Conditions for legitimate memory kernel master equation

Dariusz Chruściński

Nicolaus Copernicus University, Toruń, POLAND

60 Years Alberto Ibort Fest

The problem

- How to describe (non-relativistic) evolution of the density matrix

$$\rho \longrightarrow \rho_t$$

  beyond Markovian semi-group
- non-Markovian (or evolution with memory): how to define, how to characterize?

Quantum evolution $\longleftrightarrow$ dynamical map

$$\Lambda_t \ : \ \mathcal{D}(\mathcal{H}) \longrightarrow \mathcal{D}(\mathcal{H}) \quad ; \quad (t \geq 0)$$

$$\mathcal{D}(\mathcal{H}) \ = \ \{ \ \rho \geq 0 \ ; \quad \mathrm{Tr}\rho = 1 \ \}$$

- completely positive
- trace-preserving
- $\Lambda_0 = \mathbb{1}$

Completely positive maps

$$\Phi \ : \ \mathcal{A} \ \longrightarrow \ \mathcal{B}(\mathcal{H})$$

Stinespring 1955

Φ is completely positive iff

- there exists a Hilbert space $\mathcal{K}$
- there exists $\star$-homomorhism $\pi : \ \mathcal{A} \ \longrightarrow \ \mathcal{B}(\mathcal{K})$
- there exists $V : \mathcal{K} \longrightarrow \mathcal{H}$

$$\Phi[a] = V\pi(a)V^*$$

$$\dim \mathcal{H} = d < \infty$$

Kraus representation

$$\Phi[X] = \sum_\alpha K_\alpha X K_\alpha^\dagger$$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I}$$

unitary map $\longrightarrow$ $\Phi[X] = UXU^\dagger$

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Why complete positivity

Positive maps

$$X \geq 0 \quad \longrightarrow \quad \Phi[X] \geq 0$$

Φ_1 , Φ_2 – positive maps

$\Phi_1 \otimes \Phi_2$ – needs NOT be a positive map!!!

Why complete positivity

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completely positive maps $\subset$ positive maps

$$\Phi_1 \ , \ \Phi_2 \ \text{– CP maps} \ \longrightarrow \ \Phi_1 \otimes \Phi_2 \ \text{– is a CP map}$$

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1. Markovian semigroup
2. and beyond

Markovian semi-group

$$\frac{d}{dt}\Lambda_t = L\Lambda_t \; ; \quad \longrightarrow \quad \Lambda_t = e^{tL} \; ; \; t \geq 0$$

What is the most general L ?

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{tL}$ *is CPTP if and only if*

$$L[\rho] \;=\; -i[H,\rho] + \sum_{kl}\gamma_{kl}\left(V_k\rho V_l^\dagger - \frac{1}{2}\{V_l^\dagger V_k, \rho\}\right) \; ; \; [\gamma_{kl}] \geq 0$$

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A Brief History of the GKLS Equation

Dariusz Chruściński

Institute of Physics, Faculty of Physics, Astronomy and Informatics
Nicolaus Copernicus University, Grudziądzka 5/7, 87–100 Toruń, Poland

e-mail: darch@fizyka.umk.pl

Saverio Pascazio

Dipartimento di Fisica and MECENAS, Università di Bari, I-70126 Bari, Italy
Istituto Nazionale di Ottica (INO-CNR), I-50125 Firenze, Italy
INFN, Sezione di Bari, I-70126 Bari, Italy

e-mail: saverio.pascazio@ba.infn.it

How to go beyond Markovian semigroup ?

$$\frac{d}{dt}\Lambda_t = L\Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

$$L \longrightarrow L_t$$

$$\frac{d}{dt}\Lambda_t = L_t\Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

$$\frac{d}{dt}\Lambda_t = \int_0^t K_{t-\tau}\Lambda_\tau d\tau \quad ; \quad \Lambda_0 = \mathbb{1}$$

$K_t = \delta(t)\,L \longrightarrow$ Markovian semigroup

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$K_t = \delta(t)\, L \;\longrightarrow\;$ Markovian semigroup

Example: qubit dephasing

$$\Lambda_t[\rho] = \begin{pmatrix} \rho_{11} & \rho_{12}\cos t \\ \rho_{21}\cos t & \rho_{22} \end{pmatrix}$$

$$L_t[\rho] = \gamma(t)[\sigma_3\rho\sigma_3 - \rho]$$

$$K_t[\rho] = k(t)[\sigma_3\rho\sigma_3 - \rho]$$

Spectrum: $\lambda_1(t) = \lambda_2(t) = 1$; $\lambda_3(t) = \lambda_4(t) = \cos t$

If $\cos t$ crosses 0 the generators L_t becomes singular!!!

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$$K_t[\rho] = k(t)[\sigma_3\rho\sigma_3 - \rho]$$

$\gamma(t) = \tan t$ (singular)

$k(t) = 1$ (regular)

$$\frac{d}{dt}\Lambda_t = L_t\Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

$$L_t = ???$$

$$\frac{d}{dt}\Lambda_t = K_t * \Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

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$$\frac{d}{dt}\Lambda_t = L_t\Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

technically simpler

– –

$$\frac{d}{dt}\Lambda_t = K_t * \Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

more fundamental

Nakajima-Zwanzig projection method

$$\mathcal{H}_S \otimes \mathcal{H}_E$$

$$H = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I$$

$$\Lambda_t[\rho_S] := \mathrm{Tr}_E\left(e^{-iHt}[\rho_S \otimes \rho_E]e^{iHt}\right)$$

$$\frac{d}{dt}\Lambda_t = \int_0^t K_{t-\tau}\Lambda_\tau \, d\tau$$

memory kernel master equation is universal

Markovian semigroup

$$\frac{d}{dt}\Lambda_t = L\Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

- Born-Markov approximation
- weak coupling
- singular coupling (reservoir correlation functions $\sim \delta(t)$)
- ...

Current experiments call for more refine approach

Markovian vs. non-Markovian

Divisibility

$$\Lambda_t = V_{t,s}\Lambda_s \quad ; t \geq s$$

- CP-divisible if $V_{t,s}$ is CP
- P-divisible if $V_{t,s}$ is positive

CP-divisible $\longrightarrow$ P-divisible

Markovian semigroup $\longrightarrow$ $V_{t,s} = e^{(t-s)L}$ (CP-divisible)

Theorem (Benatti,DC,Fillipov (2017))

Λ_t is CP-divisible iff $\Lambda_t \otimes \Lambda_t$ is P-divisible

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Markovian vs. non-Markovian

- Markovianity is defined for classical stochastic processes
- Markovianity = semigroup dynamics: $\Lambda_t = e^{tL}$
- Markovianity = CP-divisibility (Rivas, Huelga, Plenio): $\Lambda_t = V_{t,s}\Lambda_s$
- Markovianity = negative information flow (Breuer, Laine, Piilo)
- Geometrical characterization of non-Markovianity (Lorenzo, Plastina, Paternostro)
- non-Markovianity via mutual information (Luo)
- non-Markovianity via channel capacity (Bylicka, DC, Maniscalco)
- non-Markovianity via channel discrimination (Bae, DC)
- ...

Suppose that Λ_t^{-1} exists

Theorem (DC, Kossakowski, Rivas (2011))

Λ_t is CP-divisible iff

$$\frac{d}{dt} \| [\mathbb{1} \otimes \Lambda_t] X \|_1 \leq 0$$

for all $X = X^\dagger \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$.

Λ_t is P-divisible iff

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Breuer-Laine-Piilo (BLP) condition – PRL 2010

Evolution is Markovian if

$$\sigma(\rho_1, \rho_2; t) := \frac{d}{dt} ||\Lambda_t(\rho_1 - \rho_2)||_1 \leq 0$$

for all pairs ρ_1 and ρ_2.

$$\frac{d}{dt} || \Lambda_t X ||_1 \leq 0 \; ; \quad X = \rho_1 - \rho_2$$

CP-divisibility $\Longrightarrow$ P-divisibility $\Longrightarrow$ BLP condition

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$$\frac{d}{dt} ||\, \Lambda_t X \,||_1 \leq 0 \; ; \quad X = \rho_1 - \rho_2$$

CP-divisibility $\Longrightarrow$ P-divisibility $\Longrightarrow$ BLP condition

Example: random unitary

$$L_t\rho = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)[\sigma_k\rho\sigma_k - \rho] = \sum_k \gamma_k(t)L_k$$

- Λ_t is CP-divisible iff

$$\gamma_1(t) \geq 0 \ ; \quad \gamma_2(t) \geq 0 \ ; \quad \gamma_3(t) \geq 0$$

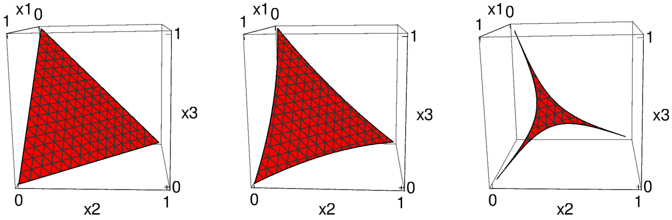
- Λ_t is P-divisible iff (P-divisible $\equiv$ BLP condition)

$$\gamma_1(t) + \gamma_2(t) \geq 0 \ ; \quad \gamma_1(t) + \gamma_3(t) \geq 0 \ ; \quad \gamma_2(t) + \gamma_3(t) \geq 0$$

DC, F. Wudarski, PLA 2013.

$\Lambda_t = x_1 e^{tL_1} + x_2 e^{tL_2} + x_3 e^{tL_3}$

- Λ_t is a Markovian semi-group if only one $x_k = 1$
- Λ_t is CP-divisible
- Λ_t is P-divisible for all x_k ($\equiv$ BLP condition)



N.Megier, DC, J.Piilo, W.Strunz, Sc. Rep. 2017

Memory effects via a memory kernel

Problem: physically admissible kernels

$$\frac{d}{dt}\Lambda_t = \int_0^t K_{t-\tau}\Lambda_\tau d\tau$$

$$K_t = ???$$

$$\frac{d}{dt}\Lambda_t = \int_0^t K_{t-\tau}\Lambda_\tau d\tau$$

Find the structure of K_t such that Λ_t is CPTP

- Barnett and Stenholm (2001): $K_t = k(t)\mathcal{L}$
- Lidar and Shabani (2005): $K_t = k(t)\mathcal{L}e^{t\mathcal{L}}$
- ...

$$k(t) = ???$$

How to construct a legitimate K_t ?

Markovian semigroup — quantum jumps representation

$$L[\rho] = -i[H,\rho] + \sum_{\alpha} \left(V_{\alpha}\rho V_{\alpha}^{\dagger} - \frac{1}{2}\{V_{\alpha}^{\dagger}V_{\alpha}, \rho\} \right)$$

$$L = B - Z$$

$$B[\rho] = \sum_{\alpha} V_{\alpha}\rho V_{\alpha}^{\dagger}$$

$$Z[\rho] = i(C\rho - \rho C^{\dagger}) \quad ; \quad C = H - \frac{i}{2}\sum_{\alpha} V_{\alpha}^{\dagger}V_{\alpha}$$

(Wigner–Weisskopf theory)

$$B[\rho] = \sum_\alpha V_\alpha \rho V_\alpha^\dagger \;;\quad C = H - \frac{i}{2}\sum_\alpha V_\alpha^\dagger V_\alpha$$

$$\dot{\Lambda}_t = (B - Z)\Lambda_t \quad ; \quad \Lambda_0 = \mathbb{1}$$

$$\dot{N}_t = -ZN_t \quad ; \quad N_0 = \mathbb{1}$$

$$N_t[\rho] = e^{-iCt}\rho e^{iC^\dagger t}$$

Dyson perturbation series

$$\Lambda_t = N_t + N_t * BN_t + N_t * BN_t * BN_t + \ldots$$

$$\Lambda_t = N_t + N_t * BN_t + N_t * BN_t * BN_t + \ldots$$

$$Q_t := BN_t$$

$$\Lambda_t = N_t + N_t * (Q_t + Q_t * Q_t + Q_t * Q_t * Q_t + \ldots)$$

N_t, Q_t –completely positive maps

quantum jump representation

$$\dot{\Lambda}_t = L\Lambda_t$$

$$\Lambda_t = e^{tL} = \mathbb{1} + tL + \frac{(tL)^2}{2!} + \frac{(tL)^3}{3!} + \dots$$

$$L = B - Z$$

$$\Lambda_t \quad = \quad N_t + N_t * Q_t + N_t * Q_t * Q_t + \dots$$

trace preservation vs. complete positivity

$L \longrightarrow K_t$

$$\dot{\Lambda}_t = L\Lambda_t$$

$$L = B - Z$$

$$\Lambda_t = N_t + N_t * BN_t + N_t * BN_t * BN_t + \dots$$

$$\dot{\Lambda}_t = K_t * \Lambda_t$$

$$K_t = B_t - Z_t$$

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$$Q_t := B_t * N_t$$

$$\Lambda_t = N_t + N_t * (Q_t + Q_t * Q_t + Q_t * Q_t * Q_t + \ldots)$$

Legitimate pair

We call $\{N_t, Q_t\}$ a legitimate pair iff

1. N_t, Q_t are CP, and $N_0 = \mathbb{1}$
2. $\mathrm{Tr}[(Q_t + \dot{N}_t)\rho] = 0$

$$\Lambda_t = N_t + N_t * (Q_t + Q_t * Q_t + Q_t * Q_t * Q_t + \ldots)$$

$$K_t = B_t - Z_t$$

$$Q_t = B_t * N_t \quad ; \quad \dot{N}_t = -Z_t * N_t$$

DC, A. Kossakowski, PRA (2016)

Barnett and Stenholm generator: $K_t = k(t)\mathcal{L}$

$$N_t = \left(1 - \int_0^t f(\tau)d\tau\right) \mathbb{1}$$

$$f(t) \geq 0 \quad ; \quad \int_0^\infty f(\tau)d\tau \leq 1$$

$Q_t = f(t)\mathcal{E}$; $\mathcal{E}$ – arbitrary quantum channel

$$K_t = k(t)(\mathcal{E} - \mathbb{1}) = k(t)\mathcal{L}$$

$$\widetilde{k}(s) = \frac{s\widetilde{f}(s)}{1 - \widetilde{f}(s)}$$

$f(t) = \gamma e^{-\gamma t} \implies k(t) = \gamma\delta(t) \longrightarrow$ Markovian semi-group

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Properties — 1

Convexity

If $\{N_t^{(k)}, Q_t^{(k)}\}$ are legitimate pairs then a convex combination

$$N_t = \sum_k p_k N_t^{(k)} \quad ; \quad Q_t = \sum_k p_k Q_t^{(k)}$$

provide a legitimate pair.

Properties — 2

Reduced pair

Suppose that $\{\mathbf{N}_t, \mathbf{Q}_t\}$ defines a legitimate pair for the evolution in $\mathcal{H} \otimes \mathcal{H}_E$. Then for arbitrary state ω in $\mathcal{H}_E$

$$N_t[\rho] = \mathrm{Tr}_E(\mathbf{N}_t[\rho \otimes \omega]),\ Q_t[\rho] = \mathrm{Tr}_E(\mathbf{Q}_t[\rho \otimes \omega]),$$

provide a legitimate pair for the evolution in $\mathcal{H}$.

Properties — 3

Gauge transformations

If $\{N_t, Q_t\}$ is a legitimate pair and $\mathcal{F}_t$ is a dynamical map, then

$$N'_t = \mathcal{F}_t N_t \; ; \quad Q'_t = \mathcal{F}_t Q_t,$$

provide a legitimate pair as well.

Inhomogeneous master equation

$$\frac{d}{dt}\rho_t = \int_0^t K_{t-\tau}\rho_\tau d\tau$$

$$\{N_t, Q_t\}$$

$$\frac{d}{dt}\rho_t = \int_0^t \mathbb{K}_{t-\tau}\rho_\tau d\tau + \frac{d}{dt}N_t\rho_0$$

ρ_0 – initial state

$$\mathbb{K}_t = \frac{d}{dt}Q_t + \delta(t)Q_0$$

Inhomogeneous master equation

$$\{N_t, Q_t\}$$

$$\frac{d}{dt}\rho_t = Q_0\rho_t + \int_0^t \left(\frac{d}{dt}\mathbb{Q}_{t-\tau}\right)\rho_\tau d\tau + \frac{d}{dt}N_t\rho_0$$

Inhomogeneous master equation

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Example

$$Q_t = f(t)\mathcal{E} \ ; \quad N_t = (1 - \int_0^t f(u)du)\mathbb{1}$$

$$\dot{\rho}_t = \gamma\mathcal{E}\rho_t + \int_0^t \dot{f}(t-\tau)\mathcal{E}[\rho_\tau]d\tau - f(t)\rho_0$$

$$\gamma := f(0)$$

What does fit our class ?

- quantum semi-Markov evolution (Breuer, Vacchini)
- collision models (Palma, Giovanetti, Lorenzo, Cicarello, Vacchini,...)
- .....

Quantum semi-Markov evolution

Q_t — quantum semi-Markov map

$\mathbf{f}_t = Q_t^\dagger[\mathbb{I}]$ — quantum waiting time operator

$\mathbf{g}_t = \mathbb{I} - \int_0^t \mathbf{f}_\tau d\tau$ — quantum survival operator

$$N_t[\rho] = \sqrt{\mathbf{g}_t}\,\rho\,\sqrt{\mathbf{g}_t}$$

$\{Q_t, N_t\}$; — legitimate pair

$\mathbf{f}_t = \Gamma e^{-\Gamma t}\,;\Gamma \geq 0 \implies$ Markovian semigroup

DC & A. Kossakowski (2017)

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$$N_t[\rho] = \sqrt{\mathbf{g}_t}\, \rho \sqrt{\mathbf{g}_t}$$

$\{Q_t, N_t\}$; — legitimate pair

$\mathbf{f}_t = \Gamma e^{-\Gamma t}\ ; \Gamma \geq 0 \implies$ Markovian semigroup

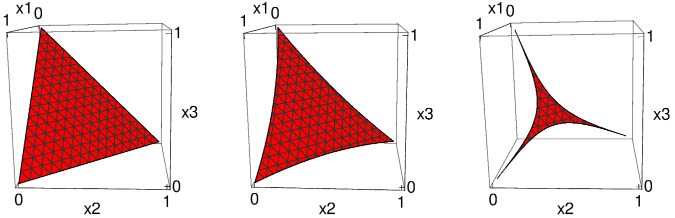
DC & A. Kossakowski (2017)

This construction via $\{Q_t, N_t\}$ covers MANY examples

But still there are examples outside this class

$\Lambda_t = x_1 e^{tL_1} + x_2 e^{tL_2} + x_3 e^{tL_3}$

- Λ_t is a Markovian semi-group if only one $x_k = 1$
- Λ_t is semi-Markov if $x_1 = x_2 = x_3 = \frac{1}{3}$
- Λ_t is CP-divisible
- Λ_t is P-divisible for all x_k



N.Megier, DC, J.Piilo, W.Strunz, Sc. Rep. 2017

New family of kernels

(Wudarski, Nalezyty, Sarbicki, and DC, PRA 2015)

$$\frac{d}{dt}\Lambda_t = \int_0^t K_{t-\tau}\Lambda_\tau \, d\tau$$

$$K_t[\rho] = \frac{1}{2}\sum_{i=1}^{3} k_i(t)(\sigma_i\rho\sigma_i - \rho) \ ; \quad K_t[\sigma_i] = \kappa_i(t)\sigma_i$$

$$\tilde{\lambda}_i(s) = \frac{1}{s - \tilde{\kappa}_i(s)} \quad i = 1, 2, 3$$

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$$\tilde{\lambda}_i(s) = \frac{1}{s - \tilde{\kappa}_i(s)} \quad i = 1, 2, 3$$

Theorem

Let $a_1, a_2, a_2 > 0$ *and* $f : [0, \infty) \to \mathbb{R}$ *such that*

$$\frac{1}{a_1} + \frac{1}{a_2} \geq \frac{1}{a_3} \ + \ \textit{cyclic perm.}$$

$$0 \leq \int_0^t f(\tau) d\tau \leq 4 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)^{-1}$$

then

$$\widetilde{\kappa}_i(s) = -\frac{s \widetilde{f}(s)}{a_i - \widetilde{f}(s)}$$

define a legitimate memory kernel K_t.

Bloch equation

$$L[\rho] = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(\sigma_k\rho\sigma_k - \rho)$$

Bloch vector $\rightarrow \quad x_k = \mathrm{Tr}(\rho\sigma_k)$

$$\frac{d}{dt}x_k = \frac{1}{T_k}x_k \; ; \qquad k = 1, 2, 3$$

$$\frac{1}{T_1} = \gamma_2 + \gamma_3 \; ; \quad \frac{1}{T_2} = \gamma_3 + \gamma_1 \; ; \quad \frac{1}{T_3} = \gamma_1 + \gamma_2$$

CP $\iff \quad \frac{1}{T_1} + \frac{1}{T_2} \geq \frac{1}{T_3}$ etc

Bloch equation

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$$\frac{d}{dt}x_k = \frac{1}{T_k}x_k \; ; \qquad k = 1, 2, 3$$

$$\frac{1}{T_1} = \gamma_2 + \gamma_3 \; ; \quad \frac{1}{T_2} = \gamma_3 + \gamma_1 \; ; \quad \frac{1}{T_3} = \gamma_1 + \gamma_2$$

CP $\iff \quad \frac{1}{T_1} + \frac{1}{T_2} \geq \frac{1}{T_3}$ etc

Multi-exponential decay of memory effects

Let $W(s)$ be a polynomial of the form

$$W(s) = (s + z_1) \dots (s + z_n)$$

and let all $z_i > 0$. If

$$\prod_{i=1}^{n} z_i \geq \frac{1}{4}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)$$

then $\widetilde{f}(s) = 1/W(s)$ satisfies the Theorem.

$$z_k \neq z_l \implies \kappa_i(t) = A_{i0}\delta(t) + A_{i1}e^{-w_{i1}t} + \ldots + A_{ir}e^{-w_{ir}t}$$

(Bernstein theorem

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Example (M. Hall, E. Anderson, ... PRA 2014)

$$L_t[\rho] = \frac{1}{2}\sum_{i=1}^{3} \gamma_i(t)(\sigma_i \rho \sigma_i - \rho)$$

$$\gamma_1 = \gamma_2 = 1 \quad ; \quad \gamma_3(t) = -\tanh t < 0$$

$$\Lambda_t = \frac{1}{2}(e^{tL_1} + e^{tL_2})$$

$$W(s) = s + 2$$

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Summary

- I provided a construction for a family of legitimate kernels in terms of legitimate pairs $\{N_t, Q_t\}$
- many known examples fit this class
- this class defines a natural generalization of classical semi-Markov evolution
- may be used to engineering quantum evolution (suppression of decoherence/dissipation)

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Alberto happy birthday!!!