Fernando Falceto
Theoretical Physics Department.
Universidad de Zaragoza

Toeplitz determinants in Mathematical Physics

Alberto Ibort fest, ICMAT, Madrid, March 5-9, 2018.

In collaboration with:
Filiberto Ares
José G. Esteve
Amilcar de Queiroz

Aim of the talk

- We review the main steps in our progress towards the understanding of Toeplitz determinants.
- We discuss connections of the latter with physics, namely: the Ising model and entanglement entropy of fermionic chains.
- We emphasize the impulse that physics has given to the development of the theory.
- Finally we present new results and conjectures on the subject.

Aim of the talk

- We review the main steps in our progress towards the understanding of Toeplitz determinants.
- We discuss connections of the latter with physics, namely: the Ising model and entanglement entropy of fermionic chains.
- We emphasize the impulse that physics has given to the development of the theory.
- Finally we present new results and conjectures on the subject.

Aim of the talk

- We review the main steps in our progress towards the understanding of Toeplitz determinants.
- We discuss connections of the latter with physics, namely: the Ising model and entanglement entropy of fermionic chains.
- We emphasize the impulse that physics has given to the development of the theory.
- Finally we present new results and conjectures on the subject.

Aim of the talk

- We review the main steps in our progress towards the understanding of Toeplitz determinants.
- We discuss connections of the latter with physics, namely: the Ising model and entanglement entropy of fermionic chains.
- We emphasize the impulse that physics has given to the development of the theory.
- Finally we present new results and conjectures on the subject.

Aim of the talk

- We review the main steps in our progress towards the understanding of Toeplitz determinants.
- We discuss connections of the latter with physics, namely: the Ising model and entanglement entropy of fermionic chains.
- We emphasize the impulse that physics has given to the development of the theory.
- Finally we present new results and conjectures on the subject.

Based on:

- P. Deift, A. Its, I. Krasovsky, *Comm.Pure Appl.Math.* 66. `arXiv:1207.4990`
- F. Ares, J. G. Esteve, F. F., *Phys. Rev.* A 90, (2014)
- F. Ares, J. G. Esteve, F. F., A. R. de Queiroz, *J. Stat. Mech.* 063104, (2017)
- F. Ares, J. G. Esteve, F. F., A. R. de Queiroz, `arXiv:1801.07043`, (2018)

Toeplitz matrices (Toeplitz 1907)

Symbol $f : S_1 \to \mathbb{C}$, $f \in L^1$

$$t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

Toeplitz matrices (Toeplitz 1907)

Symbol $f : S_1 \to \mathbb{C}$, $f \in L^1$

$$t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

Toeplitz Matrix with symbol f:

$$T_n(f) = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & \cdots & \cdots & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & t_{-2} & & & & \vdots \\ t_2 & t_1 & t_0 & t_{-1} & \ddots & & & \vdots \\ \vdots & t_2 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & t_{-2} & \vdots \\ \vdots & & & \ddots & t_1 & t_0 & t_{-1} & t_{-2} \\ \vdots & & & & t_2 & t_1 & t_0 & t_{-1} \\ t_{n-1} & \cdots & \cdots & \cdots & \cdots & t_2 & t_1 & t_0 \end{pmatrix}$$

Toeplitz determinant

Introduce the Toeplitz determinant with symbol f

$$D_n(f) = \det T_n(f)$$

Toeplitz determinant

Introduce the Toeplitz determinant with symbol f

$$D_n(f) = \det T_n(f)$$

Szegő theorem (1915):

For $f : S^1 \to \mathbb{R}^+$ continuous and $[f] = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta) \mathrm{d}\theta\right)$

$$\boxed{\lim_{n\to\infty} (D_n(f))^{1/n} = [f]}$$

Toeplitz determinant

Introduce the Toeplitz determinant with symbol f

$$D_n(f) = \det T_n(f)$$

Szegő theorem (1915):

For $f : S^1 \to \mathbb{R}^+$ continuous and $[f] = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta)\mathrm{d}\theta\right)$

$$\boxed{\lim_{n\to\infty} (D_n(f))^{1/n} = [f]}$$

Our cooperation started from a conjecture which I found. It was about a determinant considered by Toeplitz and others, formed with the Fourier-coefficients of a function f (x). I had no proof, but I published the conjecture and the young Szegő found the proof...

G. Pólya, Mathematische Annalen, 1915

Toeplitz determinant

Introduce the Toeplitz determinant with symbol f

$$D_n(f) = \det T_n(f)$$

Szegő theorem (1915):

For $f : S^1 \to \mathbb{R}^+$ continuous and $[f] = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta) \mathrm{d}\theta\right)$

$$\boxed{\lim_{n\to\infty} (D_n(f))^{1/n} = [f]}$$

In other words

$$\frac{D_n(f)}{[f]^n} = \mathrm{e}^{o(n)}$$

Toeplitz determinant

Introduce the Toeplitz determinant with symbol f

$$D_n(f) = \det T_n(f)$$

Szegő theorem (1915):

For $f : S^1 \to \mathbb{R}^+$ continuous and $[f] = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta) \mathrm{d}\theta\right)$

$$\lim_{n\to\infty} (D_n(f))^{1/n} = [f]$$

In other words

$$\frac{D_n(f)}{[f]^n} = \mathrm{e}^{o(n)}$$

Can we say something about $o(n)$?

Toeplitz determinant

Szegő Strong limit theorem (Szegő 1952, Johanson 1988):

Let $f : S^1 \to \mathbb{C}$, with $\log f \in L^1$, call

$$s_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

Hence if

$$\sum_{k=-\infty}^{\infty} |k||s_k|^2 < \infty$$

$$\boxed{\lim_{n\to\infty} \frac{D_n(f)}{\mathrm{e}^{ns_0}} = \mathrm{e}^{\sum_{k=1}^{\infty} k s_k s_{-k}}}$$

Toeplitz determinant

Szegő Strong limit theorem (Szegő 1952, Johanson 1988):

Let $f : S^1 \to \mathbb{C}$, with $\log f \in L^1$, call

$$s_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

Hence if

$$\sum_{k=-\infty}^{\infty} |k||s_k|^2 < \infty$$

$$\boxed{\lim_{n\to\infty} \frac{D_n(f)}{\mathrm{e}^{n s_0}} = \mathrm{e}^{\sum_{k=1}^{\infty} k s_k s_{-k}}}$$

Comparing with previous slide, $o(n) = \sum_{k=1}^{\infty} k s_k s_{-k} + o(1)$

Toeplitz determinant

Szegő Strong limit theorem (Szegő 1952, Johanson 1988):

Let $f : S^1 \to \mathbb{C}$, with $\log f \in L^1$, call

$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta$$

Hence if

$$\sum_{k=-\infty}^{\infty} |k||s_k|^2 < \infty$$

$$\boxed{\lim_{n\to\infty} \frac{D_n(f)}{\mathrm{e}^{ns_0}} = \mathrm{e}^{\sum_{k=1}^{\infty} k s_k s_{-k}}}$$

Comparing with previous slide, $o(n) = \sum_{k=1}^{\infty} k s_k s_{-k} + o(1)$

What if $\displaystyle\sum_{k=-\infty}^{\infty} |k||s_k|^2 = \infty$?

Ising model in two dimensions ($\sigma_{x,y}$)

Kaufman and Onsager (1949)

$$\langle \sigma_{0,0}\sigma_{n,n}\rangle = D_n(f_{\mathrm{Is}})$$

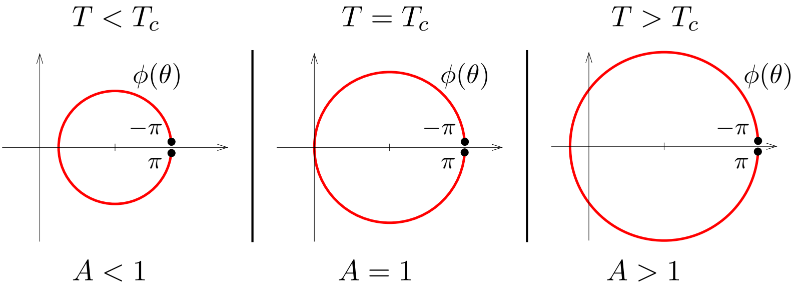
$$f_{\mathrm{Is}} = \mathrm{e}^{\mathrm{i}\mathrm{Arg}\phi}, \quad \phi(\theta) = 1 - A\mathrm{e}^{\mathrm{i}\theta}, \text{ with } \; A = (\sinh \tfrac{2J}{k_B T})^{-2}.$$

Ising model in two dimensions ($\sigma_{x,y}$)

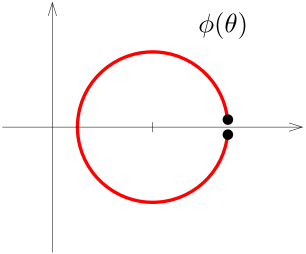
Kaufman and Onsager (1949)

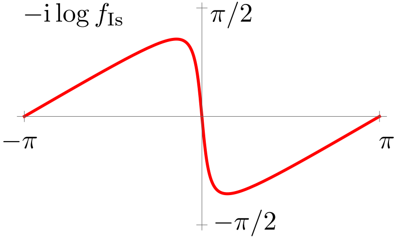
$$\langle \sigma_{0,0} \sigma_{n,n} \rangle = D_n(f_{\mathrm{Is}})$$

$$f_{\mathrm{Is}} = \mathrm{e}^{\mathrm{i}\mathrm{Arg}\phi}, \quad \phi(\theta) = 1 - A\mathrm{e}^{\mathrm{i}\theta}, \text{ with } A = \left(\sinh \frac{2J}{k_B T}\right)^{-2}.$$



Ising model, $T < T_c$ $(A < 1)$.

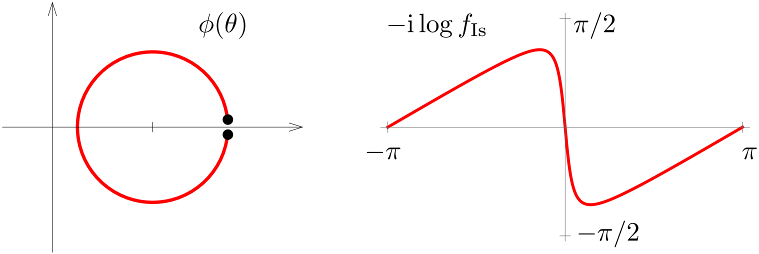




For $A < 1$, $\log f_{\mathrm{Is}} \in C^{1+\epsilon}$ $\Rightarrow$ $\sum_{k=-\infty}^{\infty} |k||s_k|^2 < \infty$ $\Rightarrow$

$\Rightarrow$ *Szegő Strong Limit Theorem* applies.

Ising model, $T < T_c$ ($A < 1$).



For $A < 1$, $\log f_{\mathrm{Is}} \in C^{1+\epsilon}$ $\Rightarrow$ $\sum_{k=-\infty}^{\infty} |k||s_k|^2 < \infty$ $\Rightarrow$

$\Rightarrow$ *Szegő Strong Limit Theorem* applies.

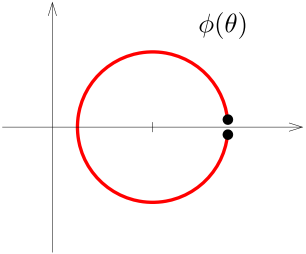
Hence

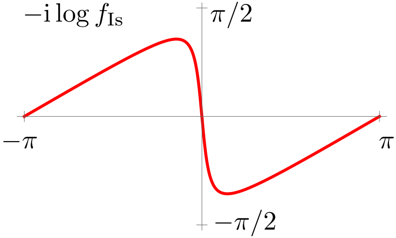
$$\lim_{n\to\infty} \frac{D_n(f_{\mathrm{Is}})}{\mathrm{e}^{ns_0}} = \mathrm{e}^{\sum_{k=1}^{\infty} k s_k s_{-k}}$$

with

$$s_0 = 0, \quad s_k = -\frac{A^{|k|}}{2k}, \qquad \sum_{k=1}^{\infty} k s_k s_{-k} = \frac{1}{4}\log(1 - A^2)$$

Ising model, $T < T_c$ $(A < 1)$.





Then

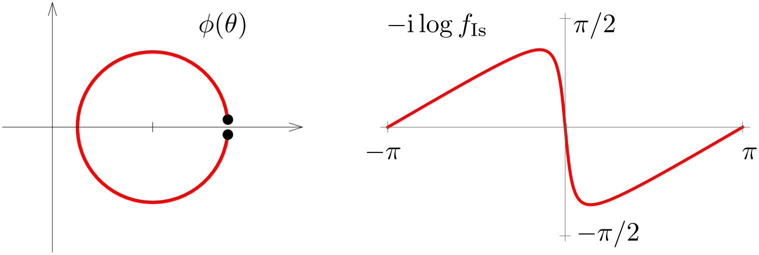
$$\lim_{n\to\infty} \langle \sigma_{0,0}\sigma_{n,n} \rangle = \lim_{n\to\infty} D_n(f_{\mathrm{Is}}) =$$

$$= \mathrm{e}^{\sum_{k=1}^{\infty} k s_k s_{-k}} = (1 - A^2)^{1/4}.$$

From which we derive the spontaneous magnetization

$$M_0 = \lim_{n\to\infty} \langle \sigma_{0,0}\sigma_{n,n} \rangle^{1/2} = (1 - A^2)^{1/8}$$

Ising model, $T < T_c$ ($A < 1$).



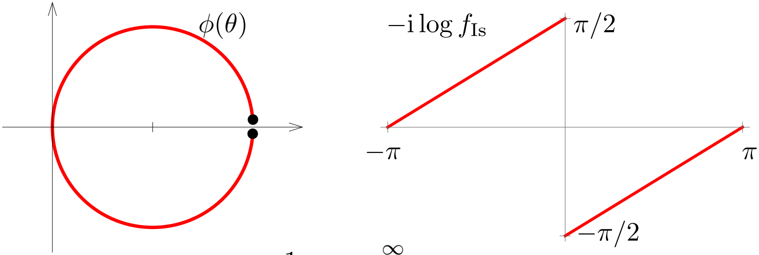
...and lo and below I found it. It was a general formula for the evaluation of Toeplitz matrices. The only thing I did not know was how to fill out the holes in the mathematics and show the epsilons and deltas and all that...

...the mathematicians got there first...

L. Onsager, 1971.

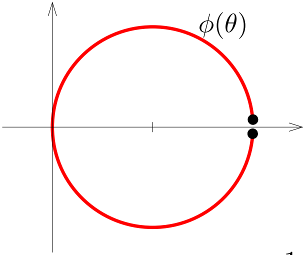
$$M_0 = \lim_{n\to\infty} \langle \sigma_{0,0}\sigma_{n,n}\rangle^{1/2} = (1 - A^2)^{1/8}$$

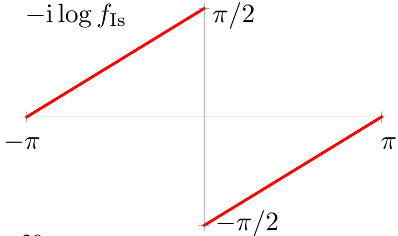
Ising model, $T = T_c$ $(A = 1)$.



f_{Is} has jumps, $s_k = -\dfrac{1}{2k} \Rightarrow \displaystyle\sum_{k=-\infty}^{\infty} |k||s_k|^2 = \infty$

Ising model, $T = T_c$ $(A = 1)$.



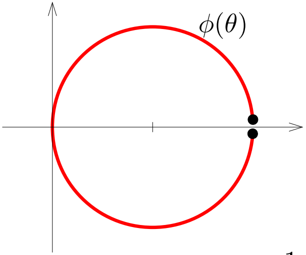


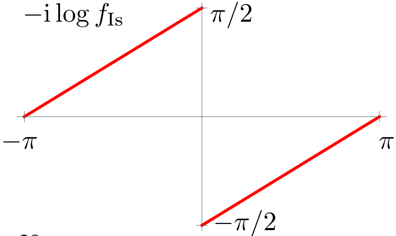
f_{Is} has jumps, $s_k = -\dfrac{1}{2k} \Rightarrow \displaystyle\sum_{k=-\infty}^{\infty} |k||s_k|^2 = \infty$

$$f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r), \quad \theta, \theta_r \in (-\pi, \pi]$$

- $V(\theta)$ periodic and smooth enough, $\quad g_\beta(\theta) = \mathrm{e}^{\mathrm{i}(\theta - \pi\,\mathrm{sgn}(\theta))\beta}$

Ising model, $T = T_c$ $(A = 1)$.



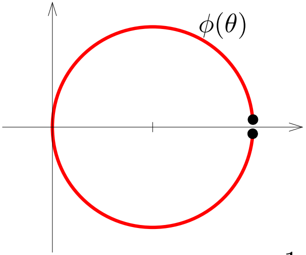


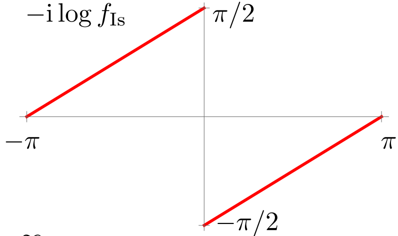
f_{Is} has jumps, $s_k = -\dfrac{1}{2k} \Rightarrow \displaystyle\sum_{k=-\infty}^{\infty} |k||s_k|^2 = \infty$

$$f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r), \quad \theta, \theta_r \in (-\pi, \pi]$$

- $V(\theta)$ periodic and smooth enough, $\quad g_\beta(\theta) = \mathrm{e}^{\mathrm{i}(\theta - \pi\,\mathrm{sgn}(\theta))\beta}$
- f has zeros and/or jump discontinuities at $\theta = \theta_r,\ r = 1, \ldots, R$.

Ising model, $T = T_c$ $(A = 1)$.





f_{Is} has jumps, $s_k = -\dfrac{1}{2k} \Rightarrow \displaystyle\sum_{k=-\infty}^{\infty} |k||s_k|^2 = \infty$

$$f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r), \quad \theta, \theta_r \in (-\pi, \pi]$$

- $V(\theta)$ periodic and smooth enough, $\quad g_\beta(\theta) = \mathrm{e}^{\mathrm{i}(\theta - \pi\,\mathrm{sgn}(\theta))\beta}$
- f has zeros and/or jump discontinuities at $\theta = \theta_r,\ r = 1, \ldots, R$.

$$f_{\mathrm{Is}} = g_{1/2}$$

Ising model, $T = T_c$ $(A = 1)$.

Fisher-Hartwig conjecture (1968), Lenard (1964), Wu (1966).

For $f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r)$

One has: $$D_n(f) = \mathrm{e}^{nV_0}\, n^{\sum_{r=1}^{R}(\alpha_r^2 - \beta_r^2)}\, E(f)(1 + o(1))$$

Ising model, $T = T_c$ $(A = 1)$.

Fisher-Hartwig conjecture (1968), Lenard (1964), Wu (1966).

For $f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r)$

One has: $D_n(f) = \mathrm{e}^{nV_0}\; n^{\sum_{r=1}^{R}(\alpha_r^2 - \beta_r^2)}\; E(f)(1 + o(1))$

$$
\begin{aligned}
E(f) &= E(\mathrm{e}^V) \prod_{r=1}^{R} \left[b_+(\mathrm{e}^{\mathrm{i}\theta_r})^{\beta_r - \alpha_r} b_-(\mathrm{e}^{\mathrm{i}\theta_r})^{-\beta_r - \alpha_r} \right] \\
&\times \prod_{r \neq r'} \left[|\mathrm{e}^{\mathrm{i}\theta_r} - \mathrm{e}^{\mathrm{i}\theta_{r'}}|^{\beta_r \beta_{r'} - \alpha_r \alpha_{r'}} g_{\alpha_r \beta_{r'}}(\theta_{r'} - \theta_r) \right] \\
&\times \prod_{r=1}^{R} \frac{G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)}
\end{aligned}
$$

Widom (1972, $\beta_r = 0$), Basor (1978), Ehrhardt (2001).

Ising model, $T = T_c$ $(A = 1)$.

Fisher-Hartwig conjecture (1968), Lenard (1964), Wu (1966).

For $f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r)$

One has: $D_n(f) = \mathrm{e}^{nV_0}\ n^{\sum_{r=1}^{R}(\alpha_r^2 - \beta_r^2)}\ E(f)(1 + o(1))$

$$\begin{aligned} E(f) &= E(\mathrm{e}^V) \prod_{r=1}^{R} \left[b_+(\mathrm{e}^{\mathrm{i}\theta_r})^{\beta_r - \alpha_r} b_-(\mathrm{e}^{\mathrm{i}\theta_r})^{-\beta_r - \alpha_r} \right] \\ &\times \prod_{r \neq r'} \left[|\mathrm{e}^{\mathrm{i}\theta_r} - \mathrm{e}^{\mathrm{i}\theta_{r'}}|^{\beta_r \beta_{r'} - \alpha_r \alpha_{r'}} g_{\alpha_r \beta_{r'}}(\theta_{r'} - \theta_r) \right] \\ &\times \prod_{r=1}^{R} \frac{G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)} \end{aligned}$$

$$V(\theta) = \sum_{k=-\infty}^{\infty} V_k \mathrm{e}^{\mathrm{i}k\theta},\ E(\mathrm{e}^V) = \mathrm{e}^{\sum_{k=1}^{\infty} k V_k V_{-k}}$$

Ising model, $T = T_c$ $(A = 1)$.

Fisher-Hartwig conjecture (1968), Lenard (1964), Wu (1966).

For $f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r)$

One has: $$D_n(f) = \mathrm{e}^{nV_0}\ n^{\sum_{r=1}^{R}(\alpha_r^2 - \beta_r^2)}\ E(f)(1 + o(1))$$

$$\begin{aligned} E(f) &= E(\mathrm{e}^V) \prod_{r=1}^{R} \left[b_+(\mathrm{e}^{\mathrm{i}\theta_r})^{\beta_r - \alpha_r} b_-(\mathrm{e}^{\mathrm{i}\theta_r})^{-\beta_r - \alpha_r} \right] \\ &\times \prod_{r \neq r'} \left[|\mathrm{e}^{\mathrm{i}\theta_r} - \mathrm{e}^{\mathrm{i}\theta_{r'}}|^{\beta_r \beta_{r'} - \alpha_r \alpha_{r'}} g_{\alpha_r \beta_{r'}}(\theta_{r'} - \theta_r) \right] \\ &\times \prod_{r=1}^{R} \frac{G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)} \end{aligned}$$

$$b_+(z) = \mathrm{e}^{\sum_{k=1}^{\infty} V_k z^k},\ b_-(z) = \mathrm{e}^{\sum_{k=1}^{\infty} V_{-k} z^{-k}}$$

Ising model, $T = T_c$ $(A = 1)$.

Fisher-Hartwig conjecture (1968), Lenard (1964), Wu (1966).

For $f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r)$

One has: $D_n(f) = \mathrm{e}^{nV_0}\, n^{\sum_{r=1}^{R}(\alpha_r^2 - \beta_r^2)}\, E(f)(1 + o(1))$

$$\begin{aligned} E(f) &= E(\mathrm{e}^{V}) \prod_{r=1}^{R} \left[b_+(\mathrm{e}^{\mathrm{i}\theta_r})^{\beta_r - \alpha_r} b_-(\mathrm{e}^{\mathrm{i}\theta_r})^{-\beta_r - \alpha_r} \right] \\ &\times \prod_{r \neq r'} \left[|\mathrm{e}^{\mathrm{i}\theta_r} - \mathrm{e}^{\mathrm{i}\theta_{r'}}|^{\beta_r \beta_{r'} - \alpha_r \alpha_{r'}} g_{\alpha_r \beta_{r'}}(\theta_{r'} - \theta_r) \right] \\ &\times \prod_{r=1}^{R} \frac{G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)} \end{aligned}$$

$$G(z) = \text{Barnes function}, \qquad G(z+1) = \Gamma(z) G(z),$$

$$G(1 - m) = 0,\ m \in \mathbb{Z}^+.$$

Ising model, $T = T_c$ $(A = 1)$.

Fisher-Hartwig conjecture (1968), Lenard (1964), Wu (1966).

For $f(\theta) = \mathrm{e}^{V(\theta)} \prod_{r=1}^{R} |\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_r}|^{2\alpha_r} \prod_{r=1}^{R} g_{\beta_r}(\theta - \theta_r)$

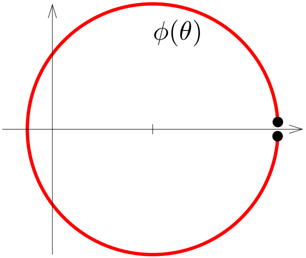
One has: $$D_n(f) = \mathrm{e}^{nV_0}\, n^{\sum_{r=1}^{R}(\alpha_r^2 - \beta_r^2)}\, E(f)(1 + o(1))$$

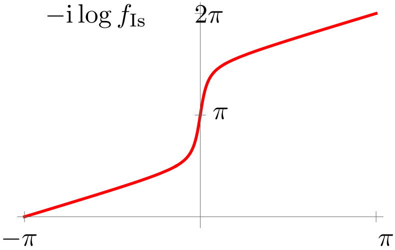
$$\begin{aligned} E(f) &= E(\mathrm{e}^V) \prod_{r=1}^{R} \left[b_+(\mathrm{e}^{\mathrm{i}\theta_r})^{\beta_r - \alpha_r} b_-(\mathrm{e}^{\mathrm{i}\theta_r})^{-\beta_r - \alpha_r} \right] \\ &\times \prod_{r \neq r'} \left[|\mathrm{e}^{\mathrm{i}\theta_r} - \mathrm{e}^{\mathrm{i}\theta_{r'}}|^{\beta_r \beta_{r'} - \alpha_r \alpha_{r'}} g_{\alpha_r \beta_{r'}}(\theta_{r'} - \theta_r) \right] \\ &\times \prod_{r=1}^{R} \frac{G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)} \end{aligned}$$

Ising model at T_c: $V = 0$, $\alpha = 0$, $\beta = 1/2 \Rightarrow$

$$\Rightarrow \langle \sigma_{0,0} \sigma_{n,n} \rangle = D_n(f) = \frac{G(3/2) G(1/2)}{n^{1/4}} (1 + o(1))$$

Ising model, $T > T_c$ $(A > 1)$.

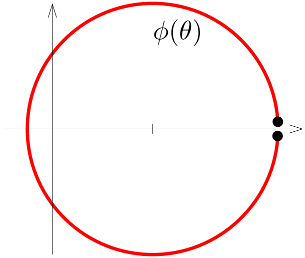


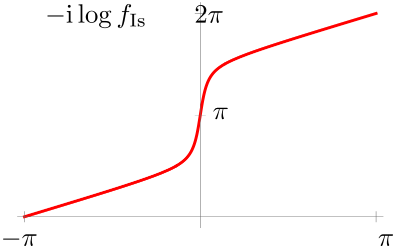


$$f_{\mathrm{Is}}(\theta) = \tilde{f}(\theta)\mathrm{e}^{\mathrm{i}\theta}, \qquad \log \tilde{f} \text{ smooth}$$

$\beta = 1 \Rightarrow G(1-\beta) = 0 \Rightarrow E(f_{\mathrm{Is}}) = 0 \Rightarrow$ F-H do not apply.

Ising model, $T > T_c$ $(A > 1)$.





$$f_{\mathrm{Is}}(\theta) = \tilde{f}(\theta)\mathrm{e}^{\mathrm{i}\theta}, \qquad \log \tilde{f} \text{ smooth}$$

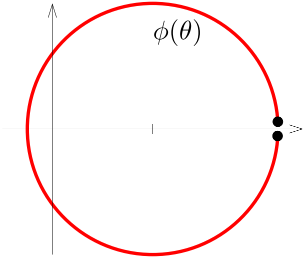
$\beta = 1 \Rightarrow G(1-\beta) = 0 \Rightarrow E(f_{\mathrm{Is}}) = 0 \Rightarrow$ F-H do not apply.

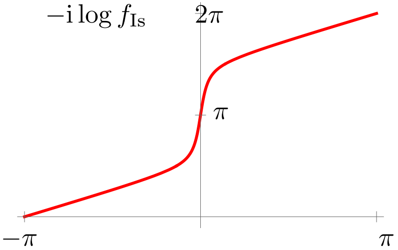
$$\boxed{D_n(f_{\mathrm{Is}}) = p_n(0) D_n(\tilde{f})}$$

$p_n(z) = z^n + \ldots$, s. t.

$$\int_{-\pi}^{\pi} p_n(\mathrm{e}^{-\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}m\theta}\tilde{f}(\theta)\mathrm{d}\theta = 0, \; 0 \le m < n.$$

Ising model, $T > T_c$ $(A > 1)$.





$$f_{\mathrm{Is}}(\theta) = \tilde{f}(\theta)\mathrm{e}^{\mathrm{i}\theta}, \qquad \log \tilde{f} \text{ smooth}$$

$\beta = 1 \Rightarrow G(1-\beta) = 0 \Rightarrow E(f_{\mathrm{Is}}) = 0 \Rightarrow$ F-H do not apply.

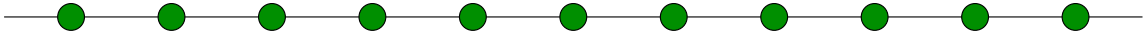
$$\boxed{D_n(f_{\mathrm{Is}}) = p_n(0)D_n(\tilde{f})}$$

$p_n(z) = z^n + \dots$, s. t.

$$\int_{-\pi}^{\pi} p_n(\mathrm{e}^{-\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}m\theta}\tilde{f}(\theta)\mathrm{d}\theta = 0, \;\; 0 \le m < n.$$

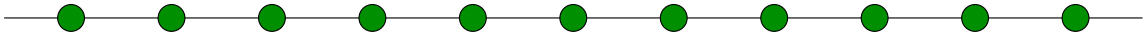
$$\langle \sigma_{0,0}\sigma_{n,n} \rangle = D_n(f_{\mathrm{Is}}) = \frac{\pi^{1/2}}{(1-A^{-2})^{1/4}}\frac{A^{-n}}{n^{1/2}}(1+o(1)), \quad A > 1$$

Fermionic chain



$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N}, \quad \{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

Fermionic chain



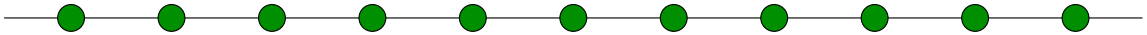
$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N}, \quad \{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

Quadratic, periodic, translational and parity invariant Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} \sum_{l=-L}^{L} \left(2A_l a_i^\dagger a_{i+l} + B_l a_i^\dagger a_{i+l}^\dagger - B_l a_i a_{i+l} \right)$$

$$A_l, B_l \in \mathbb{R}$$

Fermionic chain



$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N}, \quad \{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

Quadratic, periodic, translational and parity invariant Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} \sum_{l=-L}^{L} \left(2A_l a_i^\dagger a_{i+l} + B_l a_i^\dagger a_{i+l}^\dagger - B_l a_i a_{i+l} \right)$$

$$A_l, B_l \in \mathbb{R}$$

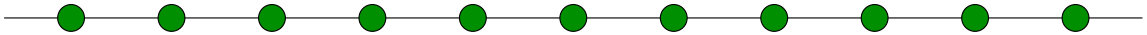
$$= \sum_{k=0}^{N-1} \Lambda(\theta_k)\, d_k^\dagger d_k, \quad \text{Bogoliubov modes.}$$

$$\Lambda(\theta) = \sqrt{\Theta(e^{i\theta})^2 - \Xi(e^{i\theta})^2},$$

$$\Theta(z) = \textstyle\sum_{-L}^{L} A_l z^l$$

$$\Xi(z) = \textstyle\sum_{-L}^{L} B_l z^l.$$

Fermionic chain



$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N}, \quad \{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

Quadratic, periodic, translational and parity invariant Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N}\sum_{l=-L}^{L} \left(2A_l a_i^\dagger a_{i+l} + B_l a_i^\dagger a_{i+l}^\dagger - B_l a_i a_{i+l}\right)$$

$$A_l, B_l \in \mathbb{R}$$

$$= \sum_{k=0}^{N-1} \Lambda(\theta_k)\, d_k^\dagger d_k, \quad \text{Bogoliubov modes.}$$

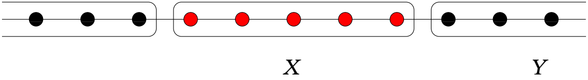
$$\Lambda(\theta) = \sqrt{\Theta(\mathrm{e}^{\mathrm{i}\theta})^2 - \Xi(\mathrm{e}^{\mathrm{i}\theta})^2},$$

$$\Theta(z) = \textstyle\sum_{-L}^{L} A_l z^l$$

$$\Xi(z) = \textstyle\sum_{-L}^{L} B_l z^l.$$

Ground state: $d_k \left|\mathrm{GS}\right\rangle = 0$

Entanglement entropy.



$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$$

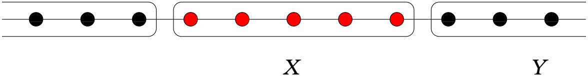
- Introduce the *reduced density matrix*

$$\rho_X = \text{Tr}_{\mathcal{H}_Y}(|\text{GS}\rangle\langle\text{GS}|).$$

- The Rényi entanglement entropy is given by

$$S_\alpha(X) = \frac{1}{1-\alpha}\log \text{Tr}(\rho_X^\alpha)$$

Entanglement entropy.



$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$$

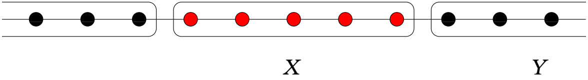
- Introduce the *reduced density matrix*

$$\rho_X = \mathsf{Tr}_{\mathcal{H}_Y}(|\mathrm{GS}\rangle\langle\mathrm{GS}|).$$

- The Rényi entanglement entropy is given by

$$S_\alpha(X) = \frac{1}{1-\alpha}\log \mathsf{Tr}(\rho_X^\alpha)$$

Entanglement entropy.



$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$$

- Introduce the *reduced density matrix*

$$\rho_X = \mathsf{Tr}_{\mathcal{H}_Y}(|\mathrm{GS}\rangle\langle\mathrm{GS}|).$$

- The Rényi entanglement entropy is given by

$$S_\alpha(X) = \frac{1}{1-\alpha}\log \mathsf{Tr}(\rho_X^\alpha)$$

Block Toeplitz matrix

Wick decomposition holds and $S_\alpha(X)$ can be obtained from the correlation matrix V_X

$$(V_X)_{ij} = \left\langle \mathrm{GS} \middle| \left[\begin{pmatrix} a_i \\ a_i^\dagger \end{pmatrix}, (a_j^\dagger, a_j) \right] \middle| \mathrm{GS} \right\rangle, \quad i, j \in X.$$

In the thermodynamic limit

$$(V_X)_{ij} = \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} M(\theta) \mathrm{e}^{\mathrm{i}\theta(i-j)} \mathrm{d}\theta.$$

A block Toeplitz matrix $T_n(M)$ with 2×2 symbol $M(\theta) = \mathcal{M}(\mathrm{e}^{\mathrm{i}\theta})$ where

$$\mathcal{M}(z) = \frac{\begin{pmatrix} \Theta(z) & \Xi(z) \\ -\Xi(z) & -\Theta(z) \end{pmatrix}}{\sqrt{\Xi(z)^2 - \Theta(z)^2}}$$

Block Toeplitz matrix

- Szegő Theorem for block Toeplitz matrices (Gyires, 1956)

$$\lim_{n\to\infty} \frac{1}{n} \log D_n(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\det M(\theta) \mathrm{d}\theta$$

Block Toeplitz matrix

- Szegő Theorem for block Toeplitz matrices (Gyires, 1956)

$$\lim_{n\to\infty} \frac{1}{n} \log D_n(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det M(\theta) \mathrm{d}\theta$$

In our case $\det M(\theta) = 1$ and the limit is 0, $o(n)$ corrections?

Block Toeplitz matrix

- Szegő Theorem for block Toeplitz matrices (Gyires, 1956)

$$\lim_{n\to\infty} \frac{1}{n} \log D_n(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\det M(\theta) \mathrm{d}\theta$$

In our case $\det M(\theta) = 1$ and the limit is 0, $o(n)$ corrections?

- Widom Theorem, 1974: For smooth enough $M(\theta)$

$$\log D_n(M) - \frac{n}{2\pi} \int_{-\pi}^{\pi} \log\det M(\theta) \mathrm{d}\theta$$

$$= \log\det \left[T(M) T(M^{-1}) \right] (1 + o(1))$$

Block Toeplitz matrix

- Szegő Theorem for block Toeplitz matrices (Gyires, 1956)

$$\lim_{n\to\infty} \frac{1}{n} \log D_n(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\det M(\theta) \mathrm{d}\theta$$

In our case $\det M(\theta) = 1$ and the limit is 0, $o(n)$ corrections?

- Widom Theorem, 1974: For smooth enough $M(\theta)$

$$\log D_n(M) - \frac{n}{2\pi} \int_{-\pi}^{\pi} \log\det M(\theta) \mathrm{d}\theta$$
$$= \log\det \left[T(M)T(M^{-1})\right] (1 + o(1))$$

It is hard to compute.

- It can be mapped into a Riemann-Hilbert problem.

- And can be solved for the non critical fermionic chain.

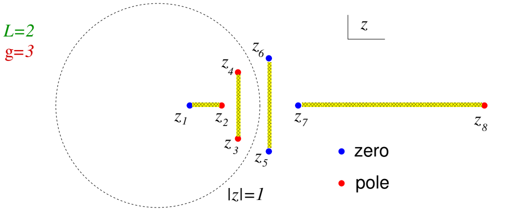
(Its, Jin, Korepin, 2007; Its, Mezzadri, Mo, 2008).

Non critical theories $(\Lambda(\theta) > 0)$

- $\mathcal{M}$: Meromorphic in a two-sheeted cover of the Riemann sphere with branch points at the zeros and poles of

$$\frac{\Xi(z)+\Theta(z)}{\Xi(z)-\Theta(z)}\,.$$

$$\mathcal{M}(z) = \frac{\begin{pmatrix} \Theta(z) & \Xi(z) \\ -\Xi(z) & -\Theta(z) \end{pmatrix}}{\sqrt{\Xi(z)^2-\Theta(z)^2}}$$



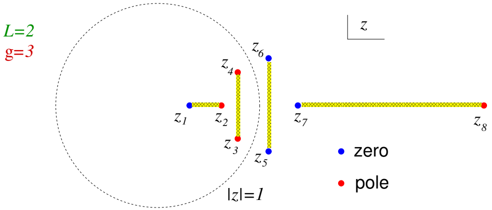
Non critical theories ($\Lambda(\theta) > 0$)

- $\mathcal{M}$: Meromorphic in a two-sheeted cover of the Riemann sphere with branch points at the zeros and poles of

$$\frac{\Xi(z)+\Theta(z)}{\Xi(z)-\Theta(z)}.$$

$$\mathcal{M}(z) = \frac{\begin{pmatrix} \Theta(z) & \Xi(z) \\ -\Xi(z) & -\Theta(z) \end{pmatrix}}{\sqrt{\Xi(z)^2 - \Theta(z)^2}}$$

- They are related by inversion and conjugation, e.g. $z_3 = \overline{z}_4 = z_6^{-1}$.



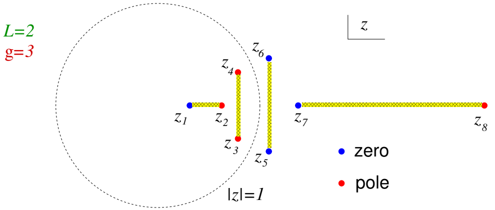
Non critical theories ($\Lambda(\theta) > 0$)

- $\mathcal{M}$: Meromorphic in a two-sheeted cover of the Riemann sphere with branch points at the zeros and poles of

$$\frac{\Xi(z) + \Theta(z)}{\Xi(z) - \Theta(z)} .$$

$$\mathcal{M}(z) = \frac{\begin{pmatrix} \Theta(z) & \Xi(z) \\ -\Xi(z) & -\Theta(z) \end{pmatrix}}{\sqrt{\Xi(z)^2 - \Theta(z)^2}}$$

- They are related by inversion and conjugation, e.g. $z_3 = \overline{z}_4 = z_6^{-1}$.



- Assume no degeneracy: $z_i \neq z_j \Rightarrow |z_i| \neq 1$.

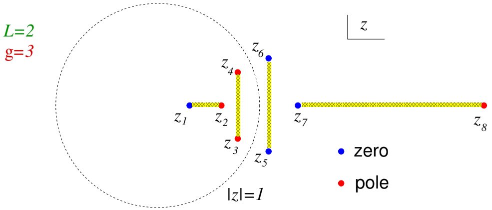
Non critical theories ($\Lambda(\theta) > 0$)

- $\mathcal{M}$: Meromorphic in a two-sheeted cover of the Riemann sphere with branch points at the zeros and poles of

$$\frac{\Xi(z) + \Theta(z)}{\Xi(z) - \Theta(z)}.$$

$$\mathcal{M}(z) = \frac{\begin{pmatrix} \Theta(z) & \Xi(z) \\ -\Xi(z) & -\Theta(z) \end{pmatrix}}{\sqrt{\Xi(z)^2 - \Theta(z)^2}}$$

- They are related by inversion and conjugation, e.g. $z_3 = \overline{z}_4 = z_6^{-1}$.



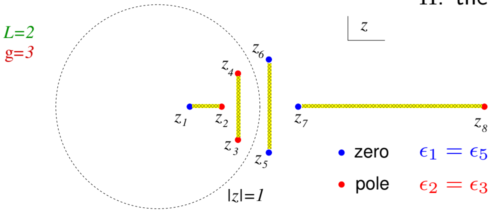
- Assume no degeneracy: $z_i \neq z_j \Rightarrow |z_i| \neq 1$.

- Riemann surface of genus $\mathrm{g} = 2L - 1$; $4L$ branch points.

Non critical theories ($\Lambda(\theta) > 0$)

$$\vartheta\left[\begin{matrix}\vec{p}\\ \vec{q}\end{matrix}\right](\vec{s}) = \sum_{\vec{n}\in\mathbb{Z}^{\mathrm{g}}} \mathrm{e}^{\pi\mathrm{i}(\vec{n}+\vec{p})\Pi\cdot(\vec{n}+\vec{p})+2\pi\mathrm{i}(\vec{s}+\vec{q})\cdot(\vec{n}+\vec{p})},$$

Π: the standard period matrix.



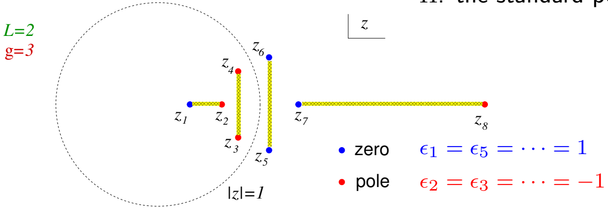
$\epsilon_1 = \epsilon_5 = \cdots = 1$

$\epsilon_2 = \epsilon_3 = \cdots = -1$

Non critical theories ($\Lambda(\theta) > 0$)

$$\vartheta\begin{bmatrix}\vec{p}\\ \vec{q}\end{bmatrix}(\vec{s}) = \sum_{\vec{n}\in\mathbb{Z}^{\mathrm{g}}} \mathrm{e}^{\pi \mathrm{i}(\vec{n}+\vec{p})\Pi\cdot(\vec{n}+\vec{p})+2\pi\mathrm{i}(\vec{s}+\vec{q})\cdot(\vec{n}+\vec{p})},$$

Π: the standard period matrix.



$$D(M) \equiv \lim_{n\to\infty} D_n(M) = \frac{\vartheta\begin{bmatrix}\vec{\mu}\\ \vec{\nu}\end{bmatrix}(\vec{e}/2)\ \vartheta\begin{bmatrix}\vec{\mu}\\ \vec{\nu}\end{bmatrix}(-\vec{e}/2)}{\vartheta\begin{bmatrix}\vec{\mu}\\ \vec{\nu}\end{bmatrix}(0)^2}$$

$$\vec{e} = (\overbrace{0,0,\ldots,0}^{L-1}\overbrace{1,1,\ldots,1}^{L}),$$

$$\mu_r = \frac{1}{4}(\epsilon_{2r+1} + \epsilon_{2r+2})$$

$$\nu_r = \frac{1}{4}\sum_{j=2}^{2r+1}\epsilon_j, \quad r = 1,\ldots,2L-1.$$

(Ares, Esteve, F.F. , Queiroz, 2017)

Möbius transformations

$$z' = \frac{az+b}{cz+d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

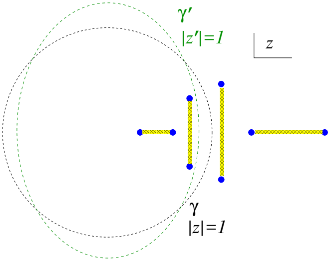
$$\mathcal{M}'(z') = \mathcal{M}(z)$$

Möbius transformations

$$z' = \frac{az+b}{cz+d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

$$\mathcal{M}'(z') = \mathcal{M}(z)$$

- Automorphisms of the Riemann surfaces.
- Preserve the period matrix $\Pi' = \Pi$ and also $\vartheta\left[\begin{smallmatrix}\vec{p}\\ \vec{q}\end{smallmatrix}\right](\vec{s})$

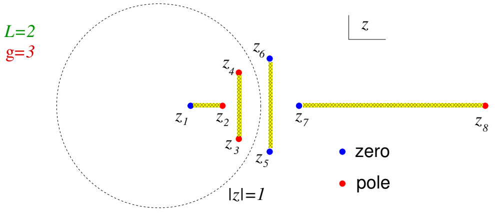


- Provided γ' can be continuously deformed to γ without crossing branch points

$$D(M) = D(M')$$

Möbius transformations

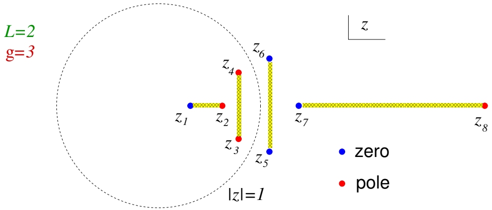
But a physical Möbius transformation should preserve the relations between the branch points, e.g $z_3 = \overline{z}_4 = z_6^{-1}$.



It implies that it should commute with inversion and conjugation, equivalently, it should preserve the real line and the unit circle.

Möbius transformations

But a physical Möbius transformation should preserve the relations between the branch points, e.g $z_3 = \overline{z}_4 = z_6^{-1}$.



It implies that it should commute with inversion and conjugation, equivalently, it should preserve the real line and the unit circle.

Therefore we are left with transformations in $SO(1,1) \subset SL(2,\mathbb{C})$

$$z' = \frac{z \cosh\zeta + \sinh\zeta}{z \sinh\zeta + \cosh\zeta}$$

Möbius transformations

$$SO(1,1) \subset SL(2,\mathbb{C}) \qquad z' = \frac{z\cosh\zeta + \sinh\zeta}{z\sinh\zeta + \cosh\zeta}$$

Möbius transformations

$$SO(1,1) \subset SL(2,\mathbb{C}) \qquad z' = \frac{z\cosh\zeta + \sinh\zeta}{z\sinh\zeta + \cosh\zeta}$$

Couplings A_l, B_l: spin L representation of $SL(2,\mathbb{C})$

$$\begin{pmatrix} A'_L \\ \vdots \\ A'_0 \\ \vdots \\ A'_{-L} \end{pmatrix} = \mathrm{e}^{\zeta \cdot (J_x)_L} \begin{pmatrix} A_L \\ \vdots \\ A_0 \\ \vdots \\ A_{-L} \end{pmatrix}, \qquad \begin{pmatrix} B'_L \\ \vdots \\ B'_0 \\ \vdots \\ B'_{-L} \end{pmatrix} = \mathrm{e}^{\zeta \cdot (J_x)_L} \begin{pmatrix} B_L \\ \vdots \\ B_0 \\ \vdots \\ B_{-L} \end{pmatrix}$$

Recall: $H = \frac{1}{2}\sum_{i=1}^{N}\sum_{l=-L}^{L}\left[A_l a_i^\dagger a_{i+l} + B_l a_i^\dagger a_{i+l}^\dagger - B_l a_i a_{i+l}\right]$

Möbius transformations

$$SO(1,1) \subset SL(2,\mathbb{C}) \qquad z' = \frac{z\cosh\zeta + \sinh\zeta}{z\sinh\zeta + \cosh\zeta}$$

Couplings A_l, B_l: spin L representation of $SL(2,\mathbb{C})$

$$\begin{pmatrix} A'_L \\ \vdots \\ A'_0 \\ \vdots \\ A'_{-L} \end{pmatrix} = \mathrm{e}^{\zeta \cdot (J_x)_L} \begin{pmatrix} A_L \\ \vdots \\ A_0 \\ \vdots \\ A_{-L} \end{pmatrix}, \quad \begin{pmatrix} B'_L \\ \vdots \\ B'_0 \\ \vdots \\ B'_{-L} \end{pmatrix} = \mathrm{e}^{\zeta \cdot (J_x)_L} \begin{pmatrix} B_L \\ \vdots \\ B_0 \\ \vdots \\ B_{-L} \end{pmatrix}$$

Recall: $H = \frac{1}{2}\sum_{i=1}^{N}\sum_{l=-L}^{L}\left[A_l a_i^\dagger a_{i+l} + B_l a_i^\dagger a_{i+l}^\dagger - B_l a_i a_{i+l}\right]$

$$\boxed{D(M) = D(M') \quad \Rightarrow \quad S_\alpha = S'_\alpha \text{ for } |X| \to \infty}$$

(Ares, Esteve, F.F. , Queiroz, 2016)

Critical theories.

When $\Lambda(\theta_r) = 0$, $M(\theta)$ has a jump discontinuity at θ_r.

Not covered by Widom theorem.

Critical theories.

When $\Lambda(\theta_r) = 0$, $M(\theta)$ has a jump discontinuity at θ_r.

Not covered by Widom theorem.

- In the scalar case we can apply the Fisher-Hartwig conjecture:

f piece wise smooth, lateral limits f_{r-} and f_{r+} at discontinuity θ_r.

$$\log D_n(f) = s_0 n + \frac{1}{4\pi^2} \sum_{r=1}^{R} \log \left(f_{r+}/f_{r-}\right)^2 \log n + O(1)$$

Critical theories.

When $\Lambda(\theta_r) = 0$, $M(\theta)$ has a jump discontinuity at θ_r.

Not covered by Widom theorem.

• In the scalar case we can apply the Fisher-Hartwig conjecture:

f piece wise smooth, lateral limits f_{r-} and f_{r+} at discontinuity θ_r.

$$\log D_n(f) = s_0 n + \frac{1}{4\pi^2} \sum_{r=1}^{R} \log\left(f_{r+}/f_{r-}\right)^2 \log n + O(1)$$

• When the symbol is a matrix M with jumps at θ_r a similar expression holds: (Ares, Esteve, F., Queiroz, 2018)

$$\log D_n(M) = s_0 n + \frac{1}{4\pi^2} \sum_{r=1}^{R} \mathrm{Tr}\left(\log\left[M_{r+}(M_{r-})^{-1}\right]\right)^2 \log n + O(1)$$

$$s_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det M(\theta)\mathrm{d}\theta, \qquad M_{r\pm} = \lim_{\theta\to\theta_r^{\pm}} M(\theta)$$

Sublogarithmic scaling

Back to the scalar symbol $f : S^1 \to \mathbb{C}$.

For piecewise smooth $f(\theta)$ with geometric average 1 ($s_0 = 0$)

$$\log D_n(f) = c \log n + O(1).$$

Sublogarithmic scaling

Back to the scalar symbol $f : S^1 \to \mathbb{C}$.

For piecewise smooth $f(\theta)$ with geometric average 1 ($s_0 = 0$)

$$\log D_n(f) = c \log n + O(1).$$

Is it possible to have

$$\log D_n(f) = c\,(\log n)^{\mu} + O(1), \quad 0 < \mu < 1$$

or

$$\log D_n(f) = c \log \log n + O(1)$$?

Sublogarithmic scaling

Back to the scalar symbol $f : S^1 \to \mathbb{C}$.

For piecewise smooth $f(\theta)$ with geometric average 1 ($s_0 = 0$)

$$\log D_n(f) = c \log n + O(1).$$

Is it possible to have

$$\log D_n(f) = c\,(\log n)^{\mu} + O(1), \quad 0 < \mu < 1$$

or

$$\log D_n(f) = c \log\log n + O(1)$$?

Motivation:

- For fun.

Sublogarithmic scaling

Back to the scalar symbol $f : S^1 \to \mathbb{C}$.

For piecewise smooth $f(\theta)$ with geometric average 1 ($s_0 = 0$)

$$\log D_n(f) = c \log n + O(1).$$

Is it possible to have

$$\log D_n(f) = c\,(\log n)^{\mu} + O(1), \quad 0 < \mu < 1$$

or

$$\log D_n(f) = c \log\log n + O(1)$$?

Motivation:

- For fun.
- It is a challenge.

Sublogarithmic scaling

Back to the scalar symbol $f : S^1 \to \mathbb{C}$.

For piecewise smooth $f(\theta)$ with geometric average 1 ($s_0 = 0$)

$$\log D_n(f) = c \log n + O(1).$$

Is it possible to have

$$\log D_n(f) = c\,(\log n)^{\mu} + O(1), \quad 0 < \mu < 1$$

or

$$\log D_n(f) = c \log\log n + O(1) \text{ ?}$$

Motivation:

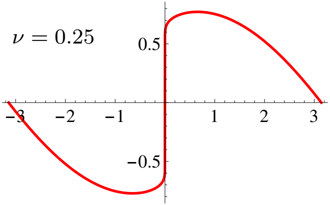
- For fun.
- It is a challenge.
- Anomalous scaling in non unitary conformal field theories.

Sublogarithmic scaling

Consider the family of functions

$$\log f_\nu(\theta) = \frac{\cos(\theta/2)\,\mathrm{sgn}(\theta)}{\left(-\log \frac{|\theta|}{2\pi}\right)^{\nu}},$$

$\theta \in (-\pi, \pi]$

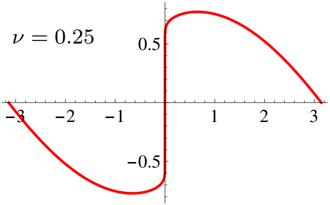


Sublogarithmic scaling

Consider the family of functions

$$\log f_\nu(\theta) = \frac{\cos(\theta/2)\,\mathrm{sgn}(\theta)}{\left(-\log \frac{|\theta|}{2\pi}\right)^\nu},$$

$$\theta \in (-\pi, \pi]$$



$$s_k \sim \frac{1}{\pi k (\log |k|)^\nu} \quad \Rightarrow \quad \sum_{k=1}^{\infty} |k||s_k|^2 = \infty \text{ for } \nu \le 0.5$$

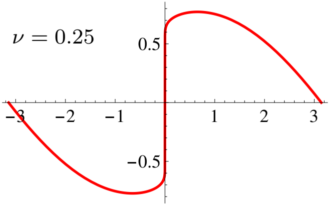
- Szegő strong limit theorem can not be applied.

Sublogarithmic scaling

Consider the family of functions

$$\log f_\nu(\theta) = \frac{\cos(\theta/2)\,\mathrm{sgn}(\theta)}{\left(-\log\frac{|\theta|}{2\pi}\right)^\nu},$$

$\theta \in (-\pi, \pi]$



$$s_k \sim \frac{1}{\pi k(\log|k|)^\nu} \quad \Rightarrow \quad \sum_{k=1}^{\infty} |k||s_k|^2 = \infty \text{ for } \nu \leq 0.5$$

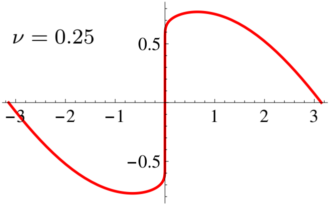
- Szegő strong limit theorem can not be applied.
- f_ν, for $\nu > 0$, is continuous and F-H formula does not apply.

Sublogarithmic scaling

Consider the family of functions

$$\log f_\nu(\theta) = \frac{\cos(\theta/2)\,\mathrm{sgn}(\theta)}{\left(-\log\frac{|\theta|}{2\pi}\right)^\nu},$$

$$\theta \in (-\pi, \pi]$$



$$s_k \sim \frac{1}{\pi k (\log|k|)^\nu} \quad \Rightarrow \quad \sum_{k=1}^{\infty} |k||s_k|^2 = \infty \text{ for } \nu \leq 0.5$$

- Szegő strong limit theorem can not be applied.
- f_ν, for $\nu > 0$, is continuous and F-H formula does not apply.

We conjecture that there are positive $Z(\nu)$ and $\delta(\nu)$, such that:

$$\log D_n(f_\nu) = \sum_{k=1}^{\lfloor n Z \rfloor} k s_k s_{-k} + o(n^{-\delta})$$

Sublogarithmic scaling

- $\log D_n(f_\nu) = \displaystyle\sum_{k=1}^{\lfloor n\,Z \rfloor} k s_k s_{-k} + o(n^{-\delta})$ implies sublogarithmic scaling:
  - $\log D_n(f_\nu) = \dfrac{1}{\pi^2(1-2\nu)}(\log n)^{1-2\nu} + o(1),\ 0 < \nu < 0.5$
  - $\log D_n(f_{0.5}) = \dfrac{1}{\pi^2}\log\log n + o(1),\ \nu = 0.5$

Sublogarithmic scaling

- $\log D_n(f_\nu) = \sum_{k=1}^{\lfloor n\, Z \rfloor} k s_k s_{-k} + o(n^{-\delta})$ implies sublogarithmic scaling:
  - $\log D_n(f_\nu) = \frac{1}{\pi^2(1-2\nu)}(\log n)^{1-2\nu} + o(1),\ 0 < \nu < 0.5$
  - $\log D_n(f_{0.5}) = \frac{1}{\pi^2} \log\log n + o(1),\ \nu = 0.5$
- For $\nu = 0$ we reproduce the Fisher-Hartwig formula with

$$\log Z(0) = 2\pi^2 \log |G(1 + \mathrm{i}/\pi)| - \gamma_{\mathrm{E}} \approx 0.9424\ldots$$

Sublogarithmic scaling

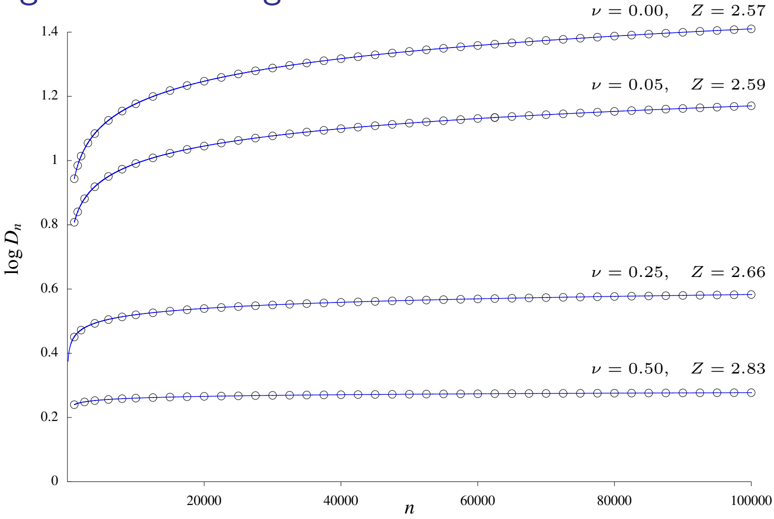
- $\log D_n(f_\nu) = \sum_{k=1}^{\lfloor n\,Z \rfloor} k s_k s_{-k} + o(n^{-\delta})$ implies sublogarithmic scaling:
  - $\log D_n(f_\nu) = \frac{1}{\pi^2(1-2\nu)}(\log n)^{1-2\nu} + o(1),\ 0 < \nu < 0.5$
  - $\log D_n(f_{0.5}) = \frac{1}{\pi^2} \log\log n + o(1),\ \nu = 0.5$

- For $\nu = 0$ we reproduce the Fisher-Hartwig formula with

  $\log Z(0) = 2\pi^2 \log |G(1 + \mathrm{i}/\pi)| - \gamma_{\mathrm{E}} \approx 0.9424\ldots$

- Supported by numerical checks.

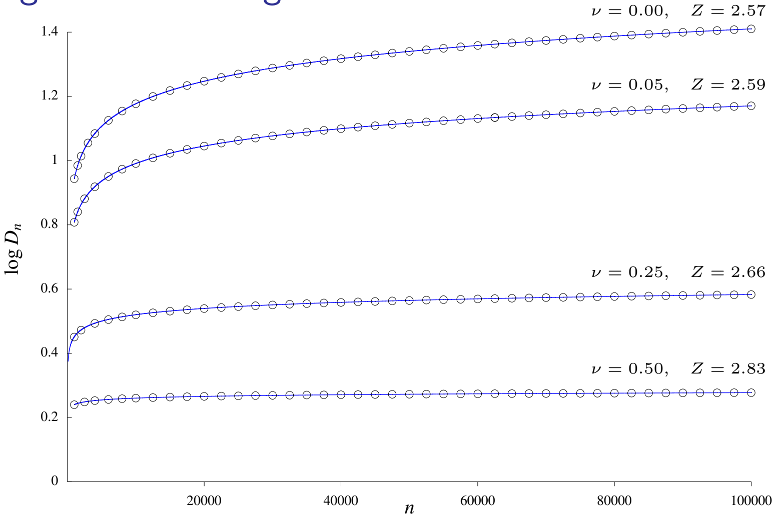
Sublogarithmic scaling



- Dots represent $\log D_n(f_\nu)$ for different values of ν and n up to 100 000.

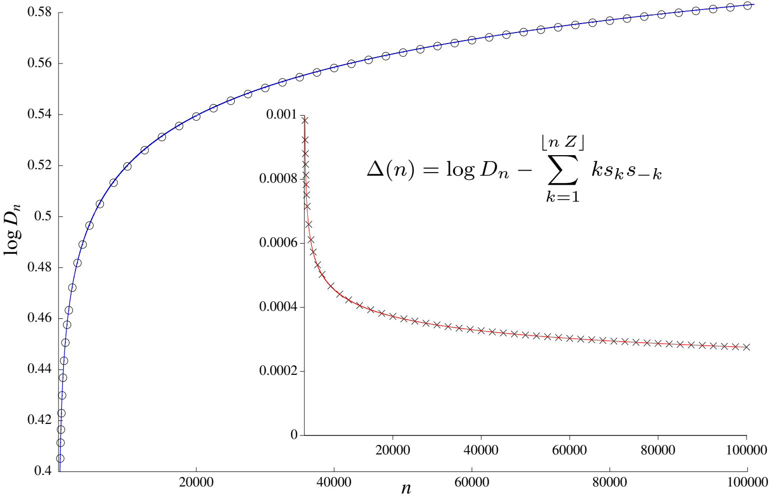
- The continuous lines are $\sum_{k=1}^{\lfloor n\,Z \rfloor} k s_k s_{-k}$ for every ν and Z from the best fit.

Sublogarithmic scaling



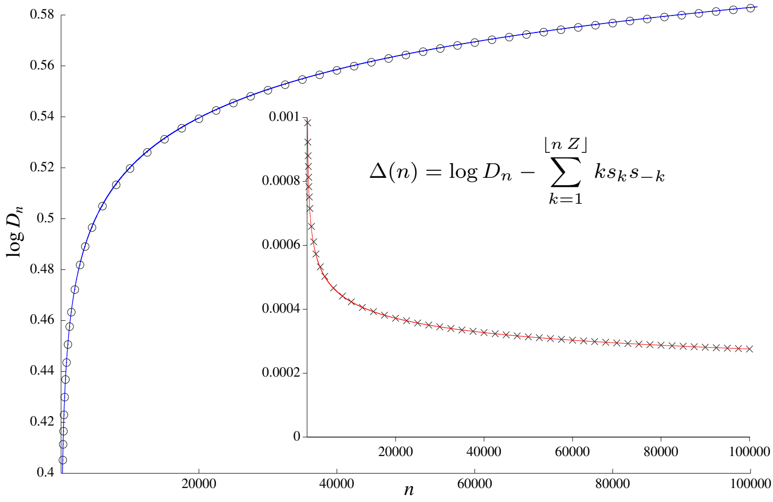
- Note that adjusting only one free parameter, Z, we obtain an excellent agreement.

Sublogarithmic scaling



Main plot: $\log D_n(f_\nu)$ for $\nu = 0.25$ (dots) and $\sum_{k=1}^{\lfloor n Z \rfloor} k s_k s_{-k}$ (continuous line).

Sublogarithmic scaling



Inset: - crosses are $\Delta(n)$, the difference between the real value and the prediction.

- continuous line is our best fit, $\Delta(n) \approx \dfrac{2.35 \times 10^{-3}}{n^{0.186}}$

Sublogarithmic scaling

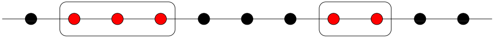
We expect this behavior in the entanglement entropy of fermionic chains with long range couplings.

$$
\begin{aligned}
H &= \sum_{i=1}^{N} \left(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i + h\ a_i^\dagger a_i \right) \\
&\quad + 2 \sum_{i=1}^{N} \sum_{l=1}^{N/2} \frac{1}{l(\log l)^\nu} (a_i^\dagger a_{i+l}^\dagger - a_i a_{i+l}).
\end{aligned}
$$

In this case, we should have

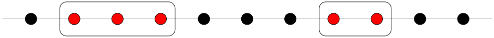
$$
\begin{aligned}
S_\alpha(X) &= c(\log |X|)^{1-2\nu} + o(1),\ \text{for } 0 \leq \nu < 0.5 \\
S_\alpha(X) &= c \log\log |X| + o(1),\ \text{for } \nu = 0.5
\end{aligned}
$$

Several Intervals.



Consider now $X = (u_1, v_1) \cup (u_2, v_2) \cup \cdots \cup (u_P, v_P)$

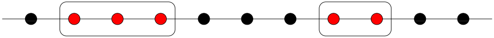
Several Intervals.



Consider now $X = (u_1, v_1) \cup (u_2, v_2) \cup \cdots \cup (u_P, v_P)$

$V_X = \langle \mathrm{GS} | \, [a_i^\dagger, a_j] \, | \mathrm{GS} \rangle \,, \; i, j \in X$ is not a Toeplitz matrix, but a principal submatrix;

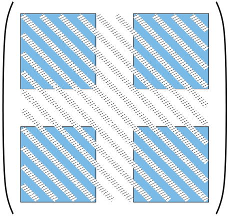
Several Intervals.



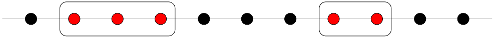
Consider now $X = (u_1, v_1) \cup (u_2, v_2) \cup \cdots \cup (u_P, v_P)$

$V_X = \langle \mathrm{GS} | [a_i^\dagger, a_j] | \mathrm{GS} \rangle , \; i, j \in X$ is not a Toeplitz matrix, but a principal submatrix;

e. g. for two intervals V_X is the shaded area of the Toeplitz matrix on the right.

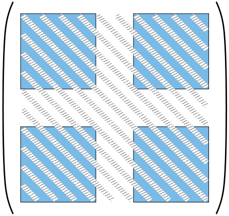


Several Intervals.



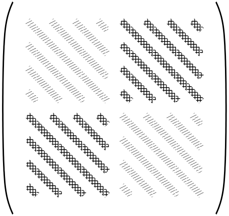
Consider now $X = (u_1, v_1) \cup (u_2, v_2) \cup \cdots \cup (u_P, v_P)$

$V_X = \langle \mathrm{GS} | [a_i^\dagger, a_j] | \mathrm{GS} \rangle \,, \; i, j \in X$ is not a Toeplitz matrix, but a principal submatrix;

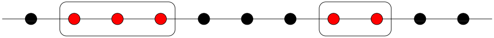


e. g. for two intervals V_X is the shaded area of the Toeplitz matrix on the right.

The correlation matrix is

$V_X =$ 

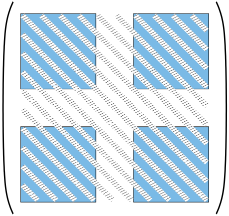
Several Intervals.



Consider now $X = (u_1, v_1) \cup (u_2, v_2) \cup \cdots \cup (u_P, v_P)$

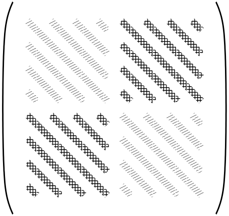
$V_X = \langle \mathrm{GS} | \, [a_i^\dagger, a_j] \, | \mathrm{GS} \rangle \, , \; i, j \in X$ is not a Toeplitz matrix, but a principal submatrix;

e. g. for two intervals V_X is the shaded area of the Toeplitz matrix on the right.



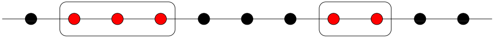
The correlation matrix is

$V_X =$



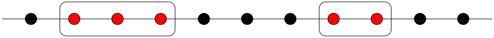
Inspired by conformal field theories, we conjecture...

Several Intervals.



...for the determinant of a principal submatrix of a Toeplitz matrix:

Several Intervals.

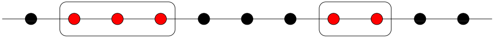


...for the determinant of a principal submatrix of a Toeplitz matrix:

$$D[\bigcup_{p=1}^{P}(u_p, v_p)] \simeq \prod_{p} D[(u_p, v_p)] \prod_{p<p'} \frac{D[(u_p, v_{p'})]D[(v_p, u_{p'})]}{D[(u_p, u_{p'})]D[(v_p, v_{p'})]},$$

$D[X] := \det V_X$

Several Intervals.

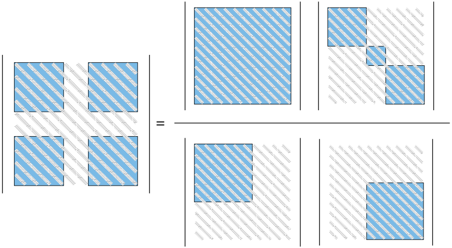


...for the determinant of a principal submatrix of a Toeplitz matrix:

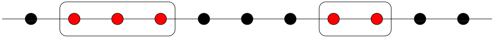
$$D[\bigcup_{p=1}^{P}(u_p, v_p)] \simeq \prod_p D[(u_p, v_p)] \prod_{p<p'} \frac{D[(u_p, v_{p'})]D[(v_p, u_{p'})]}{D[(u_p, u_{p'})]D[(v_p, v_{p'})]},$$

$$D[X] := \det V_X$$

Pictorially, for $P = 2$:



Several Intervals.

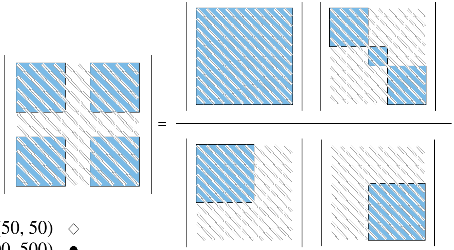


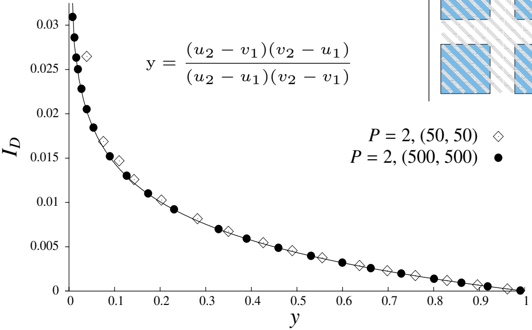
...for the determinant of a principal submatrix of a Toeplitz matrix:

$$D[\bigcup_{p=1}^{P}(u_p, v_p)] \simeq \prod_{p} D[(u_p, v_p)] \prod_{p<p'} \frac{D[(u_p, v_{p'})]D[(v_p, u_{p'})]}{D[(u_p, u_{p'})]D[(v_p, v_{p'})]},$$

$D[X] := \det V_X$

Pictorially, for $P = 2$:





Remarkable agreement!!!

(Ares, Esteve, F., 2014)

Conclusions

- We have shown how the theory of Toeplitz determinants has been boosted by physicists' demands.
- We obtained a compact expression for the determinant of the correlation function of the fermionic chain with finite range coupling.
- We discussed the role of Möbius transformations as symmetries of the Toeplitz determinants and its implications for the fermionic chain.
- We presented a conjecture on the sublogarithmic scaling of Toeplitz determinants and showed its numerical accuracy.
- Based on the results for Conformal Field Theory we have proposed an asymptotic formula for the determinant of a principal subamtrix of the Toeplitz matrix.

Conclusions

- We have shown how the theory of Toeplitz determinants has been boosted by physicists' demands.

- We obtained a compact expression for the determinant of the correlation function of the fermionic chain with finite range coupling.

- We discussed the role of Möbius transformations as symmetries of the Toeplitz determinants and its implications for the fermionic chain.

- We presented a conjecture on the sublogarithmic scaling of Toeplitz determinants and showed its numerical accuracy.

- Based on the results for Conformal Field Theory we have proposed an asymptotic formula for the determinant of a principal subamtrix of the Toeplitz matrix.

Conclusions

- We have shown how the theory of Toeplitz determinants has been boosted by physicists' demands.
- We obtained a compact expression for the determinant of the correlation function of the fermionic chain with finite range coupling.
- We discussed the role of Möbius transformations as symmetries of the Toeplitz determinants and its implications for the fermionic chain.
- We presented a conjecture on the sublogarithmic scaling of Toeplitz determinants and showed its numerical accuracy.
- Based on the results for Conformal Field Theory we have proposed an asymptotic formula for the determinant of a principal subamtrix of the Toeplitz matrix.

Conclusions

- We have shown how the theory of Toeplitz determinants has been boosted by physicists' demands.
- We obtained a compact expression for the determinant of the correlation function of the fermionic chain with finite range coupling.
- We discussed the role of Möbius transformations as symmetries of the Toeplitz determinants and its implications for the fermionic chain.
- We presented a conjecture on the sublogarithmic scaling of Toeplitz determinants and showed its numerical accuracy.
- Based on the results for Conformal Field Theory we have proposed an asymptotic formula for the determinant of a principal subamtrix of the Toeplitz matrix.

Conclusions

- We have shown how the theory of Toeplitz determinants has been boosted by physicists' demands.
- We obtained a compact expression for the determinant of the correlation function of the fermionic chain with finite range coupling.
- We discussed the role of Möbius transformations as symmetries of the Toeplitz determinants and its implications for the fermionic chain.
- We presented a conjecture on the sublogarithmic scaling of Toeplitz determinants and showed its numerical accuracy.
- Based on the results for Conformal Field Theory we have proposed an asymptotic formula for the determinant of a principal subamtrix of the Toeplitz matrix.

Conclusions

- We have shown how the theory of Toeplitz determinants has been boosted by physicists' demands.
- We obtained a compact expression for the determinant of the correlation function of the fermionic chain with finite range coupling.
- We discussed the role of Möbius transformations as symmetries of the Toeplitz determinants and its implications for the fermionic chain.
- We presented a conjecture on the sublogarithmic scaling of Toeplitz determinants and showed its numerical accuracy.
- Based on the results for Conformal Field Theory we have proposed an asymptotic formula for the determinant of a principal subamtrix of the Toeplitz matrix.

HAPPY BIRTHDAY

ALBERTO

Möbius transformations in critical theories: $\Lambda(\theta_r) = 0$.

$SO(1,1) \subset SL(2,\mathbb{C}) \qquad z' = \dfrac{z\cosh\zeta + \sinh\zeta}{z\sinh\zeta + \cosh\zeta}$

For critical theories the Toeplitz determinant is not invariant.

Conjecture For the fermionic chain it transforms as an homogeneous function.

- M has jump discontinuities at θ_r. Call $u_r = \mathrm{e}^{\mathrm{i}\theta_r}$.
- $M_{r\pm}$ lateral limits at θ_r.
- $\delta_r = \frac{1}{4\pi^2}\,\mathrm{Tr}\left(\log\left[M_{r+}(M_{r-})^{-1}\right]\right)^2$

$$D_n(M') = \prod_r \left(\frac{\partial u'_r}{\partial u_r}\right)^{\delta_r} D_n(M)(1+o(1))$$

Checked analytically in particular cases and in numerical simulations.