Overview

1. Introduction: Amenability and paradoxical decompositions in groups.
2. Amenability and paradoxicality for discrete metric spaces.
3. Amenability and paradoxical decompositions in algebra.
4. Følner sequences for operators and Følner C*-algebras.
5. Roe C*-algebras as an unifying picture.

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1. Introduction: Amenability in discrete groups

Paradoxical decomposition of an “orange” $B$:

Reasons:

- The free group on two generators acts on $B$: $\mathbb{F}_2 \leq SO(3) \curvearrowright B$
- $\mathbb{F}_2 = \langle a, b, a^{-1}, b^{-1} \rangle$ is itself paradoxical. Denote $W(a)$ are \textbf{reduced} words beginning with $a$. Then
  
  $\mathbb{F}_2 = \{e\} \sqcup W(a) \sqcup W(b) \sqcup W(a^{-1}) \sqcup W(b^{-1})$.
  
  $\mathbb{F}_2 = W(a) \sqcup aW(a^{-1}) = W(b) \sqcup bW(b^{-1})$.
- The paradoxicality of $\mathbb{F}_2$ induces (+axiom of choice) the paradoxical decomposition of $B$.
- The resolution of this apparent paradox is the theorem by Tarski: There is no finitely additive probability measure which is $\mathbb{F}_2$-invariant.
Amenability for discrete finitely generated groups:

Von Neumann ('29) realized that $\mathbb{F}_2$ lacks to have the property of amenability!

- Recall: discrete group $\Gamma$ is **amenable** if $\ell_\infty(\Gamma)$ has a $\Gamma$-invariant state (i.e., a positive, normalized, $\Gamma$-invariant functional $\psi: \ell_\infty(\Gamma) \to \mathbb{C}$).

An alternative approach to this circle of ideas:

A **Følner net** for $\Gamma$ is a net of non-empty finite subsets $\Gamma_i \subset \Gamma$ such that

$$\lim_{i} \frac{|(\gamma \Gamma_i) \triangle \Gamma_i|}{|\Gamma_i|} = 0 \quad \text{for all} \quad \gamma \in \Gamma,$$

where $\triangle$ is the symmetric difference and $|\Gamma_i|$ is the cardinality of $\Gamma_i$. If the net is increasing and $\Gamma = \bigcup_i \Gamma_i$ it is called a **proper Følner net**.

- Every finite group $F$ has a Følner sequence!
  - Just take the constant sequence $\Gamma_n = F$, $n \in \mathbb{N}$.

**Theorem**

$\Gamma$ is amenable iff $\Gamma$ is NOT paradoxical iff $\Gamma$ has a Følner net.
Summary:

What properties do Følner nets have?

- Følner nets provide an “inner” approximation of the group $\Gamma$ via finite subsets $\Gamma_i$.
- The finite sets $\Gamma_i$ grow “moderately” with respect to multiplication. Asymptotically

$$|\gamma \Gamma_i| \text{ is “small” compared with } |\Gamma_i|$$

The dynamics (group multiplication) is central to the analysis.
- In the context of groups given a Følner sequence one can construct another proper Følner sequence.

Følner nets are the “bridge” to address amenability issues beyond groups!

- Amenable structures are “reasonable” (i.e., not paradoxical) extensions of finite structures.
2. Amenability for metric spaces

Let \((X, d)\) be a **discrete metric space with bounded geometry**:

- Uniformly discrete: \(\inf \{d(x, y) \mid x, y \in X\} \geq d > 0\).
- Uniformly locally finite: for any radius \(R > 0\), \(\sup_{x \in X} |B_R(x)| < \infty\).

**Example:** \(\Gamma\) a finitely generated discrete group with the word length metric is a metric space with bounded geometry.
Definition (Block-Weinberger '92)

\((X, d)\) is amenable if there exists a Følner sequence \(\{F_n\} \subseteq X\) of finite, non-empty subsets of \(X\) such that

\[
\lim_{n \to \infty} \frac{|\partial_R F_n|}{|F_n|} = 0, \quad R > 0,
\]

where \(\partial_R F\) is the “double collar” around the boundary of \(F\).

The Følner sequence is proper if it is increasing and \(X = \bigcup_n F_n\). In this case we call the space properly amenable.

\(\partial_R F\) = “double collar around the finite set \(F \subseteq X\).”
Examples:

- If $|X| < \infty$, then $(X, d)$ is amenable. Take $F_n = X$ so that
  \[
  \frac{|\partial_R F_n|}{|F_n|} = \frac{|\partial_R X|}{|X|} = 0.
  \]

- $\Gamma$ is amenable as a group iff $\Gamma$ is amenable as a metric space (with the word length metric).

- As in the group case: $(X, d)$ is amenable iff it is properly amenable.
  
  - To see a difference between amenable and proper amenable
    generalize to extended metric spaces (i.e., $d: X \times X \to \mathbb{R} \cup \{\infty\}$)
    and analyze the structure of coarse connected components.

  **Example:** Consider $X = Y_1 \sqcup Y_2$, with $|Y_1| < \infty$, $Y_2$ non-amenable
  and $d(Y_1, Y_2) = \infty$. Then $X$ is amenable (take the constant
  sequence $F_n = Y_1$), but not properly amenable.
What is a paradoxical in this context? What dynamics?

**Definition**

Let \((X, d)\) a metric space with bounded geometry. A **partial translation** on \(X\) is a triple \((A, B, t)\), where \(A, B \subset X\), \(t: A \to B\) is a bijection with

\[
\sup_{a \in A} \{d(a, t(a))\} < \infty.
\]

\(X\) is **paradoxical** if there exists a partition \(X = X_1 \sqcup X_2\) and partial translations \(t_i: X \to X_i\), \(i = 1, 2\). The set of all partial translations is an inverse semigroup.

**Theorem (Grigorchuk, Ceccherini et al., '99)**

Let \((X, d)\) a metric space with bounded geometry. TFAE

- \((X, d)\) is amenable.
- \(X\) has NO paradoxical decompositions.
- There exists a finitely additive probability measure \(\mu\) on \(\mathcal{P}(X)\) invariant under partial translations (i.e., \(\mu(A) = \mu(B)\)).
3. Amenability and paradoxical decompositions in algebra

To address questions of amenability in the context of algebra:

- Need to give up the cardinality $|\cdot|$ to measure sizes.
- Take finite-dimensional subspaces as approximation and $\dim(\cdot)$ to measure the size of the subspaces.
- For today: $\mathcal{A}$ is a unital $\mathbb{C}$-algebra, but everything works also for any commutative field $\mathbb{K}$.

**Definition (Gromov '99)**

A unital algebra $\mathcal{A}$ is **amenable** if there is a Følner net $\{W_i\}_{i \in I}$ of non-zero finite dimensional subspaces such that

$$\lim_{i} \frac{\dim(aW_i + W_i)}{\dim(W_i)} = 1, \quad a \in \mathcal{A}.$$

If the $W_i$ are exhausting, then $\mathcal{A}$ is **properly algebraically amenable**.

- The presence of a linear structure makes the difference amenable vs. proper amenable an essential point.
Examples:

- Any matrix algebra $\mathcal{A} = M_k(\mathbb{C})$ is amenable. Take a constant sequence $W_n = \mathcal{A}$ and note that $\mathcal{A} \subset a\mathcal{A} + \mathcal{A} \subset \mathcal{A}$, $a \in \mathcal{A}$, hence

  $$\frac{\dim(a\mathcal{A} + \mathcal{A})}{\dim(\mathcal{A})} = 1.$$ 

- $\Gamma$ is a discrete group and $\mathbb{C}\Gamma$ is its group algebra. Then $\Gamma$ is amenable iff $\mathbb{C}\Gamma$ is algebraically amenable. [Bartholdi ’08]

Algebraic amenable vs. proper algebraic amenable:

- If $\mathcal{I} \triangleleft_L \mathcal{A}$ is a left ideal with $\dim \mathcal{I} < \infty$, then $\mathcal{A}$ is always algebraically amenable. Take a constant sequence $W_n = \mathcal{I}$.

- Note that if $\mathcal{A}$ is NON algebraically amenable, then $\tilde{\mathcal{A}} = \mathcal{I} \oplus \mathcal{A}$ is amenable but NOT properly amenable.
What is a paradoxical decomposition in this context?

Definition (Elek '03)

A countably dimensional algebra $\mathcal{A}$ is paradoxical if for any basis $\mathcal{B} = \{f_n\}_n$ of $\mathcal{A}$ one has:

- $\exists$ partitions of the basis: $\mathcal{B} = L_1 \cup \cdots \cup L_n = R_1 \cup \cdots \cup R_k$
- $\exists$ elements of the alg. $\mathcal{A}$: $a_1, \ldots, a_n$ and $b_1, \ldots, b_k \in \mathcal{A}$, such that $a_1L_1 \cup \cdots \cup a_nL_n \cup b_1R_1 \cup \cdots \cup b_kR_k$ are linearly independent in $\mathcal{A}$.

Theorem (Elek '03, Ara, Li, Li., Wu '17)

$\mathcal{A}$ is algebraically amenable iff it is NOT paradoxical.

- Elek proved the equivalence in the context of countably generated algebras $\mathcal{A}$ without zero-divisors.
  - He used as definition proper algebraic amenability.
- Having operators and Roe algebras in mind this is too restrictive.
  - One has to work with algebraic amenability.
  - Countable generation is also not needed.
- In this more general context algebraic amenability is also equivalent to the existence of a dimension measure on the lattice of subspaces of $\mathcal{A}$:
  \[
  \mu : W \rightarrow [0,1] , \ W \leq \mathcal{A} , \text{ suitable normalization, additivity and invariance.}
  \]
4. Følner nets for operators and Følner C*-algebras

Standing assumptions and notation in this talk:

- **Spaces:**
  - $\mathcal{H}$ denotes a complex (typically $\infty$-dimensional) separable Hilbert space.

- **Operators:**
  - $\mathcal{B}(\mathcal{H})$ set of all linear, bounded operators on $\mathcal{H}$.
  - Projections in $\mathcal{B}(\mathcal{H})$: $P^2 = P = P^*$ and $\mathcal{P}_{\text{fin}}(\mathcal{H})$ is set of non-zero finite rank projections.
Quasidiagonality and Følner sequences for families of operators:

**Definition (Connes ’76, Halmos ’68)**

- Let $\mathcal{T} \subset \mathcal{B}(\mathcal{H})$. A net $\{P_i\}_{i \in \mathbb{N}} \subset \mathcal{P}_{\text{fin}}(\mathcal{H})$ of finite-rank projections is called a **Følner net** for $\mathcal{T}$ if
  \[
  \lim_{i} \frac{\|TP_i - P_i T\|_2}{\|P_i\|_2} = 0, \quad \text{for all } T \in \mathcal{T},
  \]
  (*)
  where $\| \cdot \|_2$ is the Hilbert-Schmidt norm.

- If the sequence satisfying (*) is increasing and $P_i$ converges strongly to $1$, then we say that it is a **proper Følner net** for $\mathcal{T}$.

- $\mathcal{T} \subset \mathcal{B}(\mathcal{H})$ (countable) is a **quasidiagonal set of operators**, if there exists an increasing sequence of finite-rank projections $\{P_n\}_{n \in \mathbb{N}}$, $P_n \nearrow 1$ strongly, s.t.
  \[
  \lim_n \|TP_n - P_n T\| = 0, \quad T \in \mathcal{T}.
  \]

**Remarks:**

- Quasidiagonality $\Rightarrow$ Følner.

- $\mathcal{T}$ can be a single operator ($\mathcal{T} = \{T\}$) or a concrete C*-algebra.

- Any matrix has a Følner sequence. Take $P_n = 1_{n \times n}$.

- Which operators have/have not Følner sequences? (Yakubovich, Ll. ’13).
First consequences:

Proposition

If \( \{P_n\}_{n \in \mathbb{N}} \) is a Følner sequence for a unital C*-algebra \( A \subset B(\mathcal{H}) \) iff \( A \) has an amenable trace \( \tau \), i.e. the trace \( \tau \) on \( A \) extends to a state \( \psi \) on \( B(\mathcal{H}) \) that is centralised by \( A \), i.e.

\[
\psi \upharpoonright A = \tau \quad \text{and} \quad \psi(XA) = \psi(AX) , \quad X \in B(\mathcal{H}) , \ A \in A.
\]

Remark

- The state \( \psi \) (called hypertrace) is the alg. analogue of the invariant mean used in the context of groups. Take the net of states

\[
\psi_i(X) := \frac{\text{Tr}(P_i X)}{\text{Tr}(P_i)} , \quad X \in B(\mathcal{H})
\]

whose cluster points define amenable traces. (\( \{P_i\}_i \) Følner net.)

- Useful notion as an obstruction to the existence of Følner sequences! It is the property to approach an abstract characterization of these algebras.
What is the intrinsic characterization of these notions?

**Critic:** the definitions of quasidiagonality and Følner sequences for operators require, e.g., a concrete C*-algebra $A \subset B(H)$.

Voiculescu’s approach to quasidiagonality:

**Definition**

An (abstract) unital C*-algebra $A$ is called quasidiagonal if there exists a net of unital completely positive (u.c.p.) maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ which is both asymptotically multiplicative and asymptotically isometric, i.e.,

- $\| \varphi_i(AB) - \varphi_i(A)\varphi_i(B) \| \to 0$ for all $A, B \in A$.
- $\|A\| = \lim_{n \to \infty} \|\varphi_i(A)\|$ for all $A \in A$.

In the context of Følner sequences:

**Definition (Ara, Li. ’14)**

Let $A$ be a unital C*-algebra.

We say that $A$ is a Følner C*-algebra if there exists a net of u.c.p. maps $\varphi_i : A \to M_{k(i)}(\mathbb{C})$ such that

$$\lim_i \| \varphi_i(AB) - \varphi_i(A)\varphi_i(B) \|_{2,\text{tr}} = 0 , \quad A, B \in A , \quad (*)$$

where $\| F \|_{2,\text{tr}} := \sqrt{\text{tr}(F^*F)}$, $F \in M_n(\mathbb{C})$ and $\text{tr}(\cdot)$ is the unique tracial state on the matrix algebra $M_n(\mathbb{C})$. 
Theorem (Ara,Ll. ’14)

Let $\mathcal{A}$ be a unital C*-algebra. T.F.A.E.:  

(i) $\mathcal{A}$ is a Følner C*-algebra.  
(ii) Every faithful representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has an amenable trace.  
(iii) Every faithful essential representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has a proper Følner net.

- Bédos uses the name ”weakly hypertracial“ (’95) instead of ”Følner C*-algebra“.
- How can we get the Følner projections?  
  Stinespring $\Rightarrow \varphi(a) = V^* \pi(a) V$ for some representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H}')$ and isometry $V: \mathbb{C}^k \to \mathcal{H}'$. Følner projections appear as Stinespring’s projections $P = VV^*$. 
Is there any notion in operator algebras reflecting paradoxicality?

A possibly capturing some aspects of paradoxicality is the following notion. Let $\mathcal{A}$ be a unital C*-algebra. It is called **properly infinite** if

- there exist isometries $V_1, V_2 \in \mathcal{A}$ satisfying
  - $V_1^* V_1 = V_2^* V_2 = 1$, $V_1^* V_2 = 0$ and $V_1 V_1^* + V_2 V_2^* \leq 1$.

- Idea: $V_1$ and $V_2$ map the Hilbert space $\mathcal{H}$ isometrically onto two mutually orthogonal subspaces.
5. Roe $C^*$-Algebras

- We have addressed issues around amenability and paradoxical decompositions in very different mathematical situations:
  - Groups $\Gamma$.
  - Metric spaces $(X, d)$.
  - $\mathbb{C}$-algebras $A$.
  - Operator algebras $A \subset B(H)$, i.p., $C^*$-algebras.

- We will use Roe algebras to give a unified picture of the different approaches of amenability we have presented.

- These algebras were introduced to proof an index theorem for elliptic operators on non-compact manifolds $\mathcal{M}$. The idea is to look at the coarse structure of $\mathcal{M}$ captured by a discrete space $(X, d)$ and define a $C^*$-algebra $\mathcal{R}(X)$ to define the analytical part of the index.
Construction of Roe C*-algebra

Let \((X, d)\) be a metric space with bounded geometry.

1. **Hilbert space:** \(\mathcal{H} = \ell_2(X)\) with canonical ONB: \(\{\delta_x \mid x \in X\}\).

2. **Operators:** \(T \in B(\ell_2(X))\) and \(T \cong (T_{xy})_{x,y \in X}\).

3. **Propagation of operators:** For any operator \(T \in B(\ell_2(X))\) define

   \[ p(T) := \sup\{d(x, y) \mid T_{xy} \neq 0\} . \]

   - **Examples:**
     - If \(F \subset X\) and \(Q_F\) is the characteristic function of \(F\), then \(p(Q_F) = 0\).
     - If \(X = \mathbb{N}\), \(\mathcal{H} = \ell_2\) and \(S(\delta_n) = \delta_{n+1}\) the unilateral shift, then \(p(S) = 1\).
     - If \(S_+(\delta_n) := \delta_{2n+1}\) (a generator of the Cuntz algebra), then \(p(S_+) = \infty\).
     - The laplacian \(\Delta\) on a discrete graph has \(p(\Delta) = 1\).

4. **Translation algebra:** \(\mathcal{R}_0(X) := \bigcup_{R > 0} \{ T \in B(\ell_2(X)) \mid p(T) \leq R\}\)
   (Operators with bounded propagation).

5. **Roe C*-algebra:** \(\mathcal{R}(X) := \overline{\mathcal{R}_0(X)}\).
Remarks:

- Roe C*-algebras provide a natural link between metric spaces ↔ algebra ↔ operators/operator algebras and are fundamental objects for coarse geometry.
- If $X = \Gamma$ (discrete fin. generated group), then $R(\Gamma) = \ell_\infty(\Gamma) \rtimes \Gamma$.
- Partial translations $(A, B, t)$ in $X$ ↔ partial isometries in $R_0(X)$ via

$$t \mapsto T, \quad T_{yx} := \begin{cases} 1, & \text{if } (x, y) \in \text{gra}(t) \\ 0, & \text{otherwise} \end{cases}$$

with $T^* T = P_A$ and $TT^* = P_B$.

Theorem (Ara,Li,Ll.,Wu ’18)

$(X, d)$ a discrete metric space with bdd geometry. TFAE: (selection)

1. $(X, d)$ is amenable.
2. The translation algebra $R_0(X)$ is algebraically amenable.
3. The Roe C*-algebra $R(X)$ is a Følner C*-algebra.
4. The Roe C*-algebra $R(X)$ is not properly infinite.
Some ideas to the proof:

- \((X, d)\) amenable \(\Rightarrow \mathcal{R}(X)\) is Følner C*-alg:
  
  (Use local version of amenability)

  1. Take \(T \in \mathcal{R}_0(X)\) with \(p(T) \leq R\). For \(\varepsilon > 0\) there is an finite \(F \subset X\) such that \(|\partial_R(F)| \leq \varepsilon|F|\).
  
  2. Consider the projection \(Q_F\) and note that \(\|Q_F\|^2 = |F|\).
  
  3. \(\|[T, Q_F]\|^2 = \sum_{x \in F, y \notin F} |\langle T\delta_x, \delta_y \rangle|^2 + \text{sym}\)

  \[\leq \sum_{y \in \partial_R(F)} (\|T\delta_y\|^2 + \|T^*\delta_y\|^2) \leq 2\|T\| |\partial_R(F)|.\]

- \((X, d)\) amenable \(\Rightarrow \mathcal{R}_0(X)\) is algebraically amenable:

  1. Note that \(p(Q_F) = 0\). Take as Følner subspaces

  \[W = Q_F\mathcal{R}_0(X)Q_F \subset \mathcal{R}_0(X)\]

  2. Use \(\dim(W) = |F|\).
6. Summary and outlook

- Roe algebras provide a very nice frame, where amenability, i.e., having nice finite dimensional approximations with reasonable dynamics fit together
  
groups ↔ metric spaces ↔ algebra ↔ C*-algebras

What else?

- How about this equivalence in more degenerate dynamics?
  E.g., semigroups where the dynamics can drastically shrink the set $|sF| \ll |F|$.
- How do notions of amenability aspects enter mathematical physics?
  In QFT properly infinite operator algebras are ubiquitous.
  - Construction of the field algebra out of the observables and the DHR-selection principle $\leadsto$ proper infinity!
  - Recall, e.g., that it is a fact of nature that the von Neumann algebra associated to quantum fields in certain space-time regions of four dimensional Minkowski space are hyperfinite factors of type $\text{III}_1$, $\leadsto$ proper infinity!
Happy Birthday Alberto