Waves on asymptotically AdS spacetimes and Einstein metrics with prescribed conformal infinity

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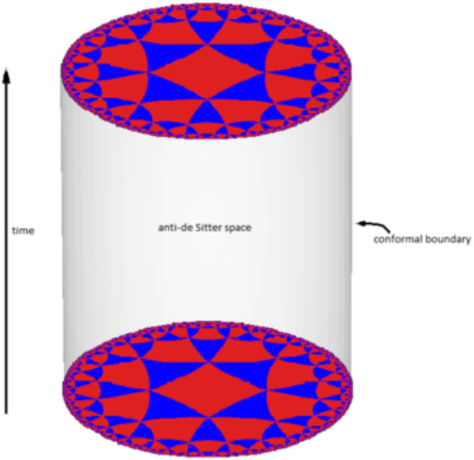
Anti-de Sitter space

AdS_{n+1} is the simply connected $(n+1)$-dimensional manifold with a Lorentzian metric of constant curvature $\mathrm{Sec} = -1$.



Anti-de Sitter space: some formulas

- $g_{\mathrm{AdS}} = -(1+r^2)dt^2 + \dfrac{dr^2}{1+r^2} + r^2 g_{\mathbb{S}^{n-1}}$ with $(t,r,\theta) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{n-1}$.
- $g_{\mathrm{AdS}} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, g_{\mathbb{S}^{n-1}}$ with $\rho := \sinh^{-1} r \in \mathbb{R}^+$.
- $g_{\mathrm{AdS}} = \dfrac{-(1+y^2/4)\,dt^2 + dy^2 + (1+y^2/4)^{-1}\, g_{\mathbb{S}^{n-1}}}{y^2}$ with

  $y := \dfrac{1}{r+\sqrt{r^2+1}} \in (0,1]$.
- Conformal boundary:

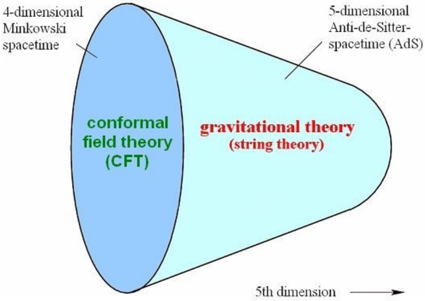
  $$\{y=0\} = \mathbb{R}\times\mathbb{S}^{n-1}\,.$$

  This is a timelike cylinder with the canonical metric $-dt^2 + g_{\mathbb{S}^{n-1}}$.
- Question: What can we say about asymptotically AdS spaces, that is, spacetimes that behave "at spacelike infinity" ("at $y=0$") essentially as AdS? Of course, they won't have constant curvature, but can they be Einstein? What do they look like?

Some motivation: the holographic principle

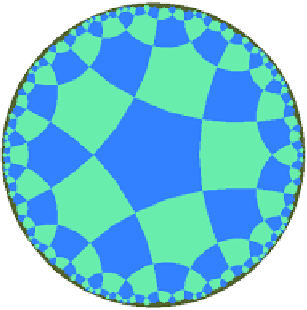
The AdS/CFT correspondence (Maldacena, 1998)

A “gravity field” in an AdS-type bulk is holographically determined by a “conformal gauge field” on the boundary and there is a certain relation between their correlation functions.



Examples of the holographic principle (borrowed from Witten 1998)

Things are explained in the Wick-rotated case: one considers the Riemannian counterpart of AdS, namely the hyperbolic space $\mathbb{H}^{n+1}$.



- $g_{\mathbb{H}} = \dfrac{4|dz|^2}{(1-|z|^2)^2}$ with $z \in \mathbb{B}^{n+1}$, the unit ball.
- $g_{\mathbb{H}} = \dfrac{(1-2y)^{-1}\,dy^2 + (1-2y)\,g_{\mathbb{S}^n}}{y^2}$ with $y \in (0, \frac{1}{2}]$, $\theta \in \mathbb{S}^n$.
- Conformal boundary: $\{y = 0\} = \mathbb{S}^n$ with the round metric $g_{\mathbb{S}^n}$.

First example: the Riemannian scalar field

- Basic model: harmonic functions in the $(n+1)$-dim'l hyperbolic space $\mathbb{H}^{n+1}$:

$$\Delta_{\mathbb{H}} \phi = 0 \,, \qquad \phi|_{\partial \mathbb{H}^{n+1}} = f \,.$$

  There is an explicit formula for the solution, given by the Poisson kernel

$$\phi(z) = \int_{\mathbb{S}^n} \left(\frac{1 - |z|^2}{|z - z'|^2} \right)^n f(z') \, d\sigma(z') \,.$$

- Harmonic functions in more general spaces, such as any open, simply connected manifold with $-a \leqslant K \leqslant -b$ (Anderson and Sullivan 1983).
  - Proof: Curvature yields sub- and supersolutions; convergence using Harnack inequalities.
- The equation

$$\Delta_g \phi - \sigma \phi = 0$$

  requires a different boundary condition:

$$y^\alpha \phi|_{y=0} = f \,, \qquad \text{with } \alpha = \alpha(\sigma).$$

Second example: the Riemannian Einstein equations

Question ("Holography for Riemannian Einstein")

Let $\widehat{g}$ be a metric on $\mathbb{S}^n$ close to the canonical one, $g_{\mathbb{S}^n}$. Is there an Einstein metric g on $\mathbb{B}^{n+1}$, close to the hyperbolic one $g_{\mathbb{H}}$, whose conformal infinity is given by $[\widehat{g}]$?

Theorem (Graham & Lee, 1990) – Riemannian Einstein metrics with prescribed conformal infinity

For any $\epsilon > 0$ there exists $\delta > 0$ such that, if $\widehat{g}$ is a metric on $\mathbb{S}^n$ with $\|\widehat{g} - g_{\mathbb{S}^n}\|_{C^{k,\gamma}} < \delta$, there is an Einstein metric g on $\mathbb{B}^{n+1}$ satisfying

$$\mathrm{Ric}_g = -ng\,,$$

with $\|g - g_{\mathbb{H}}\|_{C^{k,\gamma}_{\alpha}} < \epsilon$ and such that its conformal infinity is $[h]$.

Holography for the (Lorentzian) Einstein equations

Setting for the equations

- Einstein equations with a negative cosmological constant:

$$R_{\mu\nu} = -n g_{\mu\nu} \,.$$

- Initial data satisfying the constraint equations:
  - An n-dimensional asymptotically hyperbolic Riemannian manifold $(M, \widetilde{g}_{ij})$. This means that there is a "boundary defining function" y on M as before such that $y^2 \, \widetilde{g}_{ij}$ is smooth enough up to the boundary ∂M.
  - Symmetric tensor K_{ij} on M (the second fundamental form) such that $y^2 K_{ij}$ is smooth enough up to ∂M.
- Boundary datum: A compatible smooth enough Lorentzian metric $\widehat{g}$ on $\mathbb{R} \times \partial M$.

Basic example: In AdS_{n+1}, $M = \mathbb{B}^n$, $\widetilde{g}$ is the hyperbolic metric, $K = 0$, and $\widehat{g} = -dt^2 + g_{\mathbb{S}^{n-1}}$ is the canonical Lorentzian metric on the cylinder.

Are there any solutions to the compatibility and constraint equations?

Compatibility is easy, but the constraint equations are subtle ...

AH Solutions to the constraint equations

The Riemannian metric $\widetilde{g}$ and the symmetric tensor K must satisfy

$$\widetilde{R} - K_{ij}K^{ij} + (\mathrm{tr}_{\widetilde{g}}\, K)^2 = -n(n-1)\,,$$
$$\widetilde{\nabla}_j(K^{ij} - \widetilde{g}^{ij}\,\mathrm{tr}_{\widetilde{g}}\, K) = 0\,.$$

Difficulty: The AH metric $\widetilde{g}_{ij}$ is not smooth up to the boundary of M: $y^2\widetilde{g}_{ij}$ is in $C^2(\overline{M})$, and the metric diverges as an inverse square as one approaches ∂M.

An additional assumption permits to decouple both equations: $\widetilde{\nabla}_i\,\mathrm{tr}_{\widetilde{g}}\, K = 0$. With this extra assumption and loosely speaking:

Theorem (Andersson–Chrusciel 1996: Existence of solutions)

Solutions to the above system are "parametrized" by traceless symmetric tensor $A_{ij} \in C^\infty(\overline{M})$: for each A_{ij} as above there is a unique solution "in" $C^{n-1}(\overline{M})$, which are in fact in $C^{n-1} \cap C^\infty_{\mathrm{polyhom}}$ (and this is natural).

- For an open, dense set of $A_{ij} \in C^\infty(\overline{M})$, the solution is in $C^{n-1}(\overline{M})\backslash C^n(\overline{M})$.
- There is a "large set" of non-generic A_{ij} for which the solution is in $C^\infty(\overline{M})$.

$C^\infty_{\mathrm{polyhom}} := \{\psi : (y\partial_y)^j\partial_x^k\psi \in C^0(\overline{M})$ for all $j, k\}$, where $(x_1, \ldots, x_{n-1})$ are local coordinates on ∂M.

aAdS solutions to the Einstein equations (I)

Using conformal methods, Friedrich obtained the following breakthrough result:

Theorem (Friedrich 1995: Local WP with smooth data)

Suppose that the spacetime dimension is $n + 1 = 4$ and take any the initial data $(\widetilde{g}_{ij}, K_{ij})$ and boundary metric $\widehat{g}_{\alpha\beta}$ satisfying the contraint equations and the compatibility condition.

If $y^2 \widetilde{g}_{ij}$ and $y^2 K_{ij}$ are in $C^\infty(\overline{M})$ and $\widehat{g}_{\alpha\beta} \in C^\infty(\mathbb{R} \times \partial M)$, there is a unique (local in time) $C^\infty(\overline{M})$ solution to the Einstein equations.

- The proof extends to any even spacetime dimension and the C^∞ regularity assumption can be relaxed somewhat (say C^k with $k = k(n)$ large but finite).
- Two major drawbacks (oversimplifying a little: the regularity assumptions are “not natural”, not just “not sharp”):
  1. It does not work in odd spacetime dimensions (by the log terms in the Fefferman–Graham expansion).
  2. By Andersson–Chrusciel, generic initial data satisfying the constraint equations do not satisfy the regularity assumptions.

aAdS solutions to the Einstein equations (II)

Using a standard PDE approach, our goal is to prove the following:

Theorem (E. & Kamran, 2014: Local WP with polyhomogenous data)

In any spacetime dimension, take any the initial data $(\widetilde{g}_{ij}, K_{ij})$ and boundary metric $\widehat{g}_{\alpha\beta}$ satisfying the contraint equations and the compatibility condition. If $y^2\widetilde{g}_{ij}$ and $y^2 K_{ij}$ are in $C^{n-1} \cap C^\infty_{\text{polyhom}}(\overline{M})$ and $\widehat{g}_{\alpha\beta} \in C^\infty(\mathbb{R} \times \partial M)$, there is a unique (local in time) $C^{n-1} \cap C^\infty_{\text{polyhom}}$ solution to the Einstein equations.

- In fact, it suffices to assume that the initial and boundary data have a certain number $k = k(n)$ of (polyhomogenous) derivatives, but we will not elaborate on this. The number k we obtain is by no means sharp.
- The regularity we require matches the a priori regularity provided by the constraint equations

Which kind of difficulties does one encounter?

Since the metric behaves as y^{-2} at the boundary, there are basically three difficulties one finds when dealing with the equations:

1. Boundary conditions are hard to impose for the Einstein equations: But here we'll have $g =: g_{large} + g_{small}$, where the large part includes the boundary conditions and is obtained by "peeling off" layers of the metric in an algebraic way and the small part "has no boundary conditions".
2. The system of PDEs is not quasidiagonal: although the Einstein equation is of the form (after gauge-fixing)

$$\Box_g \, g_{\mu\nu} + \text{l.o.t.} = 0 \,,$$

   here we must consider not only the terms with second-order derivatives but also the critically singular terms. Hence the system is no longer quasidiagonal, for all practical purposes we must deal with equations of the form:

$$\Box_{\bar{g}} \bar{g}_{\mu\nu} + \frac{1}{y} A_{\mu\nu}^{\lambda\rho} \, \partial_y \bar{g}_{\lambda\rho} + \frac{1}{y^2} \widetilde{A}_{\mu\nu}^{\lambda\rho} \, \bar{g}_{\lambda\rho} + \text{l.o.t.} = 0$$

3. Effect of singular coefficients on the scalar estimates: There are terms that become unbounded at $y = 0$. To fix the problem, we use weights and twisted derivatives, developing a functional framework adapted to the geometry.

Dumbing down the problem:

Scalar fields on aAdS spaces

Setting for the simplified equations

- Klein–Gordon equation for a scalar field:

$$\Box_g \phi - \sigma\phi = 0$$

  (Or nonlinear versions of this equation, as considered by many authors.)
- $\Box_g$ is the wave operator on an $(n+1)$-dimensional manifold with an asymptotically AdS metric g (which is not globally hyperbolic).
- σ is a mass parameter.

What do the equations look like? Well, the metric is regular away from the boundary ("$y = 0$") and close to the boundary the metric is roughly like

$$\frac{-dt^2 + dy^2 + h_x}{y^2},$$

with h_x a metric on ∂M (where we take local coordinates x), so you expect the equation to be like

$$-\partial_{tt}\phi + \partial_{yy}\phi - \frac{n-1}{y}\,\partial_y\phi - \frac{\sigma}{y^2}\,\phi + \Delta_x\phi = \text{l.o.t.}\,,$$

A spectral-theoretic argument on static aAdS

When the aAdS space is static, the equation can be understood using spectral theory, as considered in detail by Ishibashi & Wald (2003). To see how, we take the model problem

$$-\partial_{tt}\phi + \partial_{yy}\phi - \frac{n-1}{y}\,\partial_y\phi - \frac{\sigma}{y^2}\,\phi + \Delta_x\phi = 0$$

with x in some compact manifold and $y \in (0,1]$. We impose some harmless Dirichlet boundary conditions at $y = 1$, since the aAdS boundary is not located there. (In a real application, say to AdS_{n+1}, we wouldn't need any.)

Now we observe that the elliptic operator

$$-\mathcal{A}\phi := \partial_{yy}\phi - \frac{n-1}{y}\,\partial_y\phi - \frac{\sigma}{y^2}\,\phi + \Delta_x\phi$$

is nonnegative and formally self-adjoint on $C_0^\infty \subset L^2(y^{n+1}\,dx\,dy)$ and

$$\begin{cases} \sigma \in (-\infty, -\frac{n^2}{4}] & \to \text{no self-adjoint extensions} \\ \sigma \in (-\frac{n^2}{4}, 1-\frac{n^2}{4}) & \to \text{several self-adjoint extensions} \\ \sigma \in [1-\frac{n^2}{4}, \infty) & \to \text{one self-adjoint extension} \end{cases}$$

The dynamics of the equation

Then, depending on the value of the mass parameter σ in the equation

$$-\partial_{tt}\phi + \partial_{yy}\phi - \frac{n-1}{y}\,\partial_y\phi - \frac{\sigma}{y^2}\,\phi + \Delta_x\phi = 0\,,$$

the following can happen:

- If $\sigma \in (-\infty, \sigma_0]$, the dynamics isn't well defined in the energy space.
- If $\sigma \in [\sigma_1, \infty)$, there is a unique s-a extension A (Friedrichs's) and the dynamics is unique defined.
- If $\sigma \in (\sigma_0, \sigma_1)$, one has to impose boundary conditions on ϕ (e.g., Dirichlet or Neumann) to determine which s-a extension A one chooses.

In the last two cases, one then writes the equation as

$$\partial_{tt}\phi + A\phi = 0\,, \qquad (\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1)\,,$$

so the only solution is

$$\phi(t) = \cos(tA^{1/2})\,\phi_0 + \frac{\sin(tA^{1/2})}{A^{1/2}}\,\phi_1\,.$$

Can one do this in a more robust way?

Twisted derivatives and twisted Sobolev spaces

A robust way to analyze the equation was found by Warnick (2013), who introduced a very convenient functional framework adapted to waves on aAdS spaces. For convenience, in the equation

$$-\partial_{tt}\phi + \partial_{yy}\phi - \frac{n-1}{y}\,\partial_y\phi - \frac{\sigma}{y^2}\,\phi + \Delta_x\phi = 0$$

one can set

$$u := y^{-\frac{n}{2}}\phi\,,$$

in terms of which the equation reads as

$$0 = -\partial_{tt}u + \partial_{yy}u + \frac{\partial_y u}{y} + \Delta_x u - \frac{\alpha^2 u}{y^2} = -\partial_{tt}u - \mathbf{D}^*_{\alpha,y}\mathbf{D}_{\alpha,y}u + \Delta_x u\,,$$

where $\alpha := (n^2/4 + \sigma)^{1/2}$,

$$\mathbf{D}_{\alpha,y}u := y^{-\alpha}\partial_y\big(y^\alpha u\big) = u_y + \frac{\alpha}{y}u\,,$$

and

$$\mathbf{D}^*_{\alpha,y}u := -y^{\alpha-1}\partial_y\big(y^{1-\alpha}u\big) = -u_y + \frac{\alpha-1}{y}u$$

is its formal adjoint with respect to the scalar product $\mathbf{L}^2 := L^2(y\,dx\,dy)$.

How to solve the KG equation?

One can use the above formulation to exploit the renormalized energy

$$E[u](t) := \int \left(u_t^2 + (\mathbf{D}_{\alpha,y} u)^2 + |\nabla_x u|^2\right) y\, dx\, dy\,.$$

The Hilbert space defined by the norm

$$\|u\|^2_{\mathbf{H}^1_\alpha} := \int \left((\mathbf{D}_{\alpha,y} u)^2 + |\nabla_x u|^2\right) y\, dx\, dy$$

is what one calls $\mathbf{H}^1_\alpha$ (with Dirichlet boundary conditions, in the range $0 < \alpha < 1$). If u satisfies our model equation

$$-\partial_{tt} u - \mathbf{D}^*_{\alpha,y}\mathbf{D}_{\alpha,y} u + \Delta_x u = 0\,, \qquad (u, u_t)|_{t=0} = (u_0, u_1)\,,$$

the energy is easily seen to be conserved, and without much trouble this implies well-posedness in the energy class:

Theorem

If $(u_1, u_0) \in \mathbf{H}^1_\alpha \times \mathbf{L}^2$ (and Dirichlet, if needed), there is a unique solution with

$$\|u\|_{L^\infty \mathbf{H}^1_\alpha} + \|u_t\|_{L^\infty \mathbf{L}^2} \lesssim \|u_0\|_{\mathbf{H}^1_\alpha} + \|u_1\|_{\mathbf{L}^2}\,.$$

Beyond the model problem

The point now is that this method is robust, so we can get finer estimates and apply it in more complicated settings (and with other boundary conditions, but we are not interested in that here):

Theorem (Warnick 2013)

If $\Box_g \phi - \sigma\phi = F$ in an aAdS manifold (not necessarily static, but with Dirichlet BCs if necessary) and $u := y^{-\frac{n}{2}}\phi$, there is a unique solution in the energy class and

$$\|u(t)\|_{\mathbf{H}^1_\alpha} + \|u_t(t)\|_{\mathbf{L}^2} \leqslant C\, e^{Ct}\left(\|u_0\|_{\mathbf{H}^1_\alpha} + \|u_1\|_{\mathbf{L}^2} + \int_0^t \|F(\tau)\|_{\mathbf{L}^2}\, d\tau\right).$$

- One can also take nonzero BCs, possibly other than Dirichlet, and prove higher regularity estimates.

Local WP for scalar nonlinear equations

Twisted Sobolev spaces are well suited too to deal with nonlinear equations. A model result would be a KG equation with a nonlinearity that vanishes fast enough at the boundary $y = 0$:

Theorem (Holography for KG: E. & Kamran, 2015)

In an aAdS space $\mathbb{R} \times M$, set $Q(\phi, \phi) := y^q g(\nabla\phi, \nabla\phi)$ and consider the equation

$$\Box_g \phi - \sigma\phi = Q(\phi, \phi)\,, \qquad y^{\alpha - \frac{n}{2}}\phi|_{y=0} = f\,, \qquad (\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1)\,.$$

Locally in time, there is a unique solution with "m derivatives in twisted Sobolev spaces" provided that:

1. The initial and boundary data have $\geqslant m_1(m, \alpha)$ derivatives themselves.
2. The initial and boundary data satisfy compatibility conditions of order $\geqslant m_2(m, \alpha)$.
3. The nonlinearity vanishes fast enough at $y = 0$, meaning $q \geqslant q_1(m, \alpha)$.

At this point we do not need a precise statement, as we will recover these ideas later, but we need to know how to prove this result before we can get back to Einstein.

Step 1: Peeling

At the boundary, let's write things in terms of $u := y^{-\frac{n}{2}}\phi$, which satisfies an equation of the form

$$Pu = -\partial_{tt}u - \mathbf{D}^*_{\alpha,y}\mathbf{D}_{\alpha,y}u + \Delta_x u + \text{l.o.t.} = 0\,, \qquad y^\alpha u|_{y=0} = f\,.$$

Step 1: Peeling. To prove local existence for the nonlinear wave equation we want to use an iteration argument with "small quantities". But the "big" (nonzero) boundary terms do not allow us to do it directly. In standard linear equations, one simply subtracts them but here one cannot do that.

There are two tricks here:

1. If the nonlinearity vanishes fast enough at $y = 0$, one can forget about it and proceed as if the equation were linear.
2. For general α, however, it is not enough to simply subtract the boundary datum. One needs to find an approximate solution so that the error is small enough. This is what we call "peeling", which we state in terms of u:

Step 1: Peeling (II)

Peeling lemma (wrong formulas)

For each k, if the nonlinearity vanishes fast enough ("$q \geqslant q_1(\alpha, k)$") there is a function of the form

$$u_k = u - \sum_{j=0}^{k-1} \mathcal{O}_{\log}(y^{-\alpha+2j} f^{\leqslant 2j})$$

that satisfies an equation of the form

$$P_g u_k = Q(u_k, u_k) + +\mathcal{O}_{\log}(y^{-\alpha+2k-2} f^{\leqslant 2k})$$

and the homogeneous boundary condition $y^\alpha u_k|_{y=0} = 0$. Here

$$\mathcal{O}_{\log}(y^s f^{\leqslant k})$$

means that it vanishes as y^s (possibly with some logarithmic loss) and is controlled by at most k derivatives of the boundary datum f.

Step 1: Peeling (III)

The idea of the peeling argument in a nutshell: we get rid of the nonlinearity (which goes to zero fast at $y = 0$) and take the model problem

$$Pu := -\partial_{tt} u - \mathbf{D}^*_{\alpha,y} \mathbf{D}_{\alpha,y} u + \Delta_x u = 0\,, \qquad y^\alpha u|_{y=0} = f(x,t)\,.$$

If we set $u_1 := u - f y^{-\alpha}$ one gets ($\Box := -\partial_{tt} + \Delta_x$)

$$Pu_1 = (\Box f)\, y^{-\alpha}\,,$$

which isn't in $\mathbf{L}^2$ for large enough α. Since $\mathbf{D}^*_{\alpha,y} \mathbf{D}_{\alpha,y} y^s = (\alpha^2 - s^2) y^{s-2}$, one now sets

$$u_2 := u_1 - (2\alpha - 4)^{-1} (\Box f)\, y^{-\alpha+2}\,, \qquad Pu_2 = c_\alpha\, (\Box f)\, y^{-\alpha+2} + c'_\alpha\, (\Box\Box f)\, y^{-\alpha+4}\,.$$

and so on.

The number of derivatives of f grows in each step, but at some point the RHS vanishes fast enough at $y = 0$ to be in $\mathbf{L}^2$. The only problem one encounters is if 2α is an integer, where logarithmic terms can appear.

Step 2: The iteration

Now we can set $u = u_k + w$ ("$u = u_{\text{large}} + u_{\text{small}}$") with a large enough k. We then set an iteration to obtain w. The equation reads as

$$\widetilde{P}w = \widetilde{Q}(w) + F\,, \qquad y^{\alpha}w|_{y=0} = 0\,.$$

The iteration looks like $\widetilde{P}w_{j+1} = \widetilde{Q}(w_j) + F\,, \qquad y^{\alpha}w|_{y=0} = 0,$. What do we need?

- Enough regularity for w_{j+1} from the linear equation (in twisted Sobolev spaces) and embeddings of these spaces to deal with the nonlinearity.
- Therefore, enough regularity/decay of F (and the initial conditions). As F comes mainly from the peeling of the boundary condition, we then require a lot of regularity on the boundary data (and a fast decay at $y = 0$, as it is used in the peeling argument).

On regularity and embeddings

1. If F and the initial data satisfy some obvious compatibility conditions, one can indeed prove higher regularity estimates of the form

$$\sum_{l=0}^{m+1} \left\|\partial_t^l u(t)\right\|_{\mathbf{H}_\alpha^{m+1-l}} \leqslant Ce^{Ct}\left(\sum_{j=0}^{m} \int_0^t \|\partial_t^j F(\tau)\|_{\mathbf{H}_\alpha^{m-j}}\, d\tau + \|u_0\|_{\mathbf{H}_\alpha^{m+1}} + \|u_1\|_{\mathbf{H}_\alpha^m}\right),$$

where $\mathbf{H}_\alpha^{m+1} := \left\{v \in \mathbf{H}_\alpha^m : \nabla_{\alpha,y} v \in \mathbf{H}_\alpha^m,\ \mathbf{D}_{\alpha,y}^{(m+1)} v \in \mathbf{L}^2\right\}$ and

$$\mathbf{D}_{\alpha,y}^{(m)} v := \begin{cases} (\mathbf{D}_{\alpha,y}^* \mathbf{D}_{\alpha,y})^{\frac{m}{2}} v & \text{if } m \text{ is even,} \\ \mathbf{D}_{\alpha,y}(\mathbf{D}_{\alpha,y}^* \mathbf{D}_{\alpha,y})^{\frac{m-1}{2}} v & \text{if } m \text{ is odd.} \end{cases}$$

2. A model of the estimates needed for the nonlinearities would be standard $H^m(\mathbb{R}^n) \subset L^\infty$ if $m > \frac{n}{2}$. But the proofs (and even exponents) are different here: e.g., a basic estimate would be that if $v \in \mathbf{H}^m$ with $m > \frac{n+1}{2} + j$, then
   - If $\alpha > 1$, $|\mathbf{D}_{\alpha,y}^{(j)} v| \lesssim \|v\|_{\mathbf{H}^m}\, y^{\min\{m-j-\frac{n+1}{2},\alpha\}}$.
   - If $\alpha > 1$, $|\mathbf{D}_{\alpha,y}^{(j)} v| \lesssim \|v\|_{\mathbf{H}^m} \begin{cases} y^{-\alpha} & \text{if } j \text{ is even,} \\ y^{\alpha-1} & \text{if } j \text{ is odd.} \end{cases}$

Where does the half a derivative loss come from?

Take the easiest case: $v \in C_c^1((0,1))$. Then:

- "Standard" $H^{1/2}((0,1)) \subset L^\infty$: for all y,

$$v(y)^2 = \int_0^y (v^2)' = 2\int_0^y vv' \leqslant 2\int_0^1 |vv'| = 2\int_0^1 (|D|^{1/2}v)^2 = 2\|v\|_{H^{1/2}}^2 .$$

- "Twisted" $\mathbf{H}_\alpha^1 \subset L^\infty$: Notice that the Hardy operator

$$A_\alpha v(y) := y^{-\alpha}\int_0^y \bar{y}^\alpha v(\bar{y})\, d\bar{y}$$

is an inverse of sorts of $\mathbf{D}_{\alpha,y} = y^{-\alpha}\partial_y y^\alpha$ because

$$\mathbf{D}_{y,\alpha}(A_\alpha v) = \ v \ \text{ and "conditionally" } \ A_\alpha(\mathbf{D}_{\alpha,y}v) = v\,.$$

But one can prove $A_\alpha : \mathbf{L}^2 \to L^\infty$ is sharp, so

$$\|v\|_{L^\infty} = \|A_\alpha(\mathbf{D}_{\alpha,y}v)\|_{L^\infty} \lesssim \|\mathbf{D}_{\alpha,y}v\|_{\mathbf{L}^2} = \|v\|_{\mathbf{H}_\alpha^1}\,.$$

Back to the Einstein equations

Step 1: Modified Einstein equations

Our first goal is to get a system of nonlinear wave equations, and it is well known that that is done by replacing the Einstein equations $R_{\mu\nu} + ng_{\mu\nu} = 0$ by

$$0 = Q_{\mu\nu} := R_{\mu\nu} + ng_{\mu\nu} + \frac{1}{2}(\nabla_\mu W_\nu + \nabla_\nu W_\mu)\,,$$
$$W_\mu := g_{\mu\nu}\, g^{\lambda\rho}\,(\Gamma^\nu_{\lambda\rho} - \widetilde{\Gamma}^\nu_{\lambda\rho})$$

with $\widetilde{\Gamma}^\nu_{\lambda\rho}$ the Christoffel symbols of certain aAdS reference metric γ_0 with the same conformal infinity.

Then the equation $Q_{\mu\nu} = 0$ reads (in certain coordinates) as

$$0 = Q_{\mu\nu} = -\frac{1}{2} g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + B_{\mu\nu}(g, \partial g)$$

with initial and boundary conditions ($\bar{g} := y^2 g \in C^2(\mathbb{R} \times \overline{M})$)

$$g|_{t=0} = g_0\,, \qquad \partial_t g|_{t=0} = g_1\,, \qquad (j_{(-T,T)\times\partial M})^* \bar{g} = \widehat{g}\,.$$

Step 2: Peeling

Peeling is very different here. For starters, the nonlinear term does not go to zero fast at $y = 0$, so it does contribute and one has to consider linearizations of the equation, where one makes two assumptions: that $\bar{g} := y^2 g \in C^2(\mathbb{R} \times \overline{M})$ and that $\bar{g}^{\mu\nu}\partial_\mu y \partial_\nu y|_{\mathbb{R}\times\partial M} = 1$ ("weakly aAdS").

The linearization of $Q(g)$, which controls the peeling argument, is quite ugly. One starts by decomposing the space of symmetric tensors as

$$\mathcal{S}^2 = \mathcal{V}_0^g \oplus \mathcal{V}_1^g \oplus \mathcal{V}_2^g \oplus \mathcal{V}_3^g \,,$$

where ($y_\mu := \partial_\mu y$)

$$\begin{aligned}
\mathcal{V}_0^g &:= \left\{H \in \mathcal{S}^2 : H_{\mu\nu} = \varphi \bar{g}_{\mu\nu} \text{ with } \varphi \text{ scalar}\right\}, \\
\mathcal{V}_1^g &:= \left\{H \in \mathcal{S}^2 : H_{\mu\nu}\bar{g}^{\nu\lambda} y_\lambda = 0 \text{ and } H_{\mu\nu}\bar{g}^{\mu\nu} = 0\right\}, \\
\mathcal{V}_2^g &:= \left\{H \in \mathcal{S}^2 : H_{\mu\nu} = \varphi\left[(n+1)y_\mu y_\nu - \bar{g}_{\mu\nu}\right] \text{ with } \varphi \text{ scalar}\right\}, \\
\mathcal{V}_3^g &:= \left\{H \in \mathcal{S}^2 : H_{\mu\nu} = a_\mu y_\nu + a_\nu y_\mu \text{ with } \bar{g}^{\lambda\rho} a_\lambda y_\rho = 0\right\}.
\end{aligned}$$

Then, writing $h = h_0 + h' \in \mathcal{V}_0^g \oplus (\mathcal{V}_1^g \oplus \mathcal{V}_2^g \oplus \mathcal{V}_3^g)$,

$$(DQ)_g(h) = -\frac{1}{2}\left((\Box_g - 2n)h_0 + (\Box_g + 2)h'\right) + \text{l.o.t.}$$

Step 2: Peeling (II)

More explicitly, if $h \in \mathcal{V}_j^g$ $(0 \leqslant j \leqslant 3)$,

$$(DQ)_g(h)_{\mu\nu} = y^{-2}\, p_j(y\partial_y)\bar{h}_{\mu\nu} + \mathcal{O}(y^{-1})\,,$$

where $p_j(s) := -\frac{1}{2}(s - \frac{n}{2} + \alpha_j)(s - \frac{n}{2} - \alpha_j)$ and the roots α_j are

$$\alpha_0 := \frac{\sqrt{n(n+8)}}{2}\,, \quad \alpha_1 := \frac{n}{2}\,, \quad \alpha_2 := \alpha_0\,, \quad \alpha_3 := \frac{\sqrt{n(n+4)}}{2}\,.$$

(Oversimplified) version of peeling ($\bar{\gamma} := y^2\gamma$):

- $\bar{\gamma}_0 :=$ "reference metric with the same conformal infinity". We then write

$$Q(\gamma_0) =: y^{-1}H_1 + \mathcal{O}(y^{-2})\,, \qquad H_1 = H_{10} \oplus H_{11} \oplus H_{12} \oplus H_{13}\,, \qquad H_{1j} \in \mathcal{V}_j^{\gamma_0}\,.$$

- $\bar{\gamma}_1 := \bar{\gamma}_0 + y(c_{10}H_{10} + c_{11}H_{11} + c_{12}H_{12} + c_{13}H_{13})$ and repeat to get $\bar{\gamma}_{k+1}$.
- Since $2\alpha_1 = n$, in general we'll get logarithmic terms starting in $\bar{\gamma}_n$ with $x^n \log x$. This accounts for the polyhomogeneity of $\bar{\gamma}_k$, in general. And, of course, we lose derivatives of in each step as before.

Step 2: Peeling (III)

Theorem (Peeling, simplified version)

Let us take a nonnegative integer $l \geqslant n-1$ and a small real $\delta > 0$. Then there is a weakly asymptotically AdS metric γ_l on $\mathbb{R} \times M$ of class $C^{n-1} \cap C^{\infty}_{\text{polyhom}}$ such that:

1. The pullback to the boundary of $\bar{\gamma}_l := y^2 \gamma_l$ is the metric we want to prescribe:
$$(j_{\mathbb{R}\times\partial M})^* \bar{\gamma}_l = \widehat{g}\,.$$
2. The metric γ_l is uniformly close to $\bar{\gamma}_0$:
$$\|\bar{\gamma}_l - \bar{\gamma}_0\|_{L^\infty} < \delta\,.$$
3. The metric γ_l is a solution of the modified Einstein equation to order $l-1$:
$$Q(\gamma_l) = \mathcal{O}_{\log}(y^{l-1})\,.$$

Step 3: The iteration

Again we decompose $g =: \gamma_l + h =$ "$g_{\text{large}} + g_{\text{small}}$" and use an iteration argument to get h as an object that goes to zero at infinity.

Specifically, set $u := y^{-\frac{n}{2}} h$. Then the modified Einstein equation $Q(g) = 0$ reads

$$P_g u = \mathcal{F}_0 + \mathcal{G}(u)\,, \qquad \mathcal{F}_0 \text{ depends only on } \gamma_l\,, \quad \mathcal{G} \text{ depends on } u, \partial u\,.$$

- Iteration: $P_{g^m} u^{m+1} = \mathcal{F}_0 + \mathcal{G}(u^m)$.
- Structure of P: It is fundamentally matrix-valued. If $u \in \mathcal{V}_j^{g^m}$, $0 \leqslant j \leqslant 3$,

$$P_{g^m} u = -\partial_{tt} u - \mathbf{D}^*_{\alpha_j, y} \mathbf{D}_{\alpha_j, y} u + \Delta_x u + \text{l.o.t.}$$

Step 3: The iteration (II)

Difficulties that make the iteration quite different than in the toy model case:

1. We have to use twisted spaces of different weights α_j $(0 \leqslant j \leqslant 3)$, and the decomposition depends on the metric. In each space there is a different decay rate at $y = 0$.
   - Spaces $\mathbf{H}^k_{g;\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ built over $\mathbf{H}^k_{\alpha_j}$.
2. $\mathcal{F}_0$ and $\mathcal{G}(u)$ don't decay very fast at $y = 0$, so there is not enough regularity in twisted spaces to close the estimates.
   - Spaces with both twisted and polyhomogeneous derivatives:

$$\mathbf{H}^{m,r}_\alpha := \{m \ \alpha\text{-twisted derivatives and } r \text{ polyhomogeneous derivatives}\}\,,$$
$$\mathcal{H}^{m,r} := \{m \text{ ordinary derivatives and } r \text{ polyhomogeneous derivatives}\}\,.$$

So one needs estimates in the corresponding spaces, relationships between them, control the dependance on the metric and the decay at $y = 0$, ...

Is AdS unstable?

Large time results, especially concerning the appearance of singularities, are very hard and interesting:

Problem: Instability of AdS

If the initial and boundary data of the Einstein equations are arbitrarily close to those of AdS, can singularities appear?

Even more, do they appear generically, in the above conditions?

Exciting recent breakthrough by Moschidis (2017)!

Observability in asymptotically AdS spacetimes

Observability in bounded domains

Model result: Boundary observability in bounded domains

Given a n-manifold with boundary M, take a Riemannian metric $g \in C^2(\overline{M})$ and assume that ϕ satisfies the KG equation with Dirichlet BC's:

$$-\partial_{tt}\phi + \Delta_g \phi - \sigma\phi = 0\,, \qquad \phi|_{\mathbb{R}\times\partial M} = 0\,.$$

Then if

1. there are no trapped geodesics (i.e., all the geodesics on M intersect ∂M),
2. T is large enough (for the intersection to occur),

the H^1 energy is controlled by the Neumann datum:

$$E[\phi] \lesssim \int_0^T \int_{\partial M} (\partial_\nu \phi)^2 \, d\sigma \, dt\,.$$

These conditions are essentially necessary, as shown using Gaussian beams: given a "ray" (a geodesic of length T, possibly with reflections on the boundary), there is a sequence of solutions to the wave equation whose portion of energy outside a tube of radius ϵ is of order $\mathcal{O}(\epsilon^4)$.

What happens in AdS?

With the assumption that there are no trapped null geodesics, it suffices to prove the result in a neighborhood of the boundary. But what one finds is not trivial:

$$\mathrm{Dir}[\phi] := y^{-\frac{n}{2}+\alpha}\phi|_{y=0}\,, \qquad \mathrm{Neu}[\phi] := y^{1-2\alpha}\partial_y(y^{-\frac{n}{2}+\alpha}\phi)|_{y=0}\,.$$

- One does know how singularities propagate in aAdS spaces (Vasy, 2012). In particular, next to the boundary, if $\Delta_g\phi - \sigma\phi = 0$ and ϕ is smooth enough (say in $C^\infty_{\mathrm{polyhom}}$), $\phi \equiv 0$

$$\mathrm{Dir}[\phi] = \mathrm{Neu}(\phi) = 0 \quad \text{on } (0, T) \times \partial M\,.$$

- Carleman inequalities by Holzegel–Shao (2015) using multipliers. In particular, the result holds if ϕ has $\mathbf{H}^2_\alpha$ regularity.

Observability in a certain range of the mass

Although one would expect to get observability for free from the Carleman using standard ideas, it is not the case. In fact one needs a very different kind of Carleman estimates to prove the result:

Theorem (Observability: E., Shao & Vergara, 2017)

If $\alpha \in (\frac{1}{2}, \infty)$ and $\mathrm{Dir}[\phi] = 0$ on $(0, T) \times \partial M$, then

$$E[\phi] \lesssim \int_0^T \int_{\partial M} (\mathrm{Neu}[\phi])^2 \, d\sigma \, dt \, .$$

- Commutators with the singular terms get additional weights of the form $y^{-\beta}$. One can Hardy them out using additional regularity, but that is not enough for observability unless one can get good signs in some places, as one can do (with some work) in this range of masses.
- These difficulties are "real", meaning that null geodesics can indeed stay close to the boundary (= in the region where $y^{-\beta}$ is large).
- More to come in the future!

Thank you very much for your attention!