The space of Quantum States, relative entropies and metric tensors

Some ideas, born during an uncountable number of discussions*, Presented by:

Florio Maria Ciaglia

In occasion of:

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Classical and Quantum Physics: Geometry, Dynamics and Control

*very often without proper lunch....sometimes after too much wine....

The model quantum system

$\mathcal{H}_n = \mathbb{C}^n$ Is the Hilbert space of the system

$\mathcal{B}(\mathcal{H}_n) = M_n(\mathbb{C})$ Is the C*-algebra of the system, (linear) observables are self-adjoint matrices

$\mathcal{S} = \{\rho \in M_n(\mathbb{C})\colon \mathrm{Tr}(\rho) = 1, \quad \rho \text{ is positive } \}$ Is the space of states of the system

$\langle \mathbf{a} \rangle_\rho \equiv \mathrm{e}_{\mathbf{a}}(\rho) := \mathrm{Tr}(\mathbf{a}\,\rho)$ Is the expectation value function of the (linear) observable a on the quantum state ρ

Remark: every finite-dimensional quantum system (without superselection sectors) is isomorphic to this model. The isomorphism is not canonical.

Quantum states as non-commutative probabilities

- A state ρ on a unital C*-algebra A is a normalized positive linear functional on A ($\rho(a^2)\geq 0$ and $\rho(\mathbf{1})=1$, where $\mathbf{1}$ is the identity element in A).

- Quantum states are the states of the non-commutative C*-algebra $B(H_n)$.

- Consider the commutative C*-algebra $C_0(X)$ of continuous functions on the topological space $X=\{1,\ldots,n\}$. It may be realized as $\mathbb{C}^n$ endowed with component-wise multiplication.

- Positive normalized linear functionals on $C_0(X)$ are then fair probability distributions on X (use the fact that $\mathbb{C}^n$ is isomorphic to its dual).

- Quantum mechanics may be thought of as a non-commutative version of probability theory where quantum states replace probability distributions.

Quantum states as non-commutative probabilities

The differential geometry of the space of probability distributions on X={1,...,n} is well-developed (Cencov and Amari as prominent references).

Stochastic Markov maps $\leftrightarrow$ Fisher-Rao invariant metric tensor;

Convex structure of probabilities $\leftrightarrow$ dually related torsion-less affine connections

Questions: what can we say about the differential geometry of the space of quantum states?

What are the relevant geometrical structures of non-commutative probabilities?

QUANTUM STATES AND THE GENERAL LINEAR GROUP

The general linear group GL(n, C) acts on the space of quantum states:

$$GL(n,\,\mathbb{C}) \times \mathcal{S} \ni (\mathrm{g},\,\rho) \to \frac{\mathrm{g}\,\rho\,\mathrm{g}^\dagger}{\mathrm{Tr}(\mathrm{g}\,\rho\,\mathrm{g}^\dagger)} \in \mathcal{S}$$

REMARK: the trace term in the denominator makes this action "nonlinear".

The orbits are classified by the (matrix) rank of the quantum states.
The space of quantum states is partitioned into the disjoint union of orbits of quantum states with fixed rank:

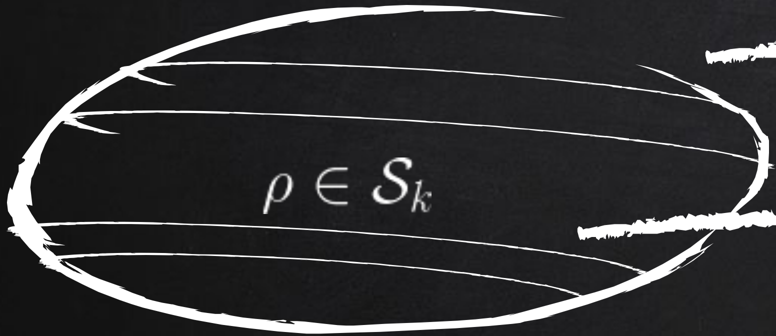
$$\mathcal{S} = \sqcup_{k=1}^{n} \mathcal{S}_k$$

QUANTUM STATES AND THE GENERAL LINEAR GROUP

The isotropy subgroup of every quantum state in S_k is a closed subgroup of GL(n, C).

Every S_k is a smooth homogeneous space of GL(n, C).

Given ρ_1, ρ_2 in S_k, there is an element in GL(n, C) connecting them.



QUANTUM STATES WITH FIXED RANK

FLOW OF THE FUNDAMENTAL VECTOR FIELDS OF THE GROUP ACTION:

$$X_A, \; A \in \mathfrak{gl}(n, \mathbb{C})$$

REMARK: k=1 → pure quantum states,
k=n (maximal rank) → invertible quantum states.

Quantum states and the unitary group

From the general linear group GL(n, C) to the unitary group U(n):

$$\frac{\mathrm{g}\,\rho\,\mathrm{g}^\dagger}{\mathrm{Tr}(\mathrm{g}\,\rho\,\mathrm{g}^\dagger)} \Longrightarrow \mathbf{U}\,\rho\,\mathbf{U}^\dagger = \mathbf{U}\,\rho\,\mathbf{U}^{-1}$$

This action preserves the spectrum of the quantum states.

Manifolds S_k as disjoint union of manifolds of isospectral quantum states.

Element in the Lie algebra of GL(n, C): $\mathfrak{gl}(n,\ \mathbb{C}) \ni \mathbf{A} = \mathbf{a} + \imath\mathbf{b},\ \ \mathbf{a} = \mathbf{a}^\dagger,\ \mathbf{b} = \mathbf{b}^\dagger$

Fundamental vector fields of U(n): $X_{\imath\mathbf{b}} \equiv X_{\mathbf{b}}$

QUANTUM STATES AND THE UNITARY GROUP

Manifolds of isospectral quantum states are (compact) Kahler manifolds.

They "are" coadjoint orbits of $U(n)$.

Konstant–Kirillov–Souriau symplectic form ω.

Killing form on $U(n)$ ⇒ invariant metric tensor g.

A complex structure J relates the symplectic form and the metric tensor:

$$\omega\left(J(\cdot),\,\cdot\right) = g\left(\cdot,\,\cdot\right)$$

REMARK: the fundamental vector fields are Hamiltonian vector fields:

$$i_{X_{\mathbf{b}}}\omega = \mathrm{d}f_{\mathbf{b}}, \quad f_{\mathbf{b}}(\rho) = \mathrm{Tr}(\mathbf{b}\rho); \qquad L_{X_{\mathbf{b}}}\omega = L_{X_{\mathbf{b}}}J = L_{X_{\mathbf{b}}}g = 0$$

QUANTUM STATES AND THE UNITARY GROUP

Gradient vector fields out of Hamiltonian vector fields using the complex structure J:

$$Y_{\mathrm{b}} := J(X_{\mathrm{b}})$$

We found that Hamiltonian and gradient vector fields associated with elements in the Lie algebra of the unitary group provide a realization of the Lie algebra of GL(n, C):

$$[X_{\mathrm{a}}\,, X_{\mathrm{b}}] = X_{\imath[\mathrm{a}\,,\mathrm{b}]}\,, \qquad [X_{\mathrm{a}}\,, Y_{\mathrm{b}}] = Y_{\imath[\mathrm{a}\,,\mathrm{b}]}\,, \qquad [Y_{\mathrm{a}}\,, Y_{\mathrm{b}}] = -X_{\imath[\mathrm{a}\,,\mathrm{b}]}$$

This realization "integrates" to an action of the general linear group GL(n, C) on the manifolds of isospectral states.

REMARK: this GL(n, C)–action is different from the "nonlinear" one defined on the manifolds of quantum states with fixed rank.

Information geometry of invertible quantum states

Dimensional constraints $\Rightarrow$ no symplectic/complex structure on the manifolds of quantum states with fixed rank (>1).

Question: Is there a metric tensor on the space of invertible quantum states which is invariant w.r.t. the action of the unitary group? Is it unique?

Answer: There is an infinite number of metric tensors on the space of invertible quantum states that are invariant with respect to the unitary group.

Remark: All these metric tensors satisfy the so-called monotonicity property (MP), that is, "invariance" w.r.t. quantum stochastic maps.

Metrics on the space of invertible quantum states

Question: How can we obtain a metric tensor satisfying the monotonicity property?

In classical Information Geometry we take a divergence function D (often it is a relative entropy), derive it twice and then evaluate the result on the diagonal:

$$D(\mathbf{x},\,\mathbf{y}) = \frac{1}{2}(x^j - y^j)^2 \implies -\left(\frac{\partial^2 D}{\partial x^j \partial y^k}\right)_{\mathbf{x}=\mathbf{y}} = \delta_{jk} \implies g = \delta_{jk}\,\mathrm{d}x^j \otimes \mathrm{d}x^k$$

We could take a quantum relative entropy and perform the same algorithm….

Remark: divergence functions are non-negative two-point functions vanishing on the diagonal.

Metrics on the space of invertible quantum states

Very often, the resulting quantum metrics satisfy the MP, however:

1) the algorithm is coordinate-based;

2) we must prove that the resulting object is a metric tensor and that it satisfies the MP.

Using the geometrical point of view we achieved the following results:

i) give a coordinate-free extraction algorithm for classical and quantum systems;

ii) prove that quantum divergence functions satisfying the data processing inequality (DPI) give rise to quantum metric tensors satisfying the MP

Covariant tensors from two-point functions

A vector field on M may be thought of as a derivation of the associative algebra of smooth functions, or as a section of the tangent bundle.

In a local chart $\{q^j\}$ on M a vector field X can be written as:

$$X = X^j(q)\,\frac{\partial}{\partial q^j}$$

Remark: Vector fields are the coordinate-free version of the derivative operator and we will use them to give a coordinate-free algorithm to extract tensor fields from two-point functions.

Covariant tensors from two-point functions

Divergence functions are defined on M x M (two-point functions):

$$
\begin{tikzcd}
\mathbb{R} & M \times M \arrow[l, "f_l = pr_l^* f"'] \arrow[r, "f_r = pr_r^* f"] \arrow[d, shift right, "pr_l"'] \arrow[d, shift left, "pr_r"] & \mathbb{R} \\
 & M \arrow[d, "f"] & \\
 & \mathbb{R} &
\end{tikzcd}
$$

Remark: The left functions f_l on M x M form a subalgebra of the algebra of smooth functions on M x M. The same is true for the right functions f_r.

Covariant tensors from two-point functions

From vector fields on M to vector fields on its double:

Left lift of a vector field on M

$$\mathfrak{X}(\mathcal{M}) \ni X \longrightarrow \mathbb{X}^l \in \mathfrak{X}(\mathcal{M}\times\mathcal{M}):$$

$$\mathbb{X}^l\, f_l = (X\, f)_l\;,\quad \mathbb{X}^l\, f_r = 0$$

$$X = X^j(q)\frac{\partial}{\partial q^j} \longrightarrow \mathbb{X}^l = X^j(x)\frac{\partial}{\partial x^j}$$

Right lift of a vector field on M

$$\mathfrak{X}(\mathcal{M}) \ni X \longrightarrow \mathbb{X}^r \in \mathfrak{X}(\mathcal{M}\times\mathcal{M}):$$

$$\mathbb{X}^r\, f_r = (X\, f)_r\;,\quad \mathbb{X}^r\, f_l = 0$$

$$X = X^j(q)\frac{\partial}{\partial q^j} \longrightarrow \mathbb{X}^r = X^j(y)\frac{\partial}{\partial y^j}$$

Proposition: for every smooth function f, and for all vector fields X, Y on M, we have:

$$[\mathbb{X}_l\,,\mathbb{Y}_l] = ([X\,,Y])_l\;,\qquad [\mathbb{X}_r\,,\mathbb{Y}_r] = ([X\,,Y])_r\;,\qquad [\mathbb{X}_l\,,\mathbb{Y}_r] = 0\,,$$

$$(f\,X)_l = f_l\mathbb{X}_l\,,\qquad (f\,X)_r = f_r\mathbb{X}_r\,,\qquad L_{\mathbb{X}_l} f_r = L_{\mathbb{X}_r} f_l = 0\,.$$

Covariant tensors from two-point functions

Question: How can we extract a covariant (0,2) tensor from a two-point function D?

To answer this question, let us consider the diagonal immersion of M into its double:

$$i_d \colon \mathcal{M} \to \mathcal{M} \times \mathcal{M}\,, \quad m \mapsto i_d(m) = (m\,, m)$$

Let X and Y be arbitrary vector fields on M, D a smooth function on its double, and define the following maps:

$$g_{ll}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_l} D\right)\,,\; g_{rr}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_r} D\right)\,,$$
$$g_{lr}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} D\right)\,,\; g_{rl}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_l} D\right)\,.$$

Covariant tensors from two-point functions

Without further assumptions on D, these maps do not define covariant (0,2) tensors.

Proposition:

1) g_{lr} and g_{rl} are covariant (0,2) tensors, and $g_{lr}(X,Y)=g_{rl}(Y,X)$;

2) g_{ll} is a symmetric covariant (0,2) tensor if and only if :

$$i_d^*\left(L_{\mathbb{X}_l} D\right) = 0 \ \ \forall X \in \mathfrak{X}(M)$$

3) g_{rr} is a symmetric covariant (0,2) tensor if and only if:

$$i_d^*\left(L_{\mathbb{X}_r} D\right) = 0 \ \ \forall X \in \mathfrak{X}(M)$$

4) if the previous two conditions hold simultaneously for D, then:

$$g_{ll} = g_{rr} = -g_{lr} = -g_{rl}$$

Covariant tensors from two-point functions

Definition: A smooth function D on the double of M such that:

$$i_d^*\left(L_{\mathbb{X}_l} D\right) = 0 \;\; \forall X \in \mathfrak{X}(M)\,, \qquad i_d^*\left(L_{\mathbb{X}_r} D\right) = 0 \;\; \forall X \in \mathfrak{X}(M)$$

is called a **potential function**, and we set $g = -g_{lr}$ for the symmetric covariant (0,2) tensor associated with D.

Definition: A smooth function D on the double of M such that:

$$D(m_1\,, m_2) \geq 0\,, \qquad D(m_1\,, m_2) = 0 \iff m_1 = m_2$$

is called a **divergence function**.

Remark: Every divergence function is a potential function, and the associated tensor g is positive-semidefinite. The converse is not true.

Covariant tensors from two-point functions

Coordinates expression: $\{q^j\}$, $\{x^j, y^j\}$ local charts on M and M x M:

$$g = \left(\frac{\partial^2 S}{\partial x^j \partial x^k}\right)\bigg|_d \mathrm{d}q^j \otimes_s \mathrm{d}q^k = \left(\frac{\partial^2 S}{\partial y^j \partial y^k}\right)\bigg|_d \mathrm{d}q^j \otimes_s \mathrm{d}q^k =$$

$$= -\left(\frac{\partial^2 S}{\partial x^j \partial y^k}\right)\bigg|_d \mathrm{d}q^j \otimes_s \mathrm{d}q^k$$

Proposition: Let D be a **potential function** on M x M, then the tensor g is positive-semidefinite if and only if every point on the diagonal is a local minimum for D. In particular, g is a metric tensor if and only if every point of the diagonal is a nondegenerate local minimum for D.

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

QUESTION: What happens when we consider smooth maps between manifolds?

$$\phi : \mathcal{N} \to \mathcal{M}$$

$$\Phi : \mathcal{N} \times \mathcal{N} \to \mathcal{M} \times \mathcal{M}$$
$$(n_1\,, n_2) \mapsto \; \Phi(n_1\,, n_2) := (\phi(n_1)\,, \phi(n_2))$$

WE CAN PROVE THAT:

$$\begin{tikzcd}
\text{Symmetric covariant tensor on M} \arrow[r, "\text{Pullback by } \phi"] & \text{Symmetric covariant tensor on N} \\
\text{Potential function on MxM} \arrow[u, "\text{extract}"'] \arrow[r, "\text{Pullback by } \Phi"] & \text{Potential function on NxN} \arrow[u, "\text{extract}"']
\end{tikzcd}$$

IS COMMUTATIVE!

Metrics on the space of invertible quantum states

We consider the family $\{S_n\}_{n\in\mathbb{N}_2}$ of manifolds of invertible quantum states, where $\mathbb{N}_2$ is the set of natural numbers without 0 and 1.

Quantum stochastic maps:

$$\phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$$

is a Quantum Stochastic map if it is a linear completely-positive trace-preserving map (CPTP map) such that:

$$\phi(\mathcal{S}_n) \subset \mathcal{S}_m$$

Remark: Quantum Stochastic maps are the quantum analogue of the Markov maps in classical theory of probability.

Metrics on the space of invertible quantum states

Monotonicity property of metric tensors

$\{g_n\}_{n\in\mathbb{N}_2}$ is a family of metric tensors on $\{\mathcal{S}_n\}_{n\in\mathbb{N}_2}$

A family of quantum metric tensors satisfies the monotonicity property (MP) if:

$$g_n(X,X) \geq (\phi^* g_m)(X,X)$$

for every n,m, for every vector field X, and for every quantum stochastic map:

$$\phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$$

Remark: The MP entails the fact that distances between quantum states do not increase under the action of Quantum Stochastic Maps.

Metrics on the space of invertible quantum states

Data processing inequality for divergence functions

$\{D_n\}_{n\in\mathbb{N}_2}$ is a family of divergence functions on $\{\mathcal{S}_n \times \mathcal{S}_n\}_{n\in\mathbb{N}_2}$

A family of divergence functions satisfies the data processing inequality (DPI) if:

$$D_n(\rho,\tilde{\rho}) \geq (\Phi^* D_m)(\rho,\tilde{\rho}) = D_m(\phi(\rho),\phi(\tilde{\rho}))$$

for every n,m, and for every quantum stochastic map:

$$\phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$$

Remarks: information can not increase under Quantum Stochastic Maps.

Proposition: The family of metric tensors extracted from a family of quantum divergence functions satisfying the DPI satisfies the MP.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

PROPOSITION: If a family of quantum divergence functions satisfies the DPI, then, the family of metric tensors we can extract from it, satisfies the MP.

PROOF:

$$D_n(\rho\,,\widetilde{\rho}) \geq D_m(\phi(\rho)\,,\phi(\widetilde{\rho})) \geq 0 \Longrightarrow D^{\phi}_{nm}(\rho\,,\widetilde{\rho}) := D_n(\rho\,,\widetilde{\rho}) - D_m(\phi(\rho)\,,\phi(\widetilde{\rho})) \geq 0$$

$D^{\phi}_{nm}(\rho\,,\widetilde{\rho})$ is a non-negative potential function vanishing on the diagonal.

g^{ϕ}_{nm} is a positive semidefinite covariant (0,2) tensor.

$$g^{\phi}_{nm}(X,Y) = -i^{*}_{d}\left(L_{\mathbb{X}_l}\, L_{\mathbb{Y}_r}(D_n - \Phi^{*}D_m)\right) = g_n(X,Y) - \phi^{*}g_m(X,Y) \geq 0$$

$$g_n(X,Y) \geq \phi^{*}g_m(X,Y)$$

Q.E.D.

Metrics on the space of invertible quantum states

We studied the family of q-z-Rényi relative entropies (Audenaert, Datta):

$$D^{q,z}_{n}(\rho,\tilde{\rho}) = \frac{1}{q(1-q)}\left(1 - \mathrm{Tr}\left[\left(\rho^{\frac{q}{z}}\,\tilde{\rho}^{\frac{1-q}{z}}\right)^{z}\right]\right), \quad q \in \mathbb{R}, \;\; z \in \mathbb{R}^{+}_{0}$$

This family of quantum divergence functions satisfies the DPI when z ≥ 0 and 0 ≤ q ≤1

REMARK: some interesting limiting cases are:

When q=z=1 we recover Von Neumann’s relative entropy.
When z=1 we recover the q-Rényi relative entropies.
When z=q we recover the q-quantum Rényi divergence.
When z=1 and q=1/2 we recover the Wigner-Yanase-Dyson skew information.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

To perform coordinate-free computations we work in the space:

$$M_n = SU(n) \times \Delta_n$$

Special unitary group

$$M_n(\mathbb{C}) \ni \mathbf{U}: \quad \mathbf{U}\,\mathbf{U}^{\dagger} = \mathbb{I}, \;\; \det(\mathbf{U}) = 1$$

Open interior of the n-dimensional simplex

$$\mathbb{R}^n \ni \vec{p}: \quad \sum_{j=1}^{n} p^j = 1, \quad p^j > 0$$

SURJECTIVE SUBMERSION:

$$\pi_n : M_n \ni (\mathbf{U}, \vec{p}) \mapsto \rho = \pi_n(\mathbf{U}, \vec{p}) = \mathbf{U}\,\rho_0\,\mathbf{U}^{\dagger} \in \mathcal{S}_n \quad \text{with } \rho_0 = \mathrm{diag}(\vec{p})$$

The kernel of the differential at each point is given by the Hermitian matrices commuting with ρ_0

Unfolding of the manifold of invertible quantum states by means of the spectral decomposition

Metrics on the space of invertible quantum states

From the q-z-Rényi relative entropies to a family of potential functions on $\{M_n\}$ by means of the pullback through π_n:

$$\mathbb{D}_n^{q,z}\left(\mathbf{U},\rho_0;\mathbf{V},\tilde{\rho}_0\right)=\frac{1}{q(1-q)}\left(1-\mathrm{Tr}\left[\left(\left(\mathbf{U}\,\rho_0\,\mathbf{U}^\dagger\right)^{\frac{q}{z}}\left(\mathbf{V}\,\tilde{\rho}_0\,\mathbf{V}^\dagger\right)^{\frac{1-q}{z}}\right)^z\right]\right)$$

Remark: M_n has a basis of globally defined vector fields and differential one-forms we use to perform computations without the need to introduce coordinates:

$$\mathrm{d}\rho=\mathbf{U}\,\mathrm{d}\rho_0\,\mathbf{U}^\dagger+\mathbf{U}\left[\mathbf{U}^\dagger\mathrm{d}\mathbf{U},\rho_0\right]\mathbf{U}^\dagger=\mathbf{U}\,\mathbf{e}_j\,\mathbf{U}^\dagger\,\mathrm{d}p^j+\mathbf{U}\left[\tau_k,\rho_0\right]\mathbf{U}^\dagger\,\theta^k$$

$$\mathrm{d}\rho_0=\mathbf{e}_j\,\mathrm{d}p^j \longrightarrow \text{One-form on } \Delta_n$$

Diagonal matrix with 1 in the j-th place of the diagonal

$$\mathbf{U}^\dagger\mathrm{d}\mathbf{U}=\tau_k\,\theta^k \longrightarrow \text{Left-invariant one-form on } SU(n)$$

Left Maurer-Cartan form

Orthonormal basis in the Lie algebra of SU(n)

Metrics on the space of invertible quantum states

After a patient calculation, we obtain the symmetric covariant (0,2) tensor:

$$g_n^{q,z} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha + \frac{z}{q(1-q)} \sum_{j,k=1}^{n^2-1}{}' \, \mathcal{C}_{jk}\, \theta^j \otimes \theta^k$$

Fisher–Rao metric: "Classical-like" contribution depending only on the eigenvalues of the quantum states

Purely quantum contribution depending on eigenvalues and phases of the quantum states; it is tangent to the orbit of the action of the unitary group

Metrics on the space of invertible quantum states

$$g_n^{q,z} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha + \frac{z}{q(1-q)} \sum_{j,k=1}^{n^2-1}{}' \mathcal{C}_{jk}\, \theta^j \otimes \theta^k$$

$\sum'$ denotes the summation over all indexes except those pertaining to the Cartan subalgebra of the Lie algebra of SU(n), and:

$$\mathcal{C}_{jk} = \sum_{\alpha,\beta=1}^{n}{}' \mathcal{E}_{\alpha\beta}\, \Re\left[M_j^{\alpha\beta} M_k^{\beta\alpha}\right] \qquad \mathcal{E}_{\alpha\beta} := \frac{(p_\alpha - p_\beta)(p_\alpha^{\frac{q}{z}} - p_\beta^{\frac{q}{z}})(p_\alpha^{\frac{1-q}{z}} - p_\beta^{\frac{1-q}{z}})}{(p_\alpha^{\frac{1}{z}} - p_\beta^{\frac{1}{z}})}$$

$$\tau_k = \sum_{\alpha,\beta=1}^{n} M_k^{\alpha\beta} e_{\alpha\beta}$$

with τ_k a basis of the Lie algebra of SU(n), and $e_{\alpha\beta}$ the matrix with 1 in the (α,β) place, and 0 elsewhere

Conclusions

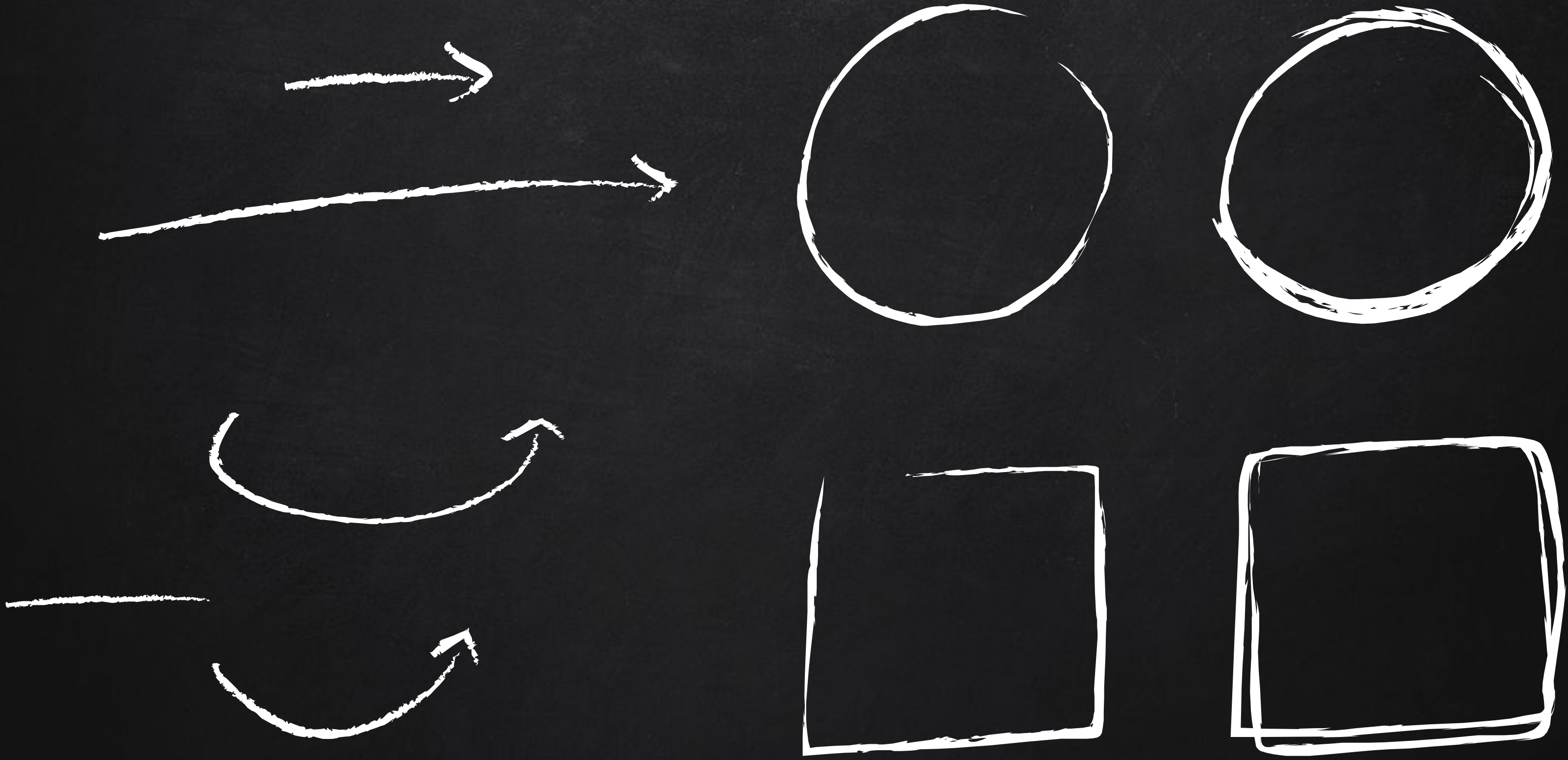
- The habit does not make the monk.....the algebraic dress of quantum mechanics hides a beautiful geometrical lingerie.
- We can give a geometrical partition of the space of quantum states by means of Lie group actions.
- Exploiting the language of differential geometry we were able to define a coordinate-free extraction algorithm for metric tensors starting from potential functions. It works for classical and quantum systems. It allows to extract skewness tensors too.
- We can successfully apply this algorithm to the quantum case, where we find that DPI implies MP, and that the MP metrics associated with q-z-Rényi relative entropies always decompose as the sum of the Fisher-Rao metric with a purely quantum contribution.

Happy Birthday!!!

There is no future.
There is no past.
Do you see?
Time is simultaneous, an intricately structured jewel that humans insist on viewing one edge at a time, when the whole design is visible in every facet.

Dr. Manhattan

Extra graphics



Thank You for your attention

“Heard joke once: Man goes to doctor. Says he's depressed. Says life seems harsh and cruel. Says he feels all alone in a threatening world where what lies ahead is vague and uncertain. Doctor says, "Treatment is simple. Great clown Pagliacci is in town tonight. Go and see him. That should pick you up." Man bursts into tears. Says, "But doctor....I am Pagliacci.”