^{(31)} - \frac{1}{2}) & (p_1^{(32)} - \frac{1}{2}) + i(p_2^{(32)} - \frac{1}{2}) & 1 - p_3^{(33)} \end{pmatrix}$$

$$\rho(1) = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix}.$$

Density matrix of ququart in terms of coin probabilities

$$\rho = \begin{pmatrix} p_3^{(44)} + p_3^{(22)} + p_3^{(33)} - 2 & (p_1^{(21)} - \frac{1}{2}) - i(p_2^{(21)} - \frac{1}{2}) & (p_1^{(31)} - \frac{1}{2}) - i(p_2^{(31)} - \frac{1}{2}) & (p_1^{(41)} - \frac{1}{2}) - i(p_2^{(41)} - \frac{1}{2}) \\ (p_1^{(21)} - \frac{1}{2}) + i(p_2^{(21)} - \frac{1}{2}) & 1 - p_3^{(22)} & (p_1^{(32)} - \frac{1}{2}) - i(p_2^{(32)} - \frac{1}{2}) & (p_1^{(42)} - \frac{1}{2}) - i(p_2^{(42)} - \frac{1}{2}) \\ (p_1^{(31)} - \frac{1}{2}) + i(p_2^{(31)} - \frac{1}{2}) & (p_1^{(32)} - \frac{1}{2}) + i(p_2^{(32)} - \frac{1}{2}) & 1 - p_3^{(33)} & (p_1^{(43)} - \frac{1}{2}) - i(p_2^{(43)} - \frac{1}{2}) \\ (p_1^{(41)} - \frac{1}{2}) + i(p_2^{(41)} - \frac{1}{2}) & (p_1^{(42)} - \frac{1}{2}) + i(p_2^{(42)} - \frac{1}{2}) & (p_1^{(43)} - \frac{1}{2}) + i(p_2^{(43)} - \frac{1}{2}) & 1 - p_3^{(44)} \end{pmatrix}$$

New entropic inequalities

$$\mathrm{Re}\,\rho_{jk} + \frac{1}{2} \geq 0, \quad \mathrm{Im}\,\rho_{jk} \leq \frac{1}{2}$$

$$\left(\frac{1}{2} - \mathrm{Im}\,\rho_{jk}\right) \ln\left[\frac{\left(\frac{1}{2} - \mathrm{Im}\,\rho_{jk}\right)}{\left(\frac{1}{2} - \mathrm{Im}\,\rho_{j'k'}\right)}\right] + \left(\frac{1}{2} + \mathrm{Im}\,\rho_{jk}\right) \ln\left[\frac{\left(\frac{1}{2} + \mathrm{Im}\,\rho_{jk}\right)}{\left(\frac{1}{2} + \mathrm{Im}\,\rho_{j'k'}\right)}\right] \geq 0$$

$$\rho_{jj} \ln\left[\frac{\rho_{jj}}{\left(\frac{1}{2} \mp \mathrm{Im}\,\rho_{j'k}\right)}\right] + (1 - \rho_{jj}) \ln\left[\frac{(1 - \rho_{jj})}{\left(\frac{1}{2} \pm \mathrm{Im}\,\rho_{j'k}\right)}\right] \geq 0.$$

$$\ln 2 \geq -\left(\frac{1}{2} \mp \mathrm{Im}\rho_{jk}\right) \ln\left(\frac{1}{2} \mp \mathrm{Im}\rho_{jk}\right) - \left(\frac{1}{2} \pm \mathrm{Im}\rho_{jk}\right) \ln\left(\frac{1}{2} \pm \mathrm{Im}\rho_{jk}\right) \geq 0$$

Quantum observables as set of classical random variables

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \qquad H^{\dagger} = H$$

$$H_{11} = z_1, \qquad H_{22} = z_2$$

$$H_{12} = x - iy, \qquad H_{21} = x + iy$$

Dichotomic random variables and coin probabilities

$$\vec{X} = \begin{pmatrix} x \\ -x \end{pmatrix}, \qquad \vec{Y} = \begin{pmatrix} y \\ -y \end{pmatrix}, \qquad \vec{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\vec{\mathcal{P}}_1 = \begin{pmatrix} p_1 \\ 1 - p_1 \end{pmatrix}, \qquad \vec{\mathcal{P}}_2 = \begin{pmatrix} p_2 \\ 1 - p_2 \end{pmatrix}, \qquad \vec{\mathcal{P}}_3 = \begin{pmatrix} p_3 \\ 1 - p_3 \end{pmatrix}$$

Dichotomic random variables and coin probabilities in quantum mechanics

$$H = \begin{pmatrix} z_1 & x - iy \\ x + iy & z_2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} p_3 & p_1 - \frac{1}{2} - i(p_2 \frac{1}{2}) \\ p_1 - \frac{1}{2} + i(p_2 \frac{1}{2}) & 1 - p_3 \end{pmatrix}$$

Quanum means in terms of classical means

$$\langle H \rangle = Tr(H\rho) =$$
$$= p_1 x + (1-p_1)(-x) + p_2 y + (1-p_2)(-y) + p_3 z_1 + (1-p_3) z_2 =$$
$$= \langle \vec{X} \rangle + \langle \vec{Y} \rangle + \langle \vec{Z} \rangle$$

Superposition principle of two states

$$|\psi_1\rangle = \begin{pmatrix} \sqrt{p_3} \\ \frac{p_1 - \frac{1}{2}}{\sqrt{p_3}} + \frac{i(p_2 - \frac{1}{2})}{\sqrt{p_3}} \end{pmatrix}$$

$$|\psi_2\rangle = \begin{pmatrix} \sqrt{\mathcal{P}_3} \\ \frac{\mathcal{P}_1 - \frac{1}{2}}{\sqrt{\mathcal{P}_3}} + \frac{i(\mathcal{P}_2 - \frac{1}{2})}{\sqrt{\mathcal{P}_3}} \end{pmatrix}$$

$$\langle\psi_1|\,\psi_2\rangle = 0$$

State vector of superposed state

$$|\psi\rangle = \sqrt{\Pi_3}\,|\psi_1\rangle + \sqrt{1-\Pi_3}\,e^{i\alpha}\,|\psi_2\rangle = \begin{pmatrix} \sqrt{\pi_3} \\ \frac{\pi_1 - \frac{1}{2}}{\sqrt{\pi_3}} + \frac{i(\pi_2 - \frac{1}{2})}{\sqrt{\pi_3}} \end{pmatrix}$$

$$\cos(\alpha) = \frac{\Pi_1 - \frac{1}{2}}{\sqrt{\Pi_3(1-\Pi_3)}}, \qquad \sin(\alpha) = \frac{\Pi_2 - \frac{1}{2}}{\sqrt{\Pi_3(1-\Pi_3)}}$$

Classical coins "interference"

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \underset{\vec{\Pi}}{\oplus} \begin{pmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \\ \mathcal{P}_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

Born rule:

$$Tr\rho_1\rho_2 = 2 + 2[p_3\mathcal{P}_3 + p_1\mathcal{P}_1 + p_2\mathcal{P}_2] - p_1 - \mathcal{P}_1 - p_2 - \mathcal{P}_2 - p_3 - \mathcal{P}_3$$

$$\pi_3 = \frac{1}{\mathcal{T}}\left\{\Pi_3 p_3 + (1-\Pi_3)\mathcal{P}_3 + 2\sqrt{p_3\mathcal{P}_3}\,(\Pi_1 - 1/2)\right\}$$

$$\begin{aligned}
\pi_1 - 1/2 = \; & \frac{1}{\mathcal{T}}\{\Pi_3(p_1 - 1/2) + (\mathcal{P}_1 - 1/2)(1-\Pi_3) + \\
& + [(\Pi_1 - 1/2)(p_1 - 1/2) + (\Pi_2 - 1/2)(p_2 - 1/2)]\sqrt{\frac{\mathcal{P}_3}{p_3}} + \\
& + [(\Pi_1 - 1/2)(\mathcal{P}_1 - 1/2) - (\Pi_2 - 1/2)(\mathcal{P}_2 - 1/2)]\sqrt{\frac{p_3}{\mathcal{P}_3}}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\pi_2 - 1/2 = \; & \frac{1}{\mathcal{T}}\{[(p_2 - 1/2)\Pi_3 + (\mathcal{P}_2 - 1/2)(1-\Pi_3)] + \\
& + \sqrt{\frac{\mathcal{P}_3}{p_3}}[(\Pi_1 - 1/2)(p_2 - 1/2) - (\Pi_2 - 1/2)(p_1 - 1/2)] + \\
& \sqrt{\frac{p_3}{\mathcal{P}_3}}[(\Pi_2 - 1/2)(\mathcal{P}_1 - 1/2) + (\Pi_1 - 1/2)(\mathcal{P}_2 - 1/2)]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T} = 1 + \frac{2}{\sqrt{p_3\mathcal{P}_3}}\{(\Pi_1 - 1/2)\,[(p_1 - 1/2)(\mathcal{P}_1 - 1/2) + (\mathcal{P}_2 - 1/2)(p_2 - 1/2) + p_3\mathcal{P}_3] + \\
+ \; (\Pi_2 - 1/2)\,[(p_2 - 1/2)(\mathcal{P}_1 - 1/2) - (p_1 - 1/2)(\mathcal{P}_2 - 1/2)]\}
\end{aligned}$$

Conclusion

1. States (density matrices) are interpreted as set of probability distributions describing positions of coins.
2. Observables — matrix elements, e.g. of Hamiltonians can be interpreted as classical dichotomic random variables corresponding to playing coins.
3. Quantum statistics — means of quantum observables, other moments can be expressed in terms of the coin probabilities and dichotomic random variables.
4. Superposition principle for quantum states is formulated as nonlinear addition rule of coin probability distributions.
5. Born rule also is formulated as another addition rule of coin probabilities which provides the transition probability between the states.
6.The qubit states can be mapped on triada of Malevich's squares and any qudit states can be mapped onto set of such triadas.
We illustrate obtained relations by the statement connected with the discussion of Bohr and Einstein "God does not play dice — God plays coins"

Happy Birthday!