Amenability notions around Roe algebras

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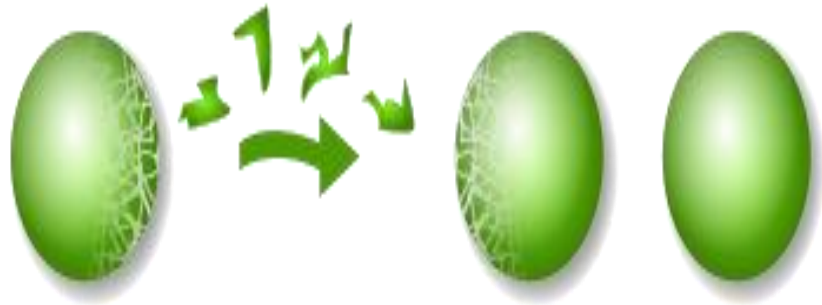
- Overview
  1. Introduction: Amenability and paradoxical decompositions in groups.
  2. Amenability and paradoxicality for discrete metric spaces.
  3. Amenability and paradoxical decompositions in algebra.
  4. Følner sequences for operators and Følner C*-algebras.
  5. Roe C*-algebras as an unifying picture.

In collaboration with with Pere Ara (UAB), Kang Li (U.Münster) and Jianchao Wu (Penn State)

1. Introduction: Amenability in discrete groups

Paradoxical decomposition of an "orange" B:



Pic: B.D. Esham

Reasons:

- The free group on two generators acts on B: $\mathbb{F}_2 \leq SO(3) \curvearrowright B$
- $\mathbb{F}_2 = \langle a, b, a^{-1}, b^{-1} \rangle$ is itself paradoxical.
  Denote $W(a)$ are **reduced** words beginning with a. Then
  - $\mathbb{F}_2 = \{e\} \sqcup W(a) \sqcup W(b) \sqcup W(a^{-1}) \sqcup W(b^{-1})$.

    $\mathbb{F}_2 = W(a) \sqcup aW(a^{-1}) = W(b) \sqcup bW(b^{-1})$.
- The paradoxicality of $\mathbb{F}_2$ induces (+axiom of choice) the paradoxical decomposition of B.
- The resolution of this apparent paradox is the theorem by Tarski:
  There is no finitely additive probability measure which is $\mathbb{F}_2$-invariant.

Amenability for discrete finitely generated groups:

Von Neumann ('29) realized that $\mathbb{F}_2$ lacks to have the property of amenability!

- Recall: discrete group Γ is **amenable** if $\ell_\infty(\Gamma)$ has a Γ-**invariant state** (i.e., a positive, normalized, Γ-invariant functional $\psi\colon \ell_\infty(\Gamma) \to \mathbb{C}$).

An alternative approach to this circle of ideas:

A **Følner net for** Γ is a net of non-empty finite subsets $\Gamma_i \subset \Gamma$ such that

$$\lim_i \frac{|(\gamma\Gamma_i)\triangle\Gamma_i|}{|\Gamma_i|} = 0 \quad \text{for all} \quad \gamma \in \Gamma\,,$$

where $\triangle$ is the symmetric difference and $|\Gamma_i|$ is the cardinality of Γ_i.
If the net is increasing and $\Gamma = \cup_i \Gamma_i$ it is called a **proper Følner net**.

- Every finite group F has a Følner sequence !
  - Just take the constant sequence $\Gamma_n = F$, $n \in \mathbb{N}$.

Theorem

Γ is amenable iff Γ is NOT paradoxical iff Γ has a Følner net.

Summary:

What properties do Følner nets have ?

- Følner nets provide an "inner" approximation of the group Γ via **finite** subsets Γ_i.
- The finite sets Γ_i grow "moderately" with respect to multiplication. Asymptotically

$$|\gamma\Gamma_i| \quad \text{is "small" compared with} \quad |\Gamma_i|$$

  The **dynamics** (group multiplication) is central to the analysis.
- In the context of groups given a Følner sequence one can construct another **proper** Følner sequence.

Følner nets are the "bridge" to address amenability issues beyond groups!

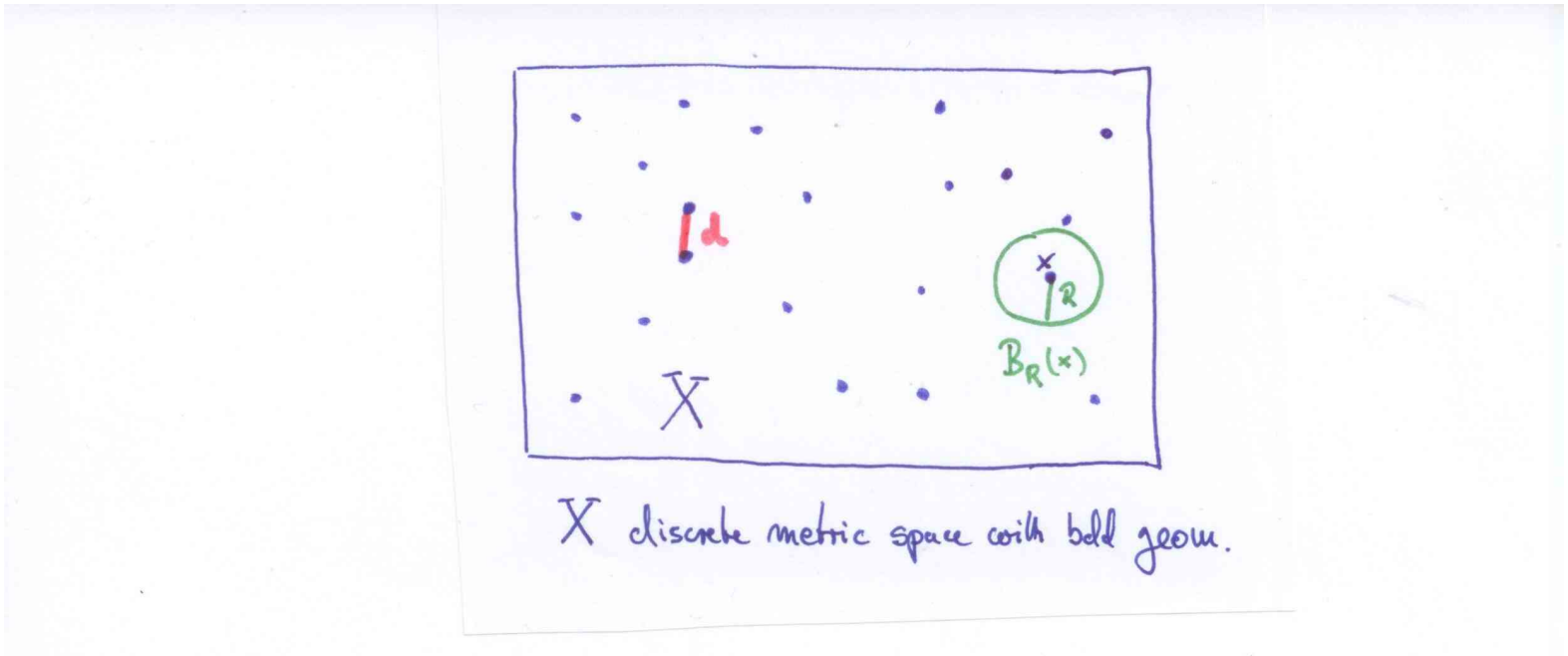
- Amenable structures are "reasonable" (i.e., not paradoxical) extensions of finite structures.

2. Amenability for metric spaces

Let (X, d) be a discrete metric space with bounded geometry:

- Uniformly discrete: $\inf\{d(x, y) \mid x, y \in X\} \geq \mathrm{d} > 0$.
- Uniformly locally finite: for any radius $R > 0$, $\sup_{x \in X} |B_R(x)| < \infty$.

Example: Γ a finitely generated discrete group with the word length metric is a metric space with bounded geometry.

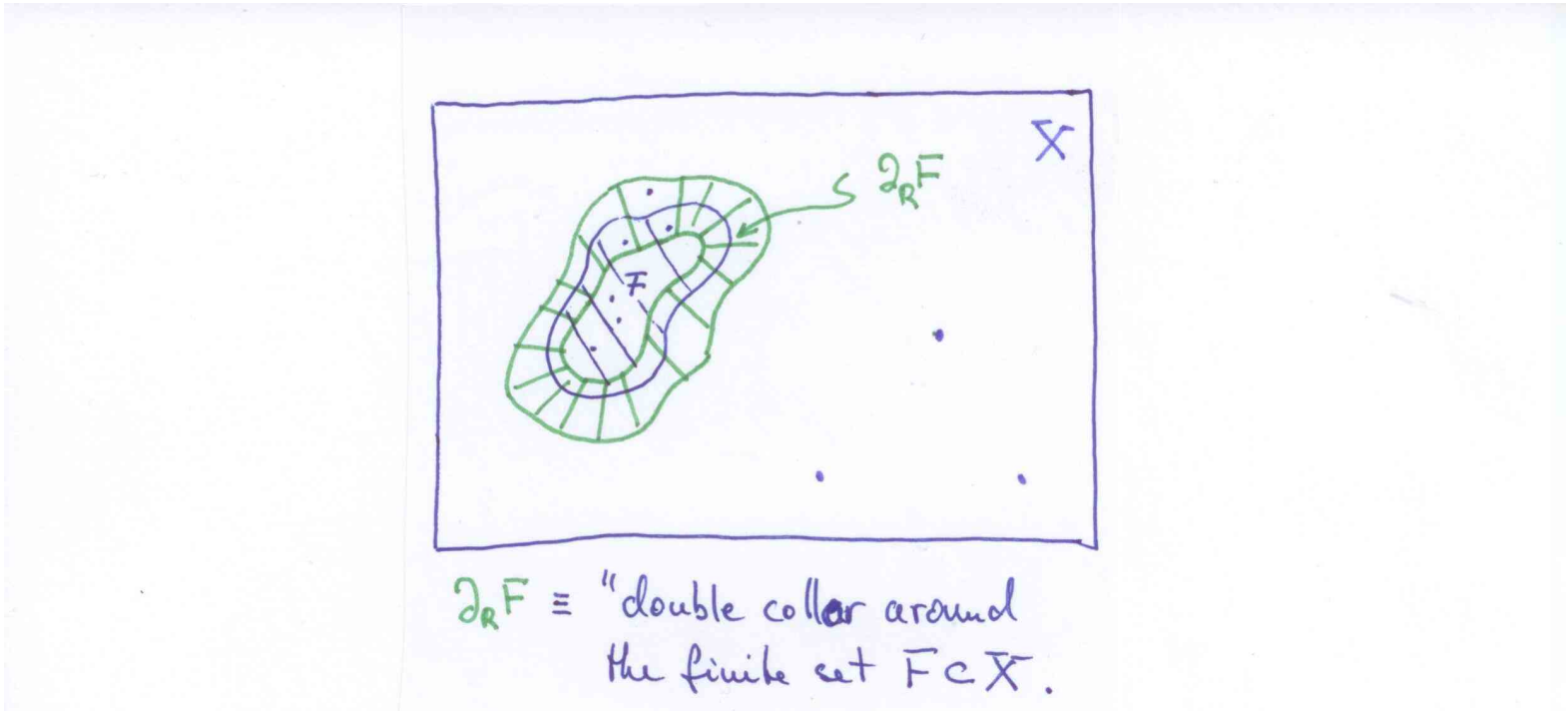


Definition (Block-Weinberger '92)

(X, d) *is* **amenable** *if there exists a* **Følner sequence** $\{F_n\} \subset X$ *of finite, non-empty subsets of* X *such that*

$$\lim_{n\to\infty} \frac{|\partial_R F_n|}{|F_n|} = 0 \,, \quad R > 0 \,,$$

where $\partial_R F$ *is the "double collar" around the boundary of* F*.*
The Følner sequence is **proper** *if it is increasing and* $X = \cup_n F_n$*. In this case we call the space* **properly amenable**.



Examples:

- If $|X| < \infty$, then (X, d) is amenable. Take $F_n = X$ so that

$$\frac{|\partial_R F_n|}{|F_n|} = \frac{|\partial_R X|}{|X|} = 0 \, .$$

- Γ is amenable as a group iff Γ is amenable as a metric space (with the word length metric).

- As in the group case: (X, d) is amenable iff it is properly amenable.

  - To see a difference between amenable and proper amenable generalize to extended metric spaces (i.e., $d \colon X \times X \to \mathbb{R} \cup \{\infty\}$) and analyze the structure of coarse connected components.

    Example: Consider $X = Y_1 \sqcup Y_2$, with $|Y_1| < \infty$, Y_2 non-amenable and $d(Y_1, Y_2) = \infty$. Then X is amenable (take the constant sequence $F_n = Y_1$), but not properly amenable.

What is a paradoxical in this context ? What dynamics ?

Definition

Let (X, d) a metric space with bounded geometry. A **partial translation on** *X is a triple (A, B, t), where $A, B \subset X$, $t\colon A \to B$ is a bijection with*

$$\sup_{a \in A} \{d(a, t(a))\} < \infty \, .$$

X is **paradoxical** *if there exists a partition $X = X_1 \sqcup X_2$ and partial translations $t_i\colon X \to X_i$, $i = 1, 2$. The set of all partial translations is an inverse semigroup.*

Theorem (Grigorchuk, Ceccherini et al., '99)

Let (X, d) a metric space with bounded geometry. TFAE

- *(X, d) is amenable.*
- *X has NO paradoxical decompositions.*
- *There exists a finitely additive probability measure μ on $\mathcal{P}(X)$ invariant under partial translations (i.e., $\mu(A) = \mu(B)$.)*

3. Amenability and paradoxical decompositions in algebra

To address questions of amenability in the context of algebra:

- Need to give up the cardinality $|\cdot|$ to measure sizes.
- Take finite-dimensional subspaces as approximation and $\dim(\cdot)$ to measure the size of the subspaces.
- For today: $\mathcal{A}$ is a unital $\mathbb{C}$-algebra, but everything works also for any commutative field $\mathbb{K}$.

Definition (Gromov '99)

A unital algebra $\mathcal{A}$ is **amenable** *if there is a* **Følner net** $\{W_i\}_{i\in I}$ *of non-zero finite dimensional subspaces such that*

$$\lim_i \frac{\dim(aW_i + W_i)}{\dim(W_i)} = 1\,, \quad a \in \mathcal{A}\,.$$

If the W_i are exhausting, then $\mathcal{A}$ is **properly algebraically amenable** .

- The presence of a linear structure makes the difference amenable vs. proper amenable an essential point.

Examples:

- Any matrix algebra $\mathcal{A} = M_k(\mathbb{C})$ is amenable. Take a constant sequence $W_n = \mathcal{A}$ and note that $\mathcal{A} \subset a\mathcal{A} + \mathcal{A} \subset \mathcal{A}$, $a \in \mathcal{A}$, hence

$$\frac{\dim(a\mathcal{A} + \mathcal{A})}{\dim(\mathcal{A})} = 1 \, .$$

- Γ is a discrete group and $\mathbb{C}\Gamma$ is its group algebra.
  Then Γ is amenable iff $\mathbb{C}\Gamma$ is algebraically amenable. [Bartholdi '08]

Algebraic amenable vs. proper algebraic amenable:

- If $\mathcal{I} \lhd_L \mathcal{A}$ is a left ideal with $\dim \mathcal{I} < \infty$, then $\mathcal{A}$ is always algebraically amenable. Take a constant sequence $W_n = \mathcal{I}$.
- Note that if $\mathcal{A}$ is NON algebraically amenable, then $\tilde{\mathcal{A}} = \mathcal{I} \oplus \mathcal{A}$ is amenable but NOT properly amenable.

What is a paradoxical decomposition in this context ?

Definition (Elek '03)

A countalby dimensional algebra $\mathcal{A}$ is paradoxical if for any basis $\mathcal{B} = \{f_n\}_n$ of $\mathcal{A}$ one has:

- *$\exists$ partitions of the basis: $\mathcal{B} = L_1 \sqcup \cdots \sqcup L_n = R_1 \sqcup \cdots \sqcup R_k$*
- *$\exists$ elements of the alg. $\mathcal{A}$: $a_1, \ldots, a_n$ and $b_1, \ldots, b_k \in \mathcal{A}$,*

such that $a_1 L_1 \cup \cdots \cup a_n L_n \cup b_1 R_1 \cup \cdots \cup b_k R_k$ are linearly independent in $\mathcal{A}$.

Theorem (Elek '03, Ara,Li,Ll.,Wu '17)

$\mathcal{A}$ is algebraically amenable iff it is NOT paradoxical.

- Elek proved the equivalence in the context of countably generated algebras $\mathcal{A}$ without zero-divisors.
  - He used as definition proper algebraic amenability.
- Having operators and Roe algebras in mind this is too restricitve.
  - One has to work with algebraic amenability.
  - Countable generation is also not needed.
- In this more general context algebraic amenability is also equivalent to the existence of a dimension measure on the lattice of subspaces of $\mathcal{A}$:

  $\mu\colon W \to [0,1]$, $W \leq \mathcal{A}$, suitable normalization, additivity and invariance.

4. Følner nets for operators and Følner C*-algebras

Standing assumptions and notation in this talk:

- Spaces:
  - $\mathcal{H}$ denotes a complex (typically ∞-dimensional) separable Hilbert space.
- Operators:
  - $\mathcal{B}(\mathcal{H})$ set of all linear, bounded operators on $\mathcal{H}$.
  - Projections in $\mathcal{B}(\mathcal{H})$: $P^2 = P = P^*$ and $\mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ is set of non-zero finite rank projections.

Quasidiagonality and Følner sequences for families of operators:

Definition (Connes '76, Halmos '68)

- *Let $\mathcal{T} \subset \mathcal{B}(\mathcal{H})$. A net $\{P_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ of finite-rank projections is called a* **Følner net for** $\mathcal{T}$ *if*

$$\lim_i \frac{\|TP_i - P_iT\|_2}{\|P_i\|_2} = 0, \quad \text{for all} \quad T \in \mathcal{T}\,, \qquad (*)$$

  where $\|\cdot\|_2$ is the Hilbert-Schmidt norm.
  - *If the sequence satisfying (*) is increasing and P_i converges strongly to $\mathbb{1}$, then we say that it is a* **proper Følner net for** $\mathcal{T}$.
- $\mathcal{T} \subset \mathcal{B}(\mathcal{H})$ *(countable) is a* **quasidiagonal set of operators**, *if there exists an increasing sequence of finite-rank projections $\{P_n\}_{n\in\mathbb{N}}$, $P_n \nearrow \mathbb{1}$ strongly, s.t. $\lim_n \|TP_n - P_nT\| = 0\,, \quad T \in \mathcal{T}\,.$*

Remarks:

- Quasidiagonality $\Rightarrow$ Følner.
- $\mathcal{T}$ can be a single operator ($\mathcal{T} = \{T\}$) or a concrete C*-algebra.
- Any matrix has a Følner sequence. Take $P_n = \mathbb{1}_{\mathtt{n}\times\mathtt{n}}$.
- Which operators have/have not Følner sequences? (Yakubovich, Ll. '13).

First consequences:

Proposition

If $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for a unital C-algebra $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$ iff $\mathcal{A}$ has an amenable trace τ, i.e. the trace τ on $\mathcal{A}$ extends to a state ψ on $\mathcal{B}(\mathcal{H})$ that is centralised by $\mathcal{A}$, i.e.*

$$\psi\restriction\mathcal{A}=\tau \quad \text{and} \quad \psi(XA)=\psi(AX)\,, \quad X\in\mathcal{B}(\mathcal{H})\,,\ A\in\mathcal{A}\,.$$

Remark

- *The state ψ (called hypertrace) is the alg. analogue of the invariant mean used in the context of groups. Take the net of states*

$$\psi_i(X):=\frac{\mathrm{Tr}(P_iX)}{\mathrm{Tr}(P_i)}\,, \quad X\in\mathcal{B}(\mathcal{H})$$

  whose cluster points define amenable traces. ($\{P_i\}_i$ Følner net.)
- *Useful notion as an obstruction to the existence of Følner sequences! It is the property to approach an abstract characterization of these algebras.*

What is the intrinsic characterization of these notions?

Critic: the definitions of quasidiagonality and Følner sequences for operators require, e.g., a concrete C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$.
Voiculescu's approach to quasidiagonality:

Definition

An (abstract) unital C-algebra $\mathcal{A}$ is called* quasidiagonal *if there exists a net of unital completely positive (u.c.p.) maps $\varphi_i : \mathcal{A} \to M_{k(i)}(\mathbb{C})$ which is both asymptotically multiplicative and asymptotically isometric, i.e.,*

- $\|\varphi_i(AB) - \varphi_i(A)\varphi_i(B)\| \to 0$ *for all* $A, B \in \mathcal{A}$.
- $\|A\| = \lim_{n\to\infty} \|\varphi_i(A)\|$ *for all* $A \in \mathcal{A}$.

In the context of Følner sequences:

Definition (Ara, Ll. '14)

Let $\mathcal{A}$ be a unital C-algebra.*

We say that $\mathcal{A}$ is a Følner C*-algebra *if there exists a net of u.c.p. maps $\varphi_i : \mathcal{A} \to M_{k(i)}(\mathbb{C})$ such that*

$$\lim_i \|\varphi_i(AB) - \varphi_i(A)\varphi_i(B)\|_{2,\mathrm{tr}} = 0\,, \quad A, B \in \mathcal{A}\,, \qquad (*)$$

where $\|F\|_{2,\mathrm{tr}} := \sqrt{\mathrm{tr}(F^*F)}$, $F \in M_n(\mathbb{C})$ and $\mathrm{tr}(\cdot)$ is the unique tracial state on the matrix algebra $M_n(\mathbb{C})$.

Theorem (Ara,Ll. '14)

Let $\mathcal{A}$ be a unital C-algebra. T.F.A.E.:* *(selection)*

(i) *$\mathcal{A}$ is a Følner C*-algebra.*

(ii) *Every faithful representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has an amenable trace.*

(iii) *Every faithful essential representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has a proper Følner net.*

- Bédos uses the name "weakly hypertracial" ('95) instead of "Følner C*-algebra".
- How can we get the Følner projections ?
  Stinespring $\Rightarrow \varphi(a) = V^*\pi(a)V$ for some representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}')$ and isometry $V\colon \mathbb{C}^k \to \mathcal{H}'$. Følner projections appear as Stinespring's projections $P = VV^*$.

Is there any notion in operator algebras reflecting paradoxicality ?

A possibly capturing some aspects of paradoxicality is the following notion. Let $\mathcal{A}$ be a unital C*-algebra. It is called **properly infinite** if

- there exist isometries $V_1, V_2 \in \mathcal{A}$ satisfying
  - $V_1^* V_1 = V_2^* V_2 = \mathbb{1}$, $V_1^* V_2 = 0$ and $V_1 V_1^* + V_2 V_2^* \leq \mathbb{1}$.
- Idea: V_1 and V_2 map the Hilbert space $\mathcal{H}$ isometrically onto two mutually orthogonal subspaces.

5. Roe C*-Algebras

- We have addressed issues around amenability and paradoxical decompositions in very different mathematical situations:
  - Groups Γ.
  - Metric spaces (X, d).
  - $\mathbb{C}$-algebras $\mathcal{A}$.
  - Operator algebras $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, i.p., C*-algebras.

- We will use Roe algebras to give a unified picture of the different approaches of amenability we have presented.
- These algebras were introduced to proof an index theorem for elliptic operators on non-compact manifolds $\mathcal{M}$. The idea is to look at the coarse structure of $\mathcal{M}$ captured by a discrete space (X, d) and define a C*-algebra $\mathcal{R}(X)$ to define the analytical part of the index.

Construction of Roe C*-algebra

Let (X, d) be a metric space with bounded geometry.

1. Hilbert space: $\mathcal{H} = \ell_2(X)$ with canonical ONB: $\{\delta_x \mid x \in X\}$.
2. Operators: $T \in \mathcal{B}(\ell_2(X))$ and $T \cong (T_{xy})_{x,y \in X}$.
3. Propagation of operators: For any operator $T \in \mathcal{B}(\ell_2(X))$ define

$$p(T) := \sup\{d(x, y) \mid T_{xy} \neq 0\} .$$

   - Examples:
     - If $F \subset X$ and Q_F is the characteristic function of F, then $p(Q_F) = 0$.
     - If $X = \mathbb{N}$, $\mathcal{H} = \ell_2$ and $S(\delta_n) = \delta_{n+1}$ the unilateral shift, then $p(S) = 1$.
       If and $S_+(\delta_n) := \delta_{2n+1}$ (a generator of the Cuntz algebra), then $p(S_+) = \infty$.
     - The laplacian Δ on a discrete graph has $p(\Delta) = 1$.
4. Translation algebra: $\mathcal{R}_0(X) := \bigcup_{R>0} \{T \in \mathcal{B}(\ell_2(X)) \mid p(T) \leq R\}$
   (Operators with bounded propagation).
5. Roe C*-algebra: $\mathcal{R}(X) := \overline{\mathcal{R}_0(X)}$.

Remarks:

- Roe C*-algebras provide a natural link

  metric spaces $\leftrightarrow$ algebra $\leftrightarrow$ operators/operator algebras

  and are fundamental objects for coarse geometry.
- If $X = \Gamma$ (discrete fin. generated group), then $\mathcal{R}(\Gamma) = \ell_\infty(\Gamma) \rtimes \Gamma$.
- Partial translations (A, B, t) in $X \leftrightarrow$ partial isometries in $\mathcal{R}_0(X)$ via

$$t \mapsto T\,, \quad T_{yx} := \begin{cases} 1\,, & \text{if } (x,y) \in \mathrm{gra}(t) \\ 0\,, & \text{otherwise} \end{cases}$$

  with $T^*T = P_A$ and $TT^* = P_B$.

Theorem (Ara,Li,Ll.,Wu '18)

(X, d) a discrete metric space with bdd geometry. TFAE: *(selection)*

1. *(X, d) is amenable.*
2. *The translation algebra $\mathcal{R}_0(X)$ is algebraically amenable.*
3. *The Roe C*-algebra $\mathcal{R}(X)$ is a Følner C*-algebra.*
4. *The Roe C*-algebra $\mathcal{R}(X)$ is not properly infinite.*

Some ideas to the proof:

- (X, d) amenable $\Rightarrow$ $\mathcal{R}(X)$ is Følner C*-alg:
  (Use local version of amenability)

  1. Take $T \in \mathcal{R}_0(X)$ with $p(T) \leq R$. For $\varepsilon > 0$ there is an finite $F \subset X$ such that $|\partial_R(F)| \leq \varepsilon |F|$.
  2. Consider the projection Q_F and note that $\|Q_F\|_2^2 = |F|$.
  3. $\|[T, Q_F]\|_2^2 = \sum_{x \in F, y \notin F} |\langle T\delta_x, \delta_y \rangle|^2 +$ sym
     $$\leq \sum_{y \in \partial_R(F)} \left(\|T\delta_y\|^2 + \|T^*\delta_y\|^2 \right) \leq 2\|T\| |\partial_R(F)|.$$

- (X, d) amenable $\Rightarrow$ $\mathcal{R}_0(X)$ is algebraically amenable:

  1. Note that $p(Q_F) = 0$. Take as Følner subspaces
     $$W = Q_F \mathcal{R}_0(X) Q_F \subset \mathcal{R}_0(X) \, !$$
  2. Use $\dim(W) = |F|$.

6. Summary and outlook

- Roe algebras provide a very nice frame, where **amenability**, i.e., having nice **finite dimensional approximations** with **reasonable dynamics** fit together

  groups $\leftrightarrow$ metric spaces $\leftrightarrow$ algebra $\leftrightarrow$ C*-algebras

What else ?

- How about this equivalence in more degenerate dynamics ?
  E.g., semigroups where the dynamics can drastically shrink the set $|sF| \ll |F|$.
- How do notions of amenability aspects enter mathematical physics ?

  In QFT **properly infinite** operator algebras are ubiquitous.
  - Construction of the field algebra out of the observables and the DHR-selection principle $\rightsquigarrow$ proper infinity !
  - Recall, e.g., that it is a fact of nature that the von Neumann algebra associated to quantum fields in certain space-time regions of four dimensional Minkowski space are **hyperfinite** factors of type III_1, $\rightsquigarrow$ proper infinity !


Alberto
Happy
Birthday
Alberto
Felizidades Alberto
Propio
$\mathcal{L}_g(A(G)) = A(\mathcal{L}_g G)$
$QP\_PQ = i \hbar \mathbb{1}$
$\pi = (PI)$