Dimensional Deception from Noncommutative Tori

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In collaboration with A. Pinzul,

based on vintage work with R.J. Szabo and G. Landi

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Physical spacetime is (or at least appears to be) four dimensional.

Four dimensional spaces have lots of nice properties, which I will not enumerate

Nevertheless there are some who would prefer to live in two dimensions:

Those who wish to quantise gravity

The main obstacle to a quantum filed theory which includes gravity is the fact that the theory is nonrenormalizable, and therefore looses its predicitive power.

The field theory problem is an ultraviolet problem which manifests itself at high energies, where the scale is given by the Plank energy $\boxed{\sim 10^{19}\text{GeV}}$

Ideally therefore one could have a space which is four dimensional at low energies (large distances), and two dimensional at high energies (small distances)

There have been proposals in this direction. One is the Hořava-Lifshitz model

The main idea is to consider space and time scaling in an anisotropic way

$$t \to a^z t \quad \vec{x} \to a\vec{x}$$

where z is usually taken to be 3

The Euclidean Laplacian (i.e. the inverse propagator for a scalar field) on a foliation becomes dependent on a mass scale M:

$$\text{“}\Delta\text{”} = \partial_t^2 + (\partial_i \partial^i)^3 + M^2(\partial_i \partial^i)^2 + M^4(\partial_i \partial^i)$$

For the rest of this talk I will be in a Euclidean context

In this model time is treated in different way from space, and therefore Lorentz invariance is broken

In the following I will present a model for which the space is noncommutative (a NC torus), but in the limit the noncommutativity disappears, but the resulting space is two (four) dimensional at large distance, or one (two) dimensional at small distances. I will discuss in detail the two to one model, the extension being straightforward but notationally messy

The work (in progress), in collaboration with A. Pinzul, is based on some work by Elliott and Evans in 1993, and work in collaboration with Landi and Szabo in 2003/2004

Let me first of all give the definition of dimension which is most useful for our purposes. It is due to Weyl and is based on the growth of the eigenvalues of the Laplacian.

Let $\boxed{N_\triangle(\omega)}$ be the number, counting multiplicities, of eigenvalues of the Laplacian $\boxed{\triangle}$ on a Riemannian manifold, less then $\boxed{\omega}$. Then there is only one value of $\boxed{d}$ such that the following expression is finite

$$\boxed{\lim_{\omega\to\infty}\frac{N_\triangle(\omega)}{\omega^{\frac{d}{2}}} = \frac{Vol(\mathcal{M})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)}}$$

The r.h.s. can actually be used to calculate the volume, in a rather elaborate way!

Being purely spectral the above formula can be used in the noncommutative case. Clearly any noncommutative space corresponding to a finite algebra will have $\boxed{d=0}$

Let me introduce the various kinds of tori I will need. The usual torus $\mathbb{T}^2$ is generated by two elements $U := \exp(2\pi i x)$ and $V := \exp(2\pi i y)$ with $x, y \in [0, 1)$ the usual coordinates along the cycles. The algebra is

$$\forall a \in \mathcal{A} \equiv \mathcal{C}^\infty(\mathbb{T}^2), \quad a = \sum_{(l,m)\in\mathbb{Z}^2} a(l,m)\, U^l V^m$$

for some Schwartz function $a : \mathbb{Z}^2 \to \mathbb{C}$.

The passage to a noncommutative torus is done keeping the above expression but deforming

$$VU = \omega UV$$

where $\omega = e^{2\pi i\theta}$ and θ real is called the deformation parameter

For a general θ this algebra cannot be realized by finite matrices

For $\theta = p/q$ rational here is a $q \times q$ representation by the clock and shift matrices

$$\mathcal{C}_q := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \omega^{q-1} \end{pmatrix} \quad , \quad \mathcal{S}_q := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\mathcal{S}_q \mathcal{C}_q = \omega \, \mathcal{C}_q \mathcal{S}_q$

These matrices are unitary, traceless and satisfy the relations $(\mathcal{C}_q)^q = (\mathcal{S}_q)^q = \mathbb{1}_q$ hence generate the matrix algebra $\mathbb{M}_q(\mathbb{C})$ which we call the fuzzy torus

Generalize Weyl to define an effective, or scaling or deceptive, dimension.

The spectral dimension is "ultraviolet", i.e. the dimension as seen in an experiment that can probe any scale.

This is not the case in reality. Define the scaling dimension as

$$\boxed{d(\omega) := 2\frac{d \ln N_{\Delta}(\omega)}{d \ln \omega}\ .}$$

This is the dimension seen in experiments that probe the physics only up to the scale $\boxed{\omega}$. The scale is defined in terms of the spectrum of a relevant physical Laplacian, the operator controlling the dynamics

The difference between the UV-dimension and the scaling can be seen when applied to any matrix geometry, i.e. when the relevant operators have finite spectra.

The counting function in this case goes to a constant when $\boxed{\omega \to \infty}$

Any matrix geometry has a UV-dimension equal to zero. At the same time, it seems very natural that, if the spectrum is truncated at very high energy, we will not be able to tell the smooth geometry from the matrix one. Hence in any accessible experiment we will see the matrix geometry as a smooth one with some defined dimension, possibly with some "quantum" corrections. This observation makes the concept of a scaling dimension to be a very natural one.

The NC torus has two outer derivations, which are the same as the ones in the ordinary torus

$$\begin{cases} \partial_1 U = 2\pi \mathrm{i} U \ , \ \partial_1 V = 0 \\ \partial_2 U = 0 \ , \ \partial_2 V = 2\pi \mathrm{i} V \end{cases} \Leftrightarrow \begin{cases} \partial_1 a = 2\pi \mathrm{i} \sum_{(l,m)\in\mathbb{Z}^2} l\, a(l,m)\, U^l V^m \\ \partial_2 a = 2\pi \mathrm{i} \sum_{(l,m)\in\mathbb{Z}^2} m\, a(l,m)\, U^l V^m \end{cases} .$$

It is easy to see that the spectrum of the Laplacian is proportional to the integers of the kind $\boxed{n_1^2 + n_2^2}$ and hence the Weyl dimension of is $\boxed{2}$

The fuzzy torus does not have outer derivations, in particular does not have the analog of these derivations, but being a finite algebra it will anyway have dimension zero at high enough energy

Let us study a simplified model for which the number of dimensions can be deceptive.

Start with a torus with two different radii, r and $R := \mu r$, the spectrum is given by $\frac{n_1^2}{r^2} + \frac{n_1^2}{R^2}$

Introduce some sort of "1-d fuzzyness" via the operator Δ_c diagonal in the basis above, but with the spectrum truncated on the direction of V at the integer N

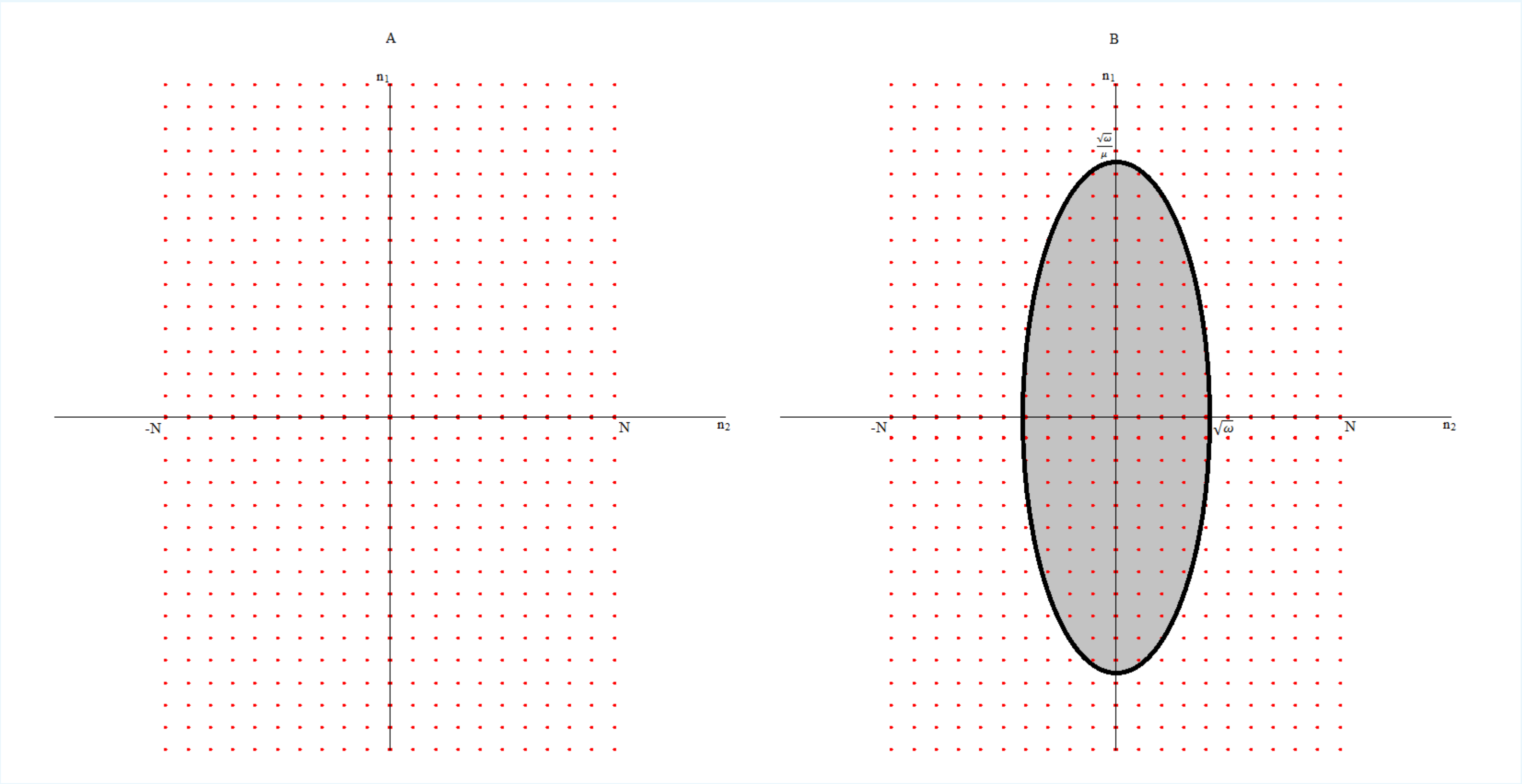
$$\Delta_c U^{n_1} = n_1^2 U^{n_1} \ , \Delta_c V^{n_2} = \begin{cases} n_2^2 V^{n_2} & |n_2| \leq N \\ 0 & |n_2| > N \end{cases}$$

Clearly Δ_c is not a differential operator

Note that the number N implicitly defines a length and therefore an energy scale. While in the R direction the Fourier series does not truncate, and therefore variation of arbitrarily small length can be taken into account, in the r direction only harmonics of width r/N contribute.

$$Spec(\triangle_{nc}) = \left\{ \frac{1}{R^2} \left(\mu^2 n_1^2 + n_2^2 \right) \ , \ n_1, n_2 \in \mathbb{Z} \ , \ |n_2| \leq N \right\}$$

The structure of a typical spectrum can be represented graphically as



A. The structure of a typical spectrum with the n_2-direction truncated at N; B. The solid curve $\mu^2 n_1^2 + n_2^2 = \omega$ represents a cut-off (we set $R = 1$). All the points of the spectrum inside the shadowed area are below the cut-off.

When $\mu \sim 1$ the low energy spectrum, up to N, is basically that of a two dimensional torus

The dimension is "deceptively" two, a low energy experiment will probe atwo dimensional torus

Then when ω reaches N a transition phase starts

The number of dimensions decreases to one

Consider now first the case $\boxed{1 \ll \omega R^2 < \mu^2 \text{ and at the same time } \omega R^2 < N^2}$

The n_1 semi-axis of the cut-off ellipse is so small that no state with $\boxed{n_1 \neq 0}$ will contribute but the number of states with non-zero n_2 is enough to allow the application of the scaling dimension formula

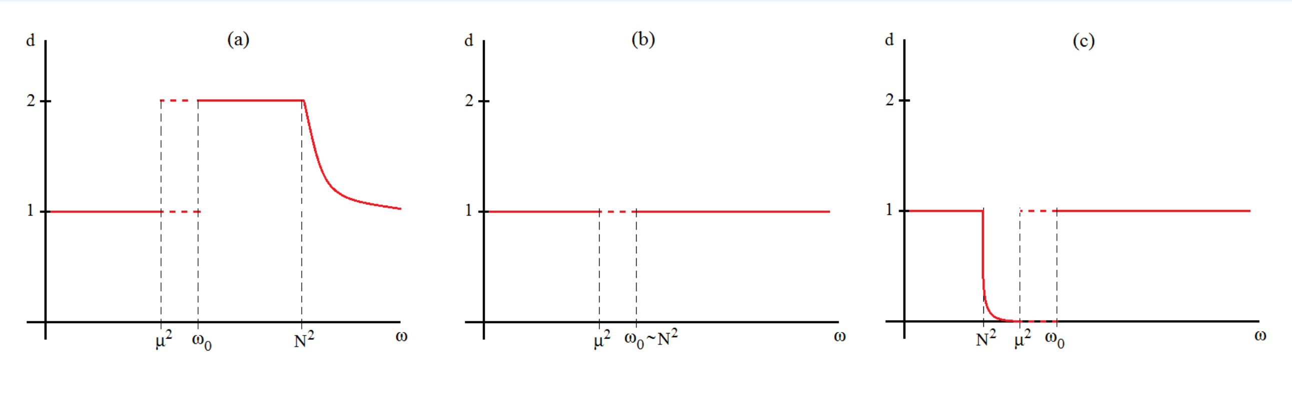
$$\boxed{N_\triangle(\omega) \sim 2\sqrt{\omega}R \;\; \Rightarrow \;\; d(\omega) = 2\frac{d \ln N_\triangle(\omega)}{d \ln \omega} = 1}$$

We arrive at a very natural and expected result: if the experiment probes the scales below the energy needed to excite the first mode it does not see the corresponding compactified dimension.

Increasing the cut-off scale ω the states with $n_1 \neq 0$ will start contributing to the counting function.

Only when a great number of them will enter, i.e. when $\omega R^2 \gg \mu^2$, (so one can pass from sum to integral) one can start using again the formula for scaling dimension to determine the dimension.

This can happen either when a) ωR^2 is still less then N^2 or b) $\omega R^2 > N^2$ (but still of the order of N) or c) $\omega R^2 \gg N^2$. This is shown below



We have seen that by changing the Laplacian it is possible to deceive the number of dimensions in a variety of ways.

In all cases however the dimension suppressed and the dimension where the original ones, and a choice has been made to suppress one of them

As in the case of Hořava-Lifshitz the fundamental symmetry of the space, which in this case is $\boxed{U(1) \times U(1)}$ acting as independent rotation on the two cycles, has been broken.

I will now present a two dimensional model for which the numebr of dimensions is again going form two to one, but the high energy space retains the fundamental symmetry of space, and the single ultraviolet dimension emerges independently form the original two.

Consider a sequence of fuzzy tori with parameters $\theta_n = \frac{p_n}{q_n} \to \theta$ with θ generic, possibly rational or even zero

Even taking the inductive limit of these algebras the resulting algebra cannot be a torus (NC or otherwise). The torus is not an approximatively finite algebra (AF). For one thing the K-theory of a torus is $\mathbb{Z} \oplus \mathbb{Z}$, while for any AF algebra is trivial.

There is however a construction, due to Elliott and Evans, which shows that the the torus $\mathbb{T}^2_\theta$ is the inductive limit of a sequence of algebras of matrices of functions on two circles

The algebra of matrices whose entries are function on a circle is Morita equivalent to the algebra of complex valued functions on the circle. It is not AF and its K-theory is $\mathbb{Z}$

Let me sketch this construction

Consider $\{\theta_n = p_n/q_n\}$, with the $q_n \to \infty$ and $n = 1 \ldots \infty$

The construction is based on the existence of a projection element of $P_{11} \in \mathbb{T}^2_\theta$, whose specific form and construction I have no time to describe.

Build P_{22} "translating" : $U \to e^{p_n/q_n} U,\ V \to V$, and iterate till $P_{q_n q_n}$

Define then P_{21} as the unitary part of $P_{22} V P_{11}$ and so on for all P_{ij}

It seems that $P_{ij}P_{kl} = \delta_{jk}P_{il}$, i.e. the P's act as a basis for $\mathbb{M}(q_n,)$, except that there is a caveat

It is possible to obtain P_{1q_n} either ar $P_{12}P_{23}\ldots P_{q_n-1\ q}$ or translating $q_n - 1$ times P_{21}

These two operators do not coincide but are related by a partial isometry z, so that the P_{ij}'s and z generate the algebra of matrix valued functions on the circle $\mathbb{M}_{q_{2n}}(\mathcal{C}^\infty(\mathbb{S}^1)) \subset \mathbb{T}^2_\theta$

Exchanging $U \leftrightarrow V$ (and after a unitary transformation) it is possible to obtain another set of matrix units and an isometry, orthogonal to the first one

I stress that all these operators are element of the original algebra, and that as $n \to \infty$ we are just building a sequence of subalgebras.

define

$$\mathsf{U}_n := \begin{pmatrix} \mathcal{C}_{q_{2n}} & 0 \\ 0 & \mathcal{S}_{q_{2n-1}}(z')^{-1} \end{pmatrix} \quad , \quad \mathsf{V}_n := \begin{pmatrix} \mathcal{S}_{q_{2n}}(z) & 0 \\ 0 & \mathcal{C}_{q_{2n-1}} \end{pmatrix}$$

with $\mathcal{C}_q$ an usual clock, but $\mathcal{S}_q(z)$ z-deformed shift matrix

$$\mathcal{C}_q := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_q & 0 & \cdots & 0 \\ 0 & 0 & \omega_q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \omega_q^{q-1} \end{pmatrix} \quad , \quad \mathcal{S}_q(z) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ z & 0 & 0 & 0 & 0 \end{pmatrix}$$

U_n and V_n generate the finite matrix algebra $\mathcal{A}_n$

$$\mathcal{A}_n \cong \mathbb{M}_{q_{2n}}(\mathcal{C}^\infty(\mathbb{S}^1)) \oplus \mathbb{M}_{q_{2n-1}}(\mathcal{C}^\infty(\mathbb{S}^1))$$

and they have a relation similar to the one of the original NCtorus, and which converges to it in the limit

$$\mathsf{V}_n\mathsf{U}_n = \omega_n \mathsf{U}_n\mathsf{V}_n \ , \quad \text{where } \omega_n = \begin{pmatrix} \omega_{q_{2n}}\mathbb{1}_{q_{2n}} & 0 \\ 0 & \omega_{q_{2n-1}}\mathbb{1}_{q_{2n-1}} \end{pmatrix}$$

Moreover, and this is the central point of the approximation

$$\lim_{n\to\infty} \|\mathsf{U}_n - U\| = \lim_{n\to\infty} \|\mathsf{V}_n - V\| = 0$$

This enables the proof that the inductive limit of the $\mathcal{A}_n$ is indeed the NCtorus $\boxed{\mathbb{T}^2_\theta}$ (details in the original papers).

Note that $\boxed{A_n}$ is not approximatively finite, and that its K-theories are $\boxed{\mathbb{Z} \oplus \mathbb{Z}}$, but, and we will discuss this later, it is Morita equivalent to two copies of functions on a 1-dimensional circle.

Also, unlike Hořava-Lifshitz and the cutoff torus earlier, the original fundamental symmetry of the torus of independently "rotate" the two cycles: $\boxed{U \to \mathrm{e}^{i\alpha_1} U \; V \to \mathrm{e}^{i\alpha_2} V}$ is still a symmetry of the high energy space

Define the truncation map

$$\boxed{\forall a \in \mathcal{A}_\theta \; , \; \Gamma_n(a) := \sum_{(l,m)\in\mathbb{Z}^2} a(l,m)\, \mathsf{U}_n^l \mathsf{V}_n^m}$$

since $(\mathcal{C}_q)^q = \mathbb{1}_q$, but $(\mathcal{S}_q(z))^q = z\mathbb{1}_q$. Defining ($[\ldots]$ the integer part)

$$a^{(n)}(m + \left[\tfrac{q}{2}\right], r; l) := \sum_{s\in\mathbb{Z}} a(sq+m, lq+r)$$

$$a'^{(n)}(m, r + \left[\tfrac{q}{2}\right]; s) := \sum_{l\in\mathbb{Z}} a(sq+m, lq+r)$$

The truncation becomes

$$\Gamma_n(a) := \left(\sum_{m,r=1}^{q_{2n}} \sum_{l\in\mathbb{Z}} a^{(n)}(m + \left[\tfrac{q_{2n}}{2}\right], r; l) z^l (\mathcal{C}_{q_{2n}})^m (\mathcal{S}_{q_{2n}}(z))^r \right) \oplus$$

$$\oplus \left(\sum_{m',r'=1}^{q_{2n-1}} \sum_{l'\in\mathbb{Z}} a'^{(n)}(m', r' + \left[\tfrac{q_{2n-1}}{2}\right]; l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\mathcal{C}_{q_{2n-1}})^{r'} \right)$$

$$=: \mathbf{a}^{(n)}(z) \oplus \mathbf{a}'^{(n)}(z')$$ where $\mathbf{a}, \mathbf{a}'$ are $q \times q$ matrices

$\mathcal{A}_n$, like the fuzzy torus, does not have an analog of ∂_i. However it does have two approximate derivatives, which close the Leibnitz rule in the limit

Using as the motivation the truncation map

$$\nabla_i \Gamma_n(a) := \Gamma_n(\partial_i a) + \text{terms which vanish as } n \to \infty \ ,$$

The choice of these terms is made in such a way as to ensure that the action is diagonal. Explicitly

$$\nabla_1 \Gamma_n(a) := 2\pi \mathrm{i} \left(\sum_{m,r=1}^{q_{2n}} \sum_{l \in \mathbb{Z}} m \, a^{(n)}(m + \left[\tfrac{q_{2n}}{2}\right], r; l) z^l (\mathcal{C}_{q_{2n}})^m (\mathcal{S}_{q_{2n}}(z))^r \right) \oplus$$

$$\oplus \left(\sum_{m',r'=1}^{q_{2n-1}} \sum_{l' \in \mathbb{Z}} (l' q_{2n-1} + m') \, a'^{(n)}(m', r' + \left[\tfrac{q_{2n-1}}{2}\right]; l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\mathcal{C}_{q_{2n-1}})^{r'} \right)$$

and an analogous expression for ∇_2

It is also useful to write these deformed derivatives as operators on the matrix-valued functions on $\mathbb{S}^1$

$$\nabla_1 \left(\mathbf{a}^{(n)}(\tau) \oplus \mathbf{a}'^{(n)}(\tau')\right) = \Sigma \mathbf{a}^{(n)}(\tau) \oplus \left(q_{2n-1} \frac{\mathsf{d}}{\mathsf{d}\tau'} \mathbf{a}'^{(n)}(\tau') + [\Theta', \mathbf{a}'^{(n)}(\tau')]\right)$$

where $z = \mathsf{e}^{2\pi \mathsf{i} \tau}$ and Θ and Σ are known matrices

In this form it is simple to see the violation of Leibnitz rule: the terms that contain the τ-derivative and commutators with Θ do respect the Leibnitz rule, the terms with the matrix multiplication by Σ don't

This exactly corresponds to throwing away the extra terms. The Leibnitz rule is recovered in the limit

We now ask what the Weyl dimension of our space is, at different scales

Define the deformed Laplacian $\Delta_{(n)}$ in the usual way

$$\Delta_{(n)} = -\nabla_1^2 - \nabla_2^2 \ ,$$

Since the general element of $\mathcal{A}_n$ can be written as $\Gamma_n(a)$ we have the eigenvalue problem:

$$-\left(\nabla_1^2 + \nabla_2^2\right)\Gamma_n(a) = \lambda\Gamma_n(a)$$

The eigenvalue problem can be rewritten as

$$4\pi^2 \left(\sum_{m,r=1}^{q_{2n}} \sum_{l\in\mathbb{Z}} \left(m^2 + (q_{2n}l + r)^2\right) \, a^{(n)}(m + \left[\tfrac{q_{2n}}{2}\right], r; l) z^l (\mathcal{C}_{q_{2n}})^m (\mathcal{S}_{q_{2n}}(z))^r \right.$$

$$\left. \oplus \sum_{m',r'=1}^{q_{2n-1}} \sum_{l'\in\mathbb{Z}} \left(r'^2 + (q_{2n-1}l' + m')^2\right) \, a'^{(n)}(m', r' + \left[\tfrac{q_{2n-1}}{2}\right]; l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\mathcal{C}_{q_{2n-1}})^{r'} \right) =$$

$$= \lambda \left(\sum_{m,r=1}^{q_{2n}} \sum_{l\in\mathbb{Z}} a^{(n)}(m + \left[\tfrac{q_{2n}}{2}\right], r; l) z^l (\mathcal{C}_{q_{2n}})^m (\mathcal{S}_{q_{2n}}(z))^r \oplus \right.$$

$$\left. \oplus \sum_{m',r'=1}^{q_{2n-1}} \sum_{l'\in\mathbb{Z}} a'^{(n)}(m', r' + \left[\tfrac{q_{2n-1}}{2}\right]; l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\mathcal{C}_{q_{2n-1}})^{r'} \right)$$

Using orthogonality relation among clock and shift we can invert to obtain, after some algebra

$$\lambda = -4\pi^2 \left(m^2 + (q_{2n}l + r)^2\right) = 4\pi^2 \left(r'^2 + (q_{2n-1}l' + m')^2\right)$$

with $l, l' \in \mathbb{Z}, \;\; 1 \le m, r \le q_{2n}, \;\; 1 \le m', r' \le q_{2n-1}$

The matching condition is Diophantine which means that is not clear that the spectrum is non empty

Fortunately there are eigenvalues: Both, $(q_{2n-1}l' + m')$ and $(q_{2n}l + r)$, are bijective maps to $\mathbb{Z}$.For every value of l', r' there is only one choice of l, r such that $(q_{2n-1}l' + m') = (q_{2n}l + r)$. Since $q_{2n-1} < q_{2n}$ then $\forall r' \quad \exists! r : r' = r$.

This shows that the spectrum is

$$4\pi^2(m^2 + s^2), \quad 1 \leq m \leq q_{2n-1}, \quad s \in \mathbb{Z}$$

We are in the same situation of the simplified model described at the beginning, except that this time we did not cut the spectrum of the Laplacian by hand.

We are ready to calculate the spectral dimension of our fuzzy geometry in two extreme limits, infrared and ultraviolet

What one should expect to see in this limits? The physical spectral dimension is the dimension as seen in the experiment that can probe the geometry up to some cut-off scale.

The IR limit should look as the commutative geometry, i.e. we expect that the spectral dimension is this case should be 2.

In the UV limit we do not have, in general, enough intuition (which is based on a commutative geometry). So, in this case the actual calculation should provide us with some hints on where the fundamental, i.e. UV, degrees of freedom really live. We will see that this is the case

IR Regime. The cut-off scale $\boxed{\omega}$ is below the characteristic quantum geometric scale. In the case of a toy model this scale was controlled by the number of the states along $\boxed{R}$-direction. In the present scale, this means that $\boxed{\omega < q_{2n-1}^2}$. Only the winding modes (from two circles) with $\boxed{l, l' = -1, 0}$ contribute. We immediately have for the counting function

$$\boxed{N_\triangle(\omega) \sim \text{degeneracy} \times \int_{m^2+s^2 \leq \frac{\omega}{4\pi^2}} dm\, ds = \text{const} \times \omega}$$

With our definition of scaling dimension we get $\boxed{d_{IR} = 2}$

This result is not unexpected, is the consequence of the fact that the *effective* radii of two $\boxed{\mathbb{S}^1}$ are very small. Although we started with all the radii of the order of 1the contribution of (l, l')-mode to the spectrum is of the order of $q^2 \gg 1$ (where q is either q_{2n} or q_{2n-1}). This effectively reduces the radii of the "internal" circles by the factor of q

UV Regime. Now many of the $\mathbb{S}^1$ winding modes are excited, $l, l' \gg 1$. The hypothetical experiment can probe the physics up to the cut-off $\omega \gg q_{2n}^2$.

In this case we have for the spectrum (in terms of l', m', r')

$$4\pi^2 \left(r'^2 + (q_{2n-1} l' + m')^2 \right) = 4\pi^2 q_{2n-1}^2 l'^2 \left(1 + \mathcal{O}\left(\frac{1}{l'} \right) \right)$$

The counting function in this limit is

$$N_\Delta(\omega) \to \text{degeneracy} \times \int^{q_{2n-1}} dm\, dr \int_{-\frac{\sqrt{\omega}}{2\pi q_{2n-1}}}^{\frac{\sqrt{\omega}}{2\pi q_{2n-1}}} dk = \text{const} \times q_{2n-1} \sqrt{\omega}$$

We get the physical dimension in ultraviolet $d_{UV} = 1$

Consider the factor q in the UV counting function $N \sim \sqrt{q_{2n-1}^2 \omega}$

From the original Weyl theorem the effective size of the UV-dimension is proportional to q, instead of being of order one or even of order of $1/q$

This "elongation" is due to the q^2 matrix degrees of freedom
This is very suggestive: in the ultraviolet the new single dimension is fundamental and the two IR dimensions of the torus have disappeared

The single reduced dimension is not one of the original two.

It is instructive to look at the UV regime from the point of view of the other representation of the deformed derivatives

In this representation the dynamics governed by the deformed Laplacian is 1-dimensional matrix model

$$\left(\nabla_1^2 + \nabla_2^2\right)\left(\mathbf{a}^{(n)}(\tau) \oplus \mathbf{a}'^{(n)}(\tau')\right) =$$

$$= \left(q_{2n}^2 \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{a}^{(n)}(\tau) + 2q_{2n}[\Theta, \tfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{a}^{(n)}(\tau)] + [\Theta, [\Theta, \mathbf{a}^{(n)}(\tau)]] + \Sigma^2\mathbf{a}^{(n)}(\tau)\right) \oplus$$

$$\oplus \left(q_{2n-1}^2 \frac{\mathrm{d}^2}{\mathrm{d}\tau'^2}\mathbf{a}'^{(n)}(\tau') + 2q_{2n-1}[\Theta', \tfrac{\mathrm{d}}{\mathrm{d}\tau'}\mathbf{a}'^{(n)}(\tau')] + [\Theta', [\Theta', \mathbf{a}'^{(n)}(\tau')]]\right.$$

$$\left. + \Sigma'^2\mathbf{a}'^{(n)}(\tau')\right) \ .$$

The leading UV term UV has two τ-derivatives, corresponding to the sum of two usual $\mathbb{S}^1$-Laplacians with the correct rescaling of the radii by $1/q$ in agreement with our previous discussion.

Conclusions

I have argued how a construction for the noncommutative torus can give a space which is effectively two dimensional at low scales, or large distances, while being at high scales, small distances, actually a two dimensional sum of tho circles

Although I have discussed a $\boxed{2 \to 1}$ reduction a $\boxed{4 \to 2}$ reduction is possible, but technically messy. I straightforward application of the above to $\boxed{\mathbb{T}^4_\theta = \mathbb{T}^2_\theta \times \mathbb{T}^2_\theta}$ gives a reduction to two two dimensional tori.

Other possibilities like reducing a four torus to four circles are possible

This is work in progress and there are several aspects, like the presence of fermions, which could unveil other interesting features

Auguri Alberto!