

Killing vector fields and quantisation of natural Hamiltonians

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Abstract

The usual canonical prescription ordinarily made for the obtention of the quantum Hamiltonian operator for a classical system leads to some ambiguities in situations beyond the simplest ones and these ambiguities arise unavoidably when the configuration space has non-zero curvature, as well as in systems in Euclidean space but with a position-dependent mass. A recently proposed method to circumvent this difficulty for natural Hamiltonians will be described. The idea is not to quantise the coordinates and their (classical) conjugate momenta (which is where the ambiguities could arise), but to work directly with Killing vector fields and associated Noether momenta in order to get in some unambiguous way the corresponding Hamiltonian operator. The examples of one-dimensional position-dependent mass systems and motions on constant curvature surfaces will be used to illustrate the method.

This is a report on previous collaborations with:

M.F. Rañada and M.Santander

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Introduction

I meet (Luis) Alberto for the first time as a student of the academic year 1976–77 of my course Mathematical Methods for Physicists II, and when finishing his studies in Physics he obtained a grant for doing his Ph. D. Thesis, entitled

Estructura geométrica de los sistemas con simetría en Mecánica Clásica y Teoría Clásica de Campos

The defence was in 1984. This was the starting point for our collaboration for almost 40 years.

Then he went for his postdoc to Paris to work in the group of Prof. Marle and Copenhagen, and a bit later he spent a year at Berkeley to collaborate with Prof. Weinstein.

Our first common scientific visit to Europe was to the 1st Workshop on Diff. Geom. Methods in Classical Mechanics, an excellent idea of Willy Sarlet to connect people of different countries with a close scientific interest



This was the beginning of a series of meetings.

The second was organised in [Jaca](#) (1987) with the collaboration of Alberto



Particularly intense was our collaboration during the

NATO ASI-XXIII GIFT Seminar “Recent Problems in Mathematical Physics

and the

XIX Int. Colloquium on Group Theoretical Methods in Physics

in Salamanca, 1992.



José F. Cariñena
Alberto Ibort
Giuseppe Marmo
Giuseppe Morandi

Geometry from Dynamics, Classical and Quantum

 Springer

A long collaboration

The book is devoted to establish a geometric approach to both **Classical and Quantum Mechanics** and the transition from Classical to Quantum Mechanics.

Symplectic geometry is the common framework for dealing with both types of systems.

The geometric framework for the description of classical mechanical systems is the theory of **Hamiltonian dynamical systems**.

A **symplectic** structure ω on a differentiable manifold M , or more generally a **Poisson** structure, is the **basic** concept.

It is then possible to define an associated **Poisson bracket** endowing the set of functions on M with a real **Lie algebra structure**.

The **dynamics** is given by the **Hamiltonian vector field** X_H defined by the Hamiltonian $H \in C^\infty(M)$ by means of

$$i(X_H)\omega = dH.$$

The system of differential equations determining the integral curves of X_H in **Darboux coordinates** are **Hamilton equations**.

A particularly interesting case is when the manifold is the cotangent bundle of the configuration space Q , $M = T^*Q$, endowed with its natural symplectic structure.

The states are the points of M , and the observables are the functions $F \in C^\infty(M)$. The measure of an observable F in a state $x \in M$ is given by the evaluation map, the result being $F(x)$

On the other side, the mathematical model for Quantum Theories is different. In Quantum Mechanics in Schrödinger picture:

- (pure) states are (rays rather than) vectors ψ of a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$
- observables are selfadjoint operators in \mathcal{H}
- the results of the measure of the observable A on the pure state ψ may be any eigenvalue of A but with probabilities such that the mean value is given by

$$e(\psi) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}$$

- dynamics is given by Schrödinger equation

The framework unifying both approaches is the theory of Hamiltonian dynamical systems.

Hamiltonian dynamical systems

A **symplectic manifold** is a pair (M, ω) where M is a differentiable manifold and ω is a symplectic form, i.e. a non-degenerate closed 2-form in M , $\omega \in Z^2(M)$: $d\omega = 0$.

Non-degeneracy of ω means that for every point $u \in M$ the map $\widehat{\omega}_u : T_u M \rightarrow T_u^* M$:

$$\langle \widehat{\omega}_u(v), v' \rangle = \omega_u(v, v') , \quad v, v' \in T_u M,$$

is a bijection. This implies that **the dimension of M is even**, $\dim M = 2n$.

$\widehat{\omega} : TM \rightarrow T^*M$ is a base-preserving fibred map, i.e. the following diagram :

$$\begin{array}{ccc} TM & \xrightarrow{\widehat{\omega}} & T^*M \\ \tau \downarrow & & \downarrow \pi \\ M & \xrightarrow{\text{id}_M} & M \end{array}$$

is commutative and **induces a \mathbb{R} -linear map between the spaces of sections of both bundles** which, with a slight abuse of notation, we also write $\widehat{\omega} : \mathfrak{X}(M) \rightarrow \bigwedge^1(M)$.

The vector fields corresponding to **closed forms** are called **locally-Hamiltonian vector fields** and those corresponding to **exact forms** are said to be **Hamiltonian vector fields**.

$$\widehat{\omega}(\mathfrak{X}_H(M, \omega)) = B^1(M), \quad \widehat{\omega}(\mathfrak{X}_{LH}(M, \omega)) = Z^1(M).$$

If $H \in C^\infty(M)$, the **Hamiltonian vector field** X_H is defined by the vector field s.t.

$$i(X_H)\omega = dH$$

(M, ω, H) is a **Hamiltonian system** whenever (M, ω) is a symplectic manifold and $H \in C^\infty(M)$: the dynamical vector field is X_H

Cartan identity, $\mathcal{L}_X = i(X) \circ d + d \circ i(X)$, shows that $X \in \mathfrak{X}_{LH}(M, \omega)$ if and only if $\mathcal{L}_X \omega = 0$.

Darboux proved that **$d\omega = 0$ and regularity of ω** imply that around each point $u \in M$ **there is a local chart (U, ϕ) such that** if $\phi = (q^1, \dots, q^n; p_1, \dots, p_n)$, then

$$\omega|_U = \sum_{i=1}^n dq^i \wedge dp_i.$$

Such coordinates are said to be Darboux coordinates. The expression of X_H in Darboux coordinates is given by

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right),$$

and therefore, the local equations determining its integral curves are similar to Hamilton equations.

$$\begin{cases} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \end{cases}$$

Define the *Poisson bracket* of two functions $f, g \in C^\infty(M)$ as being the function $\{f, g\}$ given by:

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f) .$$

In Darboux coordinates for ω the expression for $\{f, g\}$ is the usual one:

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) .$$

If X, Y are locally Hamiltonian vector fields, then $[X, Y]$ is a Hamiltonian vector field, its Hamiltonian being $\omega(Y, X)$.

It is a consequence of the relation $i(X)\mathcal{L}_Y\alpha - \mathcal{L}_Y i(X)\alpha = i([X, Y])\alpha$, which is valid for any form α . We then obtain:

$$\begin{aligned} i([X, Y])\omega &= i(X)\mathcal{L}_Y\omega - \mathcal{L}_Y i(X)\omega = -\mathcal{L}_Y i(X)\omega = \\ &= -i(Y)d[i(X)\omega] - d[i(Y)i(X)\omega] = -d[\omega(X, Y)] . \end{aligned}$$

In particular, when $X = X_f$ and $Y = X_g$ in the previous relation:

$$d\{f, g\} = -i([X_f, X_g])\omega ,$$

i.e.,

$$[X_f, X_g] = X_{\{g, f\}}.$$

This shows that the set of Hamiltonian vector fields, to be denoted $\mathfrak{X}_H(M, \omega)$, is an ideal of the Lie algebra of locally-Hamiltonian vector fields $\mathfrak{X}_{LH}(M, \omega)$ and that

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\sigma} \mathfrak{X}_H(M, \omega) \longrightarrow 0$$

with $\sigma = -\widehat{\omega}^{-1} \circ d$, is an exact sequence of Lie algebras.

An action Φ of a Lie group G on M defines a set of **fundamental vector fields** $X_a \in \mathfrak{X}(M)$, $a \in \mathfrak{g}$, by $X_a(m) = \Phi_{m*e}(-a)$ and **the map** $X : \mathfrak{g} \rightarrow \mathfrak{X}(M)$, $a \in \mathfrak{g} \rightarrow X_a$ is a **Lie algebra homomorphism**,

$$[X_a, X_b] = X_{[a,b]}.$$

If the action of G is **strongly symplectic**, $X(\mathfrak{g}) \subset \mathfrak{X}_H(M, \omega)$, then **X is a Lie algebra homomorphism** $X : \mathfrak{g} \rightarrow \mathfrak{X}_H(M, \omega)$, and then **there exists a linear map** $f : \mathfrak{g} \rightarrow C^\infty(M)$, called **comomentum map**, making commutative the following diagram:

$$\begin{array}{ccccccc} & & & & \mathfrak{g} & & \\ & & & & \downarrow X & & \\ & & f & \nearrow & & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \xrightarrow{\sigma} & \mathfrak{X}_H(M, \omega) \longrightarrow 0 \end{array}$$

The corresponding **momentum map** introduced by Souriau, is the map $P : M \rightarrow \mathfrak{g}^*$, defined by

$$\langle P(m), a \rangle = f_a(m), \quad \forall m \in M, a \in \mathfrak{g}.$$

It is not uniquely defined but **two possible comomentum map** differ by a linear map $r : \mathfrak{g} \rightarrow \mathbb{R}$,

$$f'_a(m) = f_a(m) + r(a).$$

In case of exact symplectic actions on exact symplectic manifolds (M, θ) , namely, $g^*\theta = \theta, \forall g \in G$, the fundamental vector fields X_a are Hamiltonian and a comomentum map can be defined as $f_a = -i(X_a)\theta$. This is the case of an action of a Lie group G on a cotangent bundle T^*Q lifted from an action of G on its base manifold.

Recall that if $X = \xi^i(q)\partial/\partial q^i \in \mathfrak{X}(Q)$, its cotangent lift $\tilde{X} \in \mathfrak{X}(T^*Q)$ (satisfying $\mathcal{L}_{\tilde{X}}\theta = 0$, with θ the Liouville 1-form), $\theta = p_i dq^i$, is

$$\tilde{X} = \xi^i \frac{\partial}{\partial q^i} - p_i \frac{\partial \xi^i}{\partial q^j} \frac{\partial}{\partial p_j}.$$

As an instance if the configuration space is $Q = \mathbb{R}^3$ we can consider vector fields generating translations on $Q = \mathbb{R}^3$, and their corresponding lift in $T^*\mathbb{R}^3$, the fundamental vector fields in \mathbb{R}^3 being $X_a = -a^i \partial/\partial q^i$, with canonical lifts \tilde{X}_a a $T^*\mathbb{R}^3$

$$\tilde{X}_a = -a^i \frac{\partial}{\partial q^i}.$$

The functions $f_a \in C^\infty(T^*\mathbb{R}^3)$ are defined by

$$f_a(q, p) = -[i(\tilde{X}_a)\theta](q, p) = a^i p_i.$$

Using $f_a(q, p) = \langle \vec{P}(q, p), a \rangle$ and identifying \mathbb{R}^3 , as Lie algebra of translations, with its dual, we obtain

$$\vec{P}(q, p) = \vec{p}.$$

If consider the action of $SO(3, \mathbb{R})$ on \mathbb{R}^3 and the induced action on $T^*\mathbb{R}^3$, the fundamental vector fields in \mathbb{R}^3 are

$$X_i = -\epsilon_{ijk} q^j \frac{\partial}{\partial q^k},$$

with canonical lifts

$$\tilde{X}_i = -\epsilon_{ijk} \left(q^j \frac{\partial}{\partial q^k} + p_j \frac{\partial}{\partial p_k} \right),$$

and consequently a comomentum map is

$$f_{\vec{n}}(q, p) = n_i \epsilon_{ijk} q^j p_k = \vec{n} \cdot (\vec{q} \times \vec{p}).$$

Using the identification of \mathbb{R}^3 with its dual space we see that

$$\vec{P}(q, p) = \vec{q} \times \vec{p},$$

i.e. the associated momentum is the angular momentum

Dynamical systems of mechanical type

A regular **natural Lagrangian** system is given by a **non-degenerate symmetric $(0, 2)$ -tensor field g** on the configuration space Q and a **function V on Q** : The Lagrangian function $L \in C^\infty(TQ)$ is given by

$$L_{g,V}(v) = \frac{1}{2}(\tau_Q^* g)(v, v) + \tau^* V,$$

where $\tau_Q : TQ \rightarrow Q$ is the tangent bundle.

Nondegeneracy means that the map $\hat{g} : TQ \rightarrow T^*Q$ from the tangent bundle $\tau_Q : TQ \rightarrow Q$ to the cotangent bundle $\pi_Q : T^*Q \rightarrow Q$, defined by $\langle \hat{g}(v), w \rangle = g(v, w)$, where $v, w \in T_x Q$, is **regular**.

\hat{g} is a **fibred map over the identity on Q** and induces a map between the spaces of sections $\hat{g} : \mathfrak{X}(Q) \rightarrow \Omega^1(Q)$: $\langle \hat{g}(X), Y \rangle = g(X, Y)$.

The vector field corresponding to the exact 1-form df is called **grad f** , i.e.

$$\hat{g}(\text{grad } f) = df, \quad \forall f \in C^\infty(Q).$$

The case $V = 0$ corresponds to free motion on the Riemann manifold (Q, g) , the Lagrangian then being given by the kinetic energy defined by the metric g :

$$T_g(v) = \frac{1}{2}(\tau_Q^*g)(v, v), \quad v \in TQ,$$

which can be rewritten as the following function on TQ :

$$T_g = \frac{1}{2} g(T\tau_Q \circ D, T\tau_Q \circ D),$$

with D being any second order differential equation vector field, i.e. a vector field on TQ , and therefore $\tau_{TQ} \circ D = \text{id}_{TQ}$, such that also $T\tau_Q \circ D = \text{id}_{TQ}$.

If (U, q^1, \dots, q^n) is a local chart on Q we consider the coordinate basis of $\mathfrak{X}(U)$ denoted $\{\partial/\partial q^j \mid j = 1, \dots, n\}$ and its dual basis for $\Omega^1(U)$, $\{dq^j \mid j = 1, \dots, n\}$.

A vector and a covector in a point $q \in U$ are $v = v^j (\partial/\partial q^j)_q$ and $\zeta = p_j (dq^j)_q$, with $v^j = \langle dq^j, v \rangle$ and $p_j = \langle \zeta, \partial/\partial q^j \rangle$. The local expressions for g and T_g are:

$$g = g_{ij}(q) dq^i \otimes dq^j, \quad T_g(v) = \frac{1}{2} g_{ij}(\tau_Q(v)) v^i v^j,$$

while the coordinate expression for the gradient of a function $f \in C^\infty(Q)$ is:

$$(\text{grad } f)^i = g^{ij} \frac{\partial f}{\partial q^j}, \quad \text{with} \quad \sum_{k=1}^n g^{ik}(q) g_{kj}(q) = \delta_j^i.$$

The **dynamics** is then given by a vector field Γ_L solution of the dynamical equation

$$i(\Gamma_L)\omega_L = dE_L.$$

where the **energy** E_L of the Lagrangian system is defined by $E_L = \Delta L - L$, with Δ being the **Liouville vector field**, generator of dilations along the fibres, given by

$$\Delta f(v) = \frac{d}{dt} f(e^t v)|_{t=0},$$

for all $v \in TQ$ and $f \in C^\infty(TQ)$.

As $\Delta(T_g) = 2T_g$ and $\Delta(V) = 0$, the total energy is $E_L = T_g + V$.

The **Cartan 1-form** $\theta_L = dL \circ S$, where S is **the vertical endomorphism**, gives us an exact 2-form $\omega_L = -d\theta_L$ which is non-degenerate when the Lagrangian is regular and then (TQ, ω_L, E_L) is a **Hamiltonian dynamical system**.

The coordinate expressions of $\theta_L = dL \circ S$ and ω_L are:

$$\theta_L(q, v) = g_{ij}(q) v^j dq^i, \quad \omega_L = g_{ij} dq^i \wedge dv^j + \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial q^k} v^j - \frac{\partial g_{kj}}{\partial q^i} v^j \right) dq^i \wedge dq^k.$$

To be remarked that ω_L **only depends on T_g and not on V** , and we can use ω_T instead of ω_L .

Moreover, it can be proved that if $X \in \mathfrak{X}(Q)$ and $X^c \in \mathfrak{X}(TQ)$ is its complete lift,

$$X^c T_g = T_{\mathcal{L}_X g},$$

and this property can be used to prove that the lifts of the flow of X are symplectomorphisms if and only if $\mathcal{L}_X g = 0$, i.e. X is a Killing vector field. Actually,

$$\mathcal{L}_{X^c} \omega_L = \mathcal{L}_{X^c} \omega_{T_g} = \omega_{X^c(T_g)} = \omega_{T_{\mathcal{L}_X g}},$$

and consequently, if X is a Killing vector field, $\mathcal{L}_{X^c} \omega_L = 0$. Moreover, $X^c(T_g) = 0$.

Recall also that Killing vector fields close under commutators on a Lie algebra.

Legendre transformation is regular because the $(0, 2)$ -symmetric tensor g is assumed to be non-degenerate: $v \in T_q Q \mapsto \alpha \in T_q^* Q$ such that $\langle \alpha, w \rangle = g(v, w)$.

In the above mentioned local coordinates

$$p_i = \frac{\partial T}{\partial v^i} = g_{ik}(q) v^k \iff v^i = g^{ij}(q) p_j,$$

with $g^{ik}(q) g_{kj}(q) = \delta_j^i$.

The Hamiltonian is the kinetic energy of the free system in terms of momenta:

$$H = \frac{1}{2} g(\widehat{g}^{-1}(p), \widehat{g}^{-1}(p)) = \frac{1}{2} g^{ij} p_i p_j.$$

If we consider the **exact symplectic action of a Lie group G on the bundle T^*Q defined by lifting an action of G on the base Q** , if $X_a = \xi_a^i(q) \partial/\partial q^i$ is the infinitesimal generator of $a \in \mathfrak{g}$ in the action on Q its lifting is given in the induced coordinate system by

$$\tilde{X}_a = \xi_a^i(q) \frac{\partial}{\partial q^i} - p_i \frac{\partial \xi_a^i}{\partial q^j} \frac{\partial}{\partial p_j}.$$

If \tilde{X}_a is Hamiltonian, as $\theta = p_i dq^i$ we have that **$f_a(q, p) = -p_i \xi_a^i(q)$** .

In the particular case of a **natural Lagrangian system as the complete lift $X^c \in \mathfrak{X}(TQ)$ of a Killing vector field $X \in \mathfrak{X}(Q)$ is a Hamiltonian vector field**, we have an exact action of the Lie algebra of Killing vector fields on TQ and the **momentum map $P : TQ \rightarrow \mathfrak{kill}^*$** is defined by

$$\langle P(x, v), a \rangle = f_a(x, v) = -\theta_{T_g}(X_a^c) = -g_{ij} v^j \xi_a^i, \quad (x, v) \in TQ, a \in \mathfrak{kill}.$$

and **in the corresponding Hamiltonian formalism**, $P : T^*Q \rightarrow \mathfrak{kill}^*$ by

$$\langle P(x, p), a \rangle = f_a(x, p) = -\theta_o(\tilde{X}_a) = -p_i \xi_a^i, \quad (x, p) \in T^*Q, a \in \mathfrak{kill}.$$

Note also that the lift of a Killing vector field preserves the kinetic energy

Geometric approach to Quantum Mechanics

The **Schrödinger picture** of Quantum mechanics admits a **geometric interpretation** similar to that of classical mechanics.

A separable complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ can be considered as a **real linear space**, to be then denoted $\mathcal{H}_{\mathbb{R}}$. The norm in \mathcal{H} defines a norm in $\mathcal{H}_{\mathbb{R}}$, where $\|\psi\|_{\mathbb{R}} = \|\psi\|_{\mathbb{C}}$.

The linear real space $\mathcal{H}_{\mathbb{R}}$ is endowed with a **natural symplectic structure** as follows:

$$\omega(\psi_1, \psi_2) = 2 \operatorname{Im} \langle \psi_1, \psi_2 \rangle.$$

The Hilbert $\mathcal{H}_{\mathbb{R}}$ can be considered as a **real manifold** modelled by a Banach space **admitting a global chart**.

The tangent space $T_{\phi} \mathcal{H}_{\mathbb{R}}$ at any point $\phi \in \mathcal{H}_{\mathbb{R}}$ can be identified with $\mathcal{H}_{\mathbb{R}}$ itself: the isomorphism associates $\psi \in \mathcal{H}_{\mathbb{R}}$ with the vector $\dot{\psi} \in T_{\phi} \mathcal{H}_{\mathbb{R}}$ given by:

$$\dot{\psi} f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi) \right)_{|t=0}, \quad \forall f \in C^{\infty}(\mathcal{H}_{\mathbb{R}}).$$

The **real manifold** can be endowed with a symplectic 2-form ω :

$$\omega_\phi(\dot{\psi}, \dot{\psi}') = 2 \operatorname{Im} \langle \psi, \psi' \rangle .$$

One can see that the constant symplectic structure ω in $\mathcal{H}_{\mathbb{R}}$, considered as a Banach manifold, is exact, i.e., there exists a 1-form $\theta \in \bigwedge^1(\mathcal{H}_{\mathbb{R}})$ such that $\omega = -d\theta$. Such a 1-form $\theta \in \bigwedge^1(\mathcal{H})$ is, for instance, the one defined by

$$\theta(\psi_1)[\dot{\psi}_2] = -\operatorname{Im} \langle \psi_1, \dot{\psi}_2 \rangle .$$

This shows that **the geometric framework for usual Schrödinger picture is that of symplectic mechanics**, as in the classical case.

A **continuous** vector field in $\mathcal{H}_{\mathbb{R}}$ is a **continuous** map $X: \mathcal{H}_{\mathbb{R}} \rightarrow \mathcal{H}_{\mathbb{R}}$. For instance for each $\phi \in \mathcal{H}$, the constant vector field X_ϕ defined by

$$X_\phi(\psi) = \dot{\phi} .$$

It is the generator of the one-parameter subgroup of transformations of $\mathcal{H}_{\mathbb{R}}$ given by

$$\Phi(t, \psi) = \psi + t \phi .$$

As another particular example of vector field consider the vector field X_A defined by the \mathbb{C} -linear map $A : \mathcal{H} \rightarrow \mathcal{H}$, and in particular when A is skew-selfadjoint.

With the natural identification natural of $T\mathcal{H}_{\mathbb{R}} \approx \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}}$, X_A is given by

$$X_A : \phi \mapsto (\phi, A\phi) \in \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}} .$$

When $A = I$ the vector field X_I is the Liouville generator of dilations along the fibres, $\Delta = X_I$, usually denoted Δ given by $\Delta(\phi) = (\phi, \phi)$.

Given a selfadjoint operator A in \mathcal{H} we can define a real function in $\mathcal{H}_{\mathbb{R}}$ by

$$a(\phi) = \langle \phi, A\phi \rangle ,$$

i.e.,

$$a = \langle \Delta, X_A \rangle .$$

Then,

$$\begin{aligned} da_{\phi}(\psi) &= \frac{d}{dt} a(\phi + t\psi)_{t=0} = \frac{d}{dt} [\langle \phi + t\psi, A(\phi + t\psi) \rangle]_{t=0} \\ &= 2 \operatorname{Re} \langle \psi, A\phi \rangle = 2 \operatorname{Imag} \langle -i A\phi, \psi \rangle = \omega(-i A\phi, \psi) . \end{aligned}$$

If we recall that the Hamiltonian vector field defined by the function a is such that for each $\psi \in T_{\phi}\mathcal{H} = \mathcal{H}$,

$$da_{\phi}(\psi) = \omega(X_a(\phi), \psi) ,$$

we see that

$$X_a(\phi) = -i A\phi.$$

Therefore if A is the Hamiltonian H of a quantum system, the Schrödinger equation describing time-evolution plays the rôle of 'Hamilton equations' for the Hamiltonian dynamical system (\mathcal{H}, ω, h) , where $h(\phi) = \langle \phi, H\phi \rangle$: the integral curves of X_h satisfy

$$\dot{\phi} = X_h(\phi) = -i H\phi.$$

The real functions $a(\phi) = \langle \phi, A\phi \rangle$ and $b(\phi) = \langle \phi, B\phi \rangle$ corresponding to two selfadjoint operators A and B satisfy

$$\{a, b\}(\phi) = -i \langle \phi, [A, B]\phi \rangle,$$

because

$$\{a, b\}(\phi) = [\omega(X_a, X_b)](\phi) = \omega_\phi(X_a(\phi), X_b(\phi)) = 2 \operatorname{Im} \langle A\phi, B\phi \rangle,$$

and taking into account that

$$2 \operatorname{Im} \langle A\phi, B\phi \rangle = -i [\langle A\phi, B\phi \rangle - \langle B\phi, A\phi \rangle] = -i [\langle \phi, AB\phi \rangle - \langle \phi, BA\phi \rangle],$$

we find the above result.

In particular, on the integral curves of the vector field X_h defined by a Hamiltonian H ,

$$\dot{a}(\phi) = \{a, h\}(\phi) = -i \langle \phi, [A, H]\phi \rangle,$$

what is usually known as Ehrenfest theorem:

$$\frac{d}{dt} \langle \phi, A\phi \rangle = -i \langle \phi, [A, H]\phi \rangle.$$

There is another relevant **symmetric (0, 2) tensor field** which is given by the Real part of the inner product. It endows $\mathcal{H}_{\mathbb{R}}$ with a **Riemann structure** and we have also a **complex structure** J such that

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \quad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with

$$g(Jv_1, Jv_2) = g(v_1, v_2), \quad \omega(Jv_1, Jv_2) = \omega(v_1, v_2).$$

The triplet (g, J, ω) defines a **Kähler structure** in $\mathcal{H}_{\mathbb{R}}$ and **the symmetry group of the theory must be the unitary group $U(\mathcal{H})$** whose elements preserve the inner product, or in an alternative but equivalent way (in the finite-dimensional case), by the intersection of the orthogonal group $O(2n, \mathbb{R})$ and the symplectic group $Sp(2n, \mathbb{R})$.

On the other hand, as the fundamental concept for measurements is the **expectation value** of observables, two vector fields such that

$$\frac{\langle \psi_2, A\psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} = \frac{\langle \psi_1, A\psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle}, \quad \forall A \in \text{Her}(\mathcal{H})$$

should be considered as indistinguishable.

This is **only possible when ψ_2 is proportional to ψ_1** . In fact it suffices to take as observable A the orthogonal projection on ψ_1 in the preceding relation.

Therefore we must consider **rays rather than vectors** the elements describing the quantum states.

The space of states is not \mathbb{C}^n but the projective space \mathbb{CP}^{n-1} .

It is possible to define a Kähler structure on \mathbb{CP}^{n-1} and therefore to study Lie-Kähler systems leading to superposition rules and to study time evolution in this projective space.

How to find a quantum model for a classical one?

In general the problem of 'quantisation' of a system is, given a classical Hamiltonian system (M, ω, H) , to find a Hilbert space \mathcal{H} and to choose the selfadjoint operator \hat{F} corresponding to the relevant observables F .

In the simplest case of an Euclidean configuration space, the prescription is that

- \mathcal{H} is the space of square integrable functions with respect to the Lebesgue measure
- positions \hat{x}_i have associated multiplication by x_i operators and
- the momentum operators are the differential operators

$$\hat{p}_k = -i \frac{\partial}{\partial x_k}.$$

There will be some **ordering ambiguities** for other observables because functions of x_i commute with those of p_k , but \hat{x}_i does not commute with \hat{p}_k :

$$[\hat{x}_i, \hat{p}_j] = i \delta_{ik}, \quad \text{versus} \quad \{x_i, p_k\} = \delta_{ik}.$$

Many textbooks warn that this is **only valid using Euclidean coordinates**.

This is **very restrictive**, because:

What happens in other coordinates and even worse when there are no global preferred coordinates (for instance Q is compact)?

What about position-dependent mass for which mass operator does not commute with momentum operators?

Generalising previous procedure for a more general case of a Hamiltonian dynamical system, Dirac assertion is that *Quantisation is to be understood as a map $\zeta : C^\infty(M) \rightarrow \mathcal{A}(\mathcal{H})$ such that*

$$-i\zeta(\{F, G\}) = [\zeta(F), \zeta(G)]$$

Several meanings for p_i

In Classical mechanics, if the configuration space is an Euclidean space \mathbb{R}^n , p_i denotes many different and inequivalent objects:

- p_i is the i -th coordinate of a covector in a point of the configuration space
- p_i is a real function in $T^*\mathbb{R}^n$: $p_i(\alpha) = \alpha(\partial/\partial x_i)$
- the real function p_i is the infinitesimal generator of translations along the i -th coordinate axis, which are canonical transformations for $(T^*\mathbb{R}^n, \omega_0)$

In a more general case, $Q \neq T^*\mathbb{R}^n$,

- there is not a global chart,
- the momentum coordinates are local and depend on the choice of base manifold coordinates,
- translations are not defined.

Recall however that translations and rotations are isometries of the Euclidean space.

For natural Lagrangians defined on TQ , with Q an arbitrary Riemann manifold, we can consider the isometries of the Riemann metric, whose cotangent lifts are Hamiltonian vector fields and in this way we define a strongly symplectic action of the group of isometries, with Lie algebra $\mathfrak{is}(Q)$, on T^*Q .

We are then able to define an associated momentum. The components P_a of this map are the objects to be quantised instead of the p_i which is not an intrinsic but a coordinate dependent ingredient.

This allows us to quantise functions of the momentum map by associating the function P_a with $\hat{P}_a = -i X_a$ acting on an appropriate Hilbert space.

Consequently, we can quantise functions of the momentum map.

We can consider simple models to see how this works.

Position dependent mass systems

Consider a 1-dimensional system described in terms of a coordinate x by a Lagrangian

$$L = \frac{1}{2} m(x) \dot{x}^2 - V(x), \quad x \in \mathbb{R}, \quad m(x) > 0,$$

that leads to the following nonlinear differential equation

$$m(x) \ddot{x} + \frac{1}{2} m'(x) \dot{x}^2 = 0,$$

with associated Hamiltonian H given by

$$H(x, p) = \frac{1}{2} \frac{1}{m(x)} p^2 + V(x).$$

There is an important problem with the construction of the quantum version of H , $H \rightarrow \hat{H}$ from the classical system to the quantum one, because if the mass m becomes a function of the spatial coordinate, $m = m(x)$, then the quantum version of the mass no longer commutes with the momentum.

Different forms of presenting the kinetic term in the Hamiltonian H , as for example

$$T = \frac{1}{4} \left[\frac{1}{m(x)} p^2 + p^2 \frac{1}{m(x)} \right], \quad T = \frac{1}{2} \left[\frac{1}{\sqrt{m(x)}} p^2 \frac{1}{\sqrt{m(x)}} \right], \quad T = \frac{1}{2} \left[p \frac{1}{m(x)} p \right],$$

are equivalent at the classical level but they lead to different and nonequivalent Schrödinger equations.

This problem is important mainly for two reasons.

- (i) There are a certain number of important areas, mainly related with problems on condensed-matter physics (electronic properties of semiconductors, liquid crystals, quantum dots, etc), in which the behaviour of the system depends of an effective mass that is position-dependent.
- (ii) From a more conceptual viewpoint, the ordering of factors in the transition from a commutative to a noncommutative formalism is an old question that remains as an important open problem in the theory of quantization.

- (iii) The **free motion along a simple regular curve C** looks like a position-dependent mass system. If C is given in parametric form by $\mathbf{x} : I \rightarrow \mathbb{R}^n$, $u \mapsto \mathbf{x}(u)$, its **arc-length function** $s(u)$ is an intrinsic parameter given by

$$\frac{ds}{du} = \sqrt{\frac{d\mathbf{x}}{du} \cdot \frac{d\mathbf{x}}{du}} = f(u) > 0 ,$$

and therefore

$$s(u) = \int^u \sqrt{\dot{\mathbf{x}}(\zeta) \cdot \dot{\mathbf{x}}(\zeta)} \, d\zeta .$$

The **geodesics of this metric coincide, up to reparametrisation**, with the curves solution of the Euler–Lagrange equation of the Lagrangian

$$L_0(u, v_u) = \frac{1}{2} m_0 f(u) v_u^2 ,$$

where m_0 is a constant with mass dimension, i.e.

$$\frac{d}{dt}(f(u)\dot{u}) = \frac{1}{2}f'(u)\dot{u}^2 \implies f(u)\ddot{u} + \frac{1}{2}f'(u)\dot{u}^2 = 0 ,$$

where $f'(u) = df/du$.

Classical motion on a cycloid: A case study

Consider the motion of a particle of mass m_0 moving on a gravitational field along a cycloid inverted in such a way that the origin is sited in the lowest point, namely:

$$\mathbf{x}(\vartheta) = (x(\vartheta + \pi), -y(\vartheta + \pi)) + (0, 2R) = (R(\vartheta + \pi + \sin \vartheta), R(1 - \cos \vartheta)),$$

i.e.,

$$\mathbf{x}(\vartheta) = \left(R(\vartheta + \pi + \sin \vartheta), 2R \sin^2 \frac{\vartheta}{2} \right), \quad \vartheta \in (-\pi, \pi).$$

Consequently,

$$\dot{\mathbf{x}}(\vartheta) = (R(1 + \cos \vartheta), R \sin \vartheta),$$

and then,

$$\|\dot{\mathbf{x}}\|^2 = R^2(1 + \cos^2 \vartheta + 2 \cos \vartheta + \sin^2 \vartheta) = 2R^2(1 + \cos \vartheta) = 4R^2 \cos^2 \frac{\vartheta}{2},$$

from which we obtain (recall that $\theta \in (-\pi, \pi)$ and therefore $\cos(\vartheta/2) > 0$)

$$\frac{ds}{d\vartheta} = 2R \cos \frac{\vartheta}{2} \implies s(\vartheta) = 4R \left[\sin \frac{\zeta}{2} \right]_0^{\vartheta} = 4R \sin \frac{\vartheta}{2}.$$

The expression of the metric in these coordinates is

$$g = 2R \cos \frac{\vartheta}{2} d\vartheta^2,$$

i.e. the free motion is described by the Lagrangian with a position-dependent mass given by $m(\vartheta) = 2m_0 R \cos \frac{\vartheta}{2}$:

$$L_0(\vartheta, \dot{\vartheta}) = m_0 R \cos \frac{\vartheta}{2} \dot{\vartheta}^2.$$

The potential function describing the action of the gravity in terms of the arc-length s is:

$$V(\vartheta) = m_0 g y(\vartheta) = 2m_0 g R \sin^2 \frac{\vartheta}{2} = 2m_0 g R \frac{s^2}{16R^2} = \frac{1}{2} m_0 \frac{g}{4R} s^2,$$

and then the Lagrangian is given by

$$L(\vartheta, \dot{\vartheta}) = m_0 R \cos \frac{\vartheta}{2} \dot{\vartheta}^2 - 2m_0 g R \sin^2 \frac{\vartheta}{2},$$

or in terms of the canonical coordinate

$$L(s, \dot{s}) = \frac{1}{2} m_0 \dot{s}^2 - \frac{1}{2} m_0 \frac{g}{4R} s^2.$$

Correspondingly, the tangent vector to the curve is

$$\mathbf{t}(\vartheta) = \frac{1}{2R \cos \frac{\vartheta}{2}} (R(1 + \cos \vartheta), R \sin \vartheta) = \left(\cos \frac{\vartheta}{2}, \sin \frac{\vartheta}{2} \right),$$

which shows that the tangential force is given by

$$F_t = \mathbf{F} \cdot \mathbf{t} = -m_0 g \sin \frac{\vartheta}{2} = -\frac{m_0 g}{4R} s,$$

while the tangential component of $\ddot{\mathbf{x}}$ is \ddot{s} , and then Newton's second law is

$$\ddot{s} = -\frac{m_0 g}{4R} s.$$

Both expressions show that **the motion along the inverted cycloid in terms of the arc-length s is oscillatory with a constant period function**

$$\tau = \frac{2\pi}{\omega} = 4\pi \sqrt{\frac{R}{g}}.$$

The initial position only fixes the amplitude of motion. This is the reason for the tautochronous behaviour of the motion along the cycloid.

Quantisation of motions on curves

A curve is endowed with a (local) chart given by the arc-length which turns out to be a privileged chart. We can apply the usual canonical quantisation rule using such a chart because the expressions of the Lagrangian and Hamiltonian for free motion are exactly the same expressions as for the Euclidean case, i.e.

$$L_0(s, \dot{s}) = \frac{1}{2}m_0 \dot{s}^2, \quad H_0(s, p_s) = \frac{p_s^2}{2m_0}.$$

This means that a natural prescription is to replace the momentum p_s by $-i\hbar\partial/\partial s$ as an operator acting on the Hilbert space $\mathcal{L}_0^2(C, ds)$ of square integrable functions on the curve C vanishing on the boundary of C . As the measure ds is invariant under length-displacements such operator is selfadjoint.

Quantum motion on a cycloid: A case study

The **quantum model** for a particle of mass m_0 living on a cycloid as configuration space, under the action of a gravitational force, in terms of the arc-length parameter is like that of a harmonic oscillator with mass m_0 and $\omega^2 = m_0 g/(4R)$, for $-4R \leq s \leq 4R$.

The **Hilbert space** of the corresponding quantum system will be $\mathcal{L}_0^2(-L, L)$ of square integrable functions in the interval $(-L, L)$, with $L = 4R$, **satisfying the boundary conditions** $\psi(-L) = \psi(L) = 0$, and **the quantum Hamiltonian operator is given by**

$$H = -\frac{\hbar^2}{2m_0} \frac{d^2}{ds^2} + V(s),$$

where

$$V(s) = \begin{cases} \frac{1}{2}m_0\omega^2 s^2 & \text{if } |s| \leq R \\ \infty & \text{if } |s| \geq 4R \end{cases}.$$

This problem of a confined harmonic oscillator has been studied by Ghosh. **The Hamiltonian is parity invariant** and consequently **the eigenfunctions are either even**

or odd functions. The time-independent Schrödinger equation is

$$\left(-\frac{\hbar^2}{2m_0} \frac{d^2}{ds^2} + \frac{1}{2} m_0 \omega^2 s^2 \right) \psi(s) = E \psi(s),$$

which can be rewritten as

$$\frac{d^2}{dz^2} + \left(\varepsilon - \frac{z^2}{4} \right) \psi(z) = 0,$$

where

$$z = \sqrt{\frac{2m_0\omega}{\hbar}}, \quad E = \varepsilon \hbar\omega.$$

It is common to write $\varepsilon = \nu + 1/2$, by similarity with the usual harmonic oscillator, i.e. $E = (\nu + 1/2) \hbar\omega$.

If we introduce now the change

$$\psi(z) = e^{-z^2/4} \phi(z),$$

and redefine the independent variable as $y = \frac{1}{2}z^2$, the new function $\phi(y)$ satisfies the confluent hypergeometric equation (Abramowitz and Stegun, p 504):

$$y \frac{d^2 \phi}{dy^2} + (b - y) \frac{d\phi}{dy} - a y = 0,$$

with $b = \frac{1}{2}$ and $a = -\frac{\nu}{2}$, i.e.

$$y \frac{d^2 \phi}{dy^2} + \left(\frac{1}{2} - y \right) \frac{d\phi}{dy} + \frac{\nu}{2} y.$$

The point $y = 0$ is a regular singular point while $y = \infty$ is an irregular singularity. A basis of the linear space of solutions for $b \notin \mathbb{Z}$ is given by the confluent hypergeometric function, also called **Kummer function** $M(a, b, y)$ and its related function $U(a, b, y)$ with power expansions (see e.g. Abramowitz and Stegun, p 504):

$$\begin{aligned} M(a, b, y) &= {}_1F_1(a, b, y) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{y^n}{n!} \\ U(a, b, y) &= \frac{\pi}{\sin(\pi b)} \left(\frac{M(a, b, y)}{\Gamma(1+a-b)\Gamma(b)} - y^{1-b} \frac{M(1+a-b, 2-b, y)}{\Gamma(a)\Gamma(2-b)} \right) \end{aligned}$$

where (a_n) denotes $(a_n) = a(a+1) \cdots (a+n-1)$, with $(a_0) = 1$.

The general solution $\psi(s)$ can be written as

$$\begin{aligned} \psi(s) &= e^{-m\omega L^2/2\hbar} \left[\left(A + B \frac{\sqrt{\pi}}{\Gamma((1-\nu)/2)} \right) M\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{m_0\omega s^2}{\hbar}\right) \right. \\ &\quad \left. - 2B \frac{\sqrt{\pi}}{\Gamma(-\nu/2)} \sqrt{\frac{m_0\omega}{\hbar}} s M\left(\frac{1-\nu}{2}, \frac{3}{2}, \frac{m_0\omega s^2}{\hbar}\right) \right]. \end{aligned}$$

The first term on the right-hand side is an even function and the second one is odd. Therefore, **the conditions on the parameter ν for $\psi(s)$ to be an eigenfunction** are

$$\begin{aligned} \left(A + B \frac{\sqrt{\pi}}{\Gamma((1-\nu)/2)} \right) M \left(-\frac{\nu}{2}, \frac{1}{2}, \frac{m_0 \omega L^2}{\hbar} \right) &= 0 \quad \text{for even functions} \\ \frac{\sqrt{\pi}}{\Gamma(-\nu/2)} \sqrt{\frac{m_0 \omega}{\hbar}} M \left(\frac{1-\nu}{2}, \frac{3}{2}, \frac{m_0 \omega L^2}{\hbar} \right) &= 0 \quad \text{for odd functions} \end{aligned}$$

These equations should be used to determine the energy eigenvalues.

Quantisation of position dependent mass systems

The **usual approach** make uses of **the formalism** (α, β, γ) : T has the following expression introduced by **von Roos** (generalizing a previous study by **BenDaniel et al.**

$$T_{\alpha\beta\gamma} = \frac{1}{4} \left(m^\alpha p m^\beta p m^\gamma + m^\gamma p m^\beta p m^\alpha \right), \quad \alpha + \beta + \gamma = -1.$$

It is important to remark that in order to study a quantum system (in the Schrödinger picture) we should **first fix the Hilbert space \mathcal{H} and then the (essentially) selfadjoint operators corresponding to the relevant observables to be quantized.**

Therefore the quantization of the Hamiltonian of a system means first to define the appropriate Hilbert space of pure states, and then construction of the quantum Hamiltonian.

In the problem of quantization of a Hamiltonian system with a PDM the definition of **the measure $d\mu$ defining the Hilbert space $L^2(\mathbb{R}, d\mu)$ strongly depends on the characteristics of the function $m(x)$.**

Note that the kinetic Lagrangian T for motion along a curve admits an infinitesimal symmetry given by the vector field

$$X = \frac{\partial}{\partial s} = \frac{1}{\sqrt{f(u)}} \frac{\partial}{\partial u},$$

which preserves the metric distance, because $\mathcal{L}_X(ds) = 0$.

In an analogous way, the kinetic Lagrangian T for a position dependent mass system possesses an exact Noether symmetry. The function T is not invariant under translations but under the action of the vector field X given by

$$X(x) = \frac{1}{\sqrt{m(x)}} \frac{\partial}{\partial x},$$

(displacement $\delta x = \epsilon(m(x))^{-1/2}$, in the physicists language), i.e. we have $X^t(T) = 0$, where X^t denotes the tangent lift to the velocity phase space $\mathbb{R} \times \mathbb{R}$ (that, in differential geometric terms, is the tangent bundle TQ of the configuration space $Q = \mathbb{R}$) of the vector field $X \in \mathfrak{X}(\mathbb{R})$,

$$X^t(x, v) = \frac{1}{\sqrt{m(x)}} \left(\frac{\partial}{\partial x} - \left(\frac{1}{2} \frac{m'(x)}{m(x)} \right) v \frac{\partial}{\partial v} \right).$$

Recall that given a Riemannian space (M, g) , a **Killing vector field X defined on M is a symmetry of the metric g** (in the sense that it satisfies $\mathcal{L}_X g = 0$ where \mathcal{L}_X denotes the Lie derivative with respect to X). Killing vector fields also preserve the volume Ω_g determined by the metric, that is,

$$\Omega_g = \sqrt{|g|} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \quad \mathcal{L}_X \Omega_g = 0,$$

where $|g|$ denotes the determinant of the matrix g defining the Riemann structure.

One can show that **if $T_g(x, v) = \frac{1}{2} g_{ij}(x) v^i v^j$, then**

$$X^t(T_g) = T_{\tilde{g}}, \quad \tilde{g} = \mathcal{L}_X g.$$

Consequently, **X is a Killing vector field for g iff X^t is a symmetry for the associated kinetic energy function T_g .** The above **X is a Killing vector field**. In fact it is a Killing vector field of the one-dimensional m -dependent metric

$$g = m(x) \, dx \otimes dx, \quad ds^2 = m(x) \, dx^2.$$

The line element is invariant under the flow of the vector field **$X = f(x) \partial / \partial x$** when

$$f \, m' + 2 \, m \, f' = 0,$$

and, therefore, in order to the vector field X to be a Killing vector, it should be proportional to the vector field X , which represents (the infinitesimal generator of) an exact Noether symmetry for the geodesic motion. If we denote by θ_L the 1-form

$$\theta_L = \left(\frac{\partial L}{\partial v} \right) dx = m(x)v dx ,$$

the associated Noether constant of the motion P for the free (geodesic) motion, called Noether momentum is

$$P = i(X^t) \theta_L = \sqrt{m(x)} v .$$

Quasi-regular representation

The Hilbert space for a quantum system with a configuration space M is the linear space of square integrable functions on M with respect to an appropriate measure, $L^2(M, d\mu)$.

In the case of a natural system the measure to be considered must be invariant under the the Killing vector fields of the metric. The reason is the following:

If $\Phi : G \times M \rightarrow M$ denotes the action of a Lie group G on a differentiable manifold M , then the associated quasi-regular representation is given by the following action

of G on the set of complex functions on M :

$$(U(g)\psi)(x) = \psi(\Phi(g^{-1}, x)).$$

If M admits an invariant measure $d\mu$ we can restrict the action on the set $L^2(M, d\mu)$ and if $d\mu$ is G -invariant, the representation so obtained is a unitary representation.

If a one-parameter subgroup $\gamma(t) = \exp(at)$, $a \in \mathfrak{g}$, is considered, then the fundamental vector field $X_a \in \mathfrak{g}$, which is given by

$$(X\psi)(x) = \frac{d}{dt}\psi(\Phi(\exp(-ta), x))|_{t=0},$$

when restricted to the subspace $L^2(M, d\mu)$ is a skew-selfadjoint operator provided that the measure μ is $\gamma(t)$ -invariant, because $U(\gamma(t))$ is a one-parameter group of unitary transformations.

The infinitesimal generator in the regular representation is a generator for a 1-parameter group of unitary transformations, and consequently it is skew-self-adjoint operator. Of course if we want the generators of several one-parameter groups be skew-self-adjoint, the measure defining the Hilbert space must be invariant under each 1-parameter subgroup.

For the one-dimensional PDM system, the quantum system must be described by the Hilbert space $L^2(\mathbb{R}, d\mu_x)$ of square integrable functions w.r.t. an **invariant under X measure**, $d\mu_x$, therefore determined by the metric.

The Lebesgue measure dx is not invariant under $X = f(x)\partial/\partial x$, **the invariance condition for the measure $d\mu_x = \rho(x) dx$** being

$$f \rho' + \rho f' = 0.$$

Then, **the only measure invariant under X for $f(x) = (m(x))^{-1/2}$** is a multiple of

$$d\mu_x = \sqrt{m(x)} dx.$$

This automatically implies that **the first-order linear operator X is skew-symmetric**. This means that **the operator \hat{P}** representing the quantum version of the Noether momentum P **must be selfadjoint**, not in the standard space $L^2(\mathbb{R}) \equiv L^2(\mathbb{R}, dx)$, but **in the space $L^2(\mathbb{R}, d\mu_x)$** of square integrable functions with respect the PDM measure $d\mu_x$.

Using the Legendre transformation the momentum p and velocity v are related by $p = m(x) v$, so that **the expressions of the Noether momenta and the Hamiltonian**

(kinetic term plus a potential) in the phase space are

$$P = \frac{1}{\sqrt{m(x)}} p,$$

and

$$H = \frac{1}{2} P^2 + V(x).$$

As we have pointed out, the generator of the infinitesimal 'translation' symmetry, $(1/\sqrt{m(x)}) d/dx$, is skew-Hermitian in the space $L^2(\mathbb{R}, d\mu_x)$ and therefore the transition from the classical system to the quantum one is given by defining the operator \hat{P} as follows

$$P \mapsto \hat{P} = \frac{1}{\sqrt{m(x)}} \left(-i \hbar \frac{d}{dx} \right),$$

so that

$$\frac{1}{m} p^2 \rightarrow -\hbar^2 \left(\frac{1}{\sqrt{m(x)}} \frac{d}{dx} \right) \left(\frac{1}{\sqrt{m(x)}} \frac{d}{dx} \right),$$

in such a way that the quantum Hamiltonian \hat{H} is represented by the following Hermitian (defined on the space $L^2(\mathbb{R}, d\mu_x)$) operator

$$\hat{H} = -\frac{\hbar^2}{2} \left(\frac{1}{\sqrt{m(x)}} \frac{d}{dx} \right) \left(\frac{1}{\sqrt{m(x)}} \frac{d}{dx} \right) + V(x),$$

$$= -\frac{\hbar^2}{2} \frac{1}{m(x)} \frac{d^2}{dx^2} + \frac{\hbar^2}{4} \left(\frac{m'(x)}{m^2(x)} \right) \frac{d}{dx} + V(x),$$

and then the Schrödinger equation $\hat{H} \Psi = E \Psi$ becomes

$$-\frac{\hbar^2}{2} \frac{1}{m(x)} \frac{d^2 \Psi}{dx^2} + \frac{\hbar^2}{4} \left(\frac{m'(x)}{m^2(x)} \right) \frac{d \Psi}{dx} + V(x) \Psi = E \Psi.$$

Constant curvature surfaces

The expression of the **arc length element** in geodesic polar coordinates (ρ, ϕ) on a constant curvature surface M_{κ}^2 can be written as follows

$$\begin{aligned} ds_{\kappa}^2 &= d\rho^2 + \frac{1}{\kappa} \sin^2(\sqrt{\kappa}\rho) d\phi^2, & \text{if } \kappa > 0, \\ ds_0^2 &= d\rho^2 + \rho^2 d\phi^2, & \text{if } \kappa = 0, \\ ds_{\kappa}^2 &= d\rho^2 - \frac{1}{\kappa} \sinh^2(\sqrt{-\kappa}\rho) d\phi^2, & \text{if } \kappa < 0. \end{aligned}$$

It is possible to deal with all of them in an unified way by introducing the following labelled trigonometric functions

$$C_{\kappa}(x) = \begin{cases} \cos \sqrt{\kappa} x & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh \sqrt{-\kappa} x & \text{if } \kappa < 0, \end{cases} \quad S_{\kappa}(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \text{if } \kappa < 0, \end{cases}$$

and the κ -dependent tangent function

$$T_{\kappa}(x) = S_{\kappa}(x) / C_{\kappa}(x).$$

The fundamental properties of these **curvature-dependent trigonometric functions** are

$$C_{\kappa}^2(x) + \kappa S_{\kappa}^2(x) = 1,$$

and

$$S_{\kappa}(2x) = 2 S_{\kappa}(x) C_{\kappa}(x), \quad C_{\kappa}(2x) = C_{\kappa}^2(x) - \kappa S_{\kappa}^2(x),$$

$$\frac{d}{dx} S_{\kappa}(x) = C_{\kappa}(x), \quad \frac{d}{dx} C_{\kappa}(x) = -\kappa S_{\kappa}(x),$$

and therefore

$$\frac{d}{dx} T_{\kappa}(x) = \frac{1}{C_{\kappa}^2(x)}$$

With this notation the arc length element in all the three cases is:

$$ds_{\kappa}^2 = d\rho^2 + S_{\kappa}^2(\rho) d\phi^2,$$

i.e. the only nonzero components of the metric are

$$g_{\rho\rho} = 1, \quad g_{\phi\phi} = S_{\kappa}^2(\rho) \implies g^{\rho\rho} = 1, \quad g^{\phi\phi} = \frac{1}{S_{\kappa}^2(\rho)}.$$

Note that ρ denotes the distance along a geodesic on the manifold M_κ^2 .

The Lagrangian for the geodesic (free) motion on the spaces $(S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$ is

$$L_{0\kappa}(\rho, \phi, v_\rho, v_\phi) = T_\kappa(\rho, \phi, v_\rho, v_\phi) = \frac{1}{2} (v_\rho^2 + S_\kappa^2(\rho) v_\phi^2) ,$$

and for a general mechanical type system (Riemannian metric minus a potential)

$$L_\kappa(\rho, \phi, v_\rho, v_\phi) = \frac{1}{2} (v_\rho^2 + S_\kappa^2(\rho) v_\phi^2) - V(\rho, \phi, \kappa) .$$

The corresponding Legendre transformation maps the point $(\rho, \phi, v_\rho, v_\phi)$ into $(\rho, \phi, p_\rho, p_\phi)$ with

$$p_\rho = v_\rho, \quad p_\phi = S_\kappa^2(\rho) v_\phi,$$

i.e. the Hamiltonian is

$$H = \frac{1}{2} \left(p_\rho^2 + \frac{p_\phi^2}{S_\kappa^2(\rho)} \right) + V(\rho, \phi, \kappa)$$

Under the κ -dependent change of coordinates $r = S_\kappa(\rho)$ the Lagrangian $L_{0\kappa}$ becomes

$$L_{0\kappa}(r, \phi, v_r, v_\phi) = \frac{1}{2} \left(\frac{v_r^2}{1 - \kappa r^2} + r^2 v_\phi^2 \right)$$

and, if we define $x = r \cos \phi$ and $y = r \sin \phi$, the Lagrangian becomes

$$L_{0\kappa}(x, y, v_x, v_y) = \frac{1}{2} \frac{1}{1 - \kappa r^2} \left[v_x^2 + v_y^2 - \kappa (x v_y - y v_x)^2 \right], \quad r^2 = x^2 + y^2.$$

The Hamiltonian expressed in terms of these coordinates can be found to be:

$$H_{0\kappa}(x, y, p, p_y) = \frac{1}{2} (p_x^2 + p_y^2 - \kappa (x p_y - y p_x)^2).$$

In order to look for the **Killing vector fields** for the metric we recall that $X = X_\rho \partial/\partial\rho + X_\phi \partial/\partial\phi \in \mathfrak{X}(Q)$ is a **Killing vector field** if and only if its complete lift

$$X_\rho \frac{\partial}{\partial\rho} + X_\phi \frac{\partial}{\partial\phi} + \left(\frac{\partial X_\rho}{\partial\rho} v_\rho + \frac{\partial X_\rho}{\partial\phi} v_\phi \right) \frac{\partial}{\partial v_\rho} + \left(\frac{\partial X_\phi}{\partial\rho} v_\rho + \frac{\partial X_\phi}{\partial\phi} v_\phi \right) \frac{\partial}{\partial v_\phi}$$

is a **strict symmetry of L** , i.e. $X^c L = 0$. This implies that

$$\left\{ \begin{array}{l} \frac{\partial X_\rho}{\partial\rho} = 0 \\ \frac{\partial X_\rho}{\partial\phi} + S_\kappa^2(\rho) \frac{\partial X_\phi}{\partial\rho} = 0 \\ S_\kappa(\rho) C_\kappa(\rho) X_\rho + S_\kappa^2(\rho) \frac{\partial X_\phi}{\partial\phi} = 0. \end{array} \right.$$

A solution of this system is $X_\rho \equiv 0$ and $X_\phi = 1$, i.e. the vector field $X_3 = \partial/\partial\phi$.

X_ρ only depends on ϕ . A simple derivation w.r.t. ϕ in the second equation leads to

$$\frac{\partial^2 X_\rho}{\partial \phi^2} + S_\kappa^2(\rho) \frac{\partial^2 X_\phi}{\partial \rho \partial \phi} = 0,$$

while if we derive with respect to ρ in the expression obtained from the third equation

$$\frac{\partial X_\phi}{\partial \phi} = -\frac{C_\kappa(\rho)}{S_\kappa(\rho)} X_\rho,$$

we get

$$\frac{\partial^2 X_\phi}{\partial \rho \partial \phi} = \frac{1}{S_\kappa^2(\rho)} X_\rho.$$

This relation shows that X_ρ is a solution of the equation

$$\frac{\partial^2 X_\rho}{\partial \phi^2} + X_\rho = 0,$$

and then the value of X_ϕ may be then obtained from preceding relation. In summary, we have got the **other two linearly independent Killing vector fields**

$$X_1 = \cos \phi \frac{\partial}{\partial \rho} - \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \sin \phi \frac{\partial}{\partial \phi}, \quad X_2 = \sin \phi \frac{\partial}{\partial \rho} + \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \cos \phi \frac{\partial}{\partial \phi}.$$

Note now that the vector fields X_1 , X_2 and X_3 so defined are such that

$$\begin{aligned}[X_1, X_2] &= \left[\cos \phi \frac{\partial}{\partial \rho} - \sin \phi \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \frac{\partial}{\partial \phi}, \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \frac{\partial}{\partial \phi} \right] \\ &= \frac{C_\kappa^2(\rho) - 1}{S_\kappa^2(\rho)} \frac{\partial}{\partial \phi} = -\kappa X_3,\end{aligned}$$

while

$$\begin{aligned}[X_3, X_1] &= \left[\frac{\partial}{\partial \phi}, \cos \phi \frac{\partial}{\partial \rho} - \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \sin \phi \frac{\partial}{\partial \phi} \right] = -\sin \phi \frac{\partial}{\partial \rho} - \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \cos \phi \frac{\partial}{\partial \phi} = -X_2, \\ [X_2, X_3] &= \left[\sin \phi \frac{\partial}{\partial \rho} + \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \cos \phi \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \right] = -\cos \phi \frac{\partial}{\partial \rho} + \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \sin \phi \frac{\partial}{\partial \phi} = -X_1,\end{aligned}$$

and therefore they close on a real Lie algebra isomorphic either to $\mathfrak{so}(3)$, if $\kappa > 0$, $\mathfrak{so}(2, 1)$, when $\kappa < 0$, or $\mathfrak{e}(2)$ when $\kappa = 0$, i.e. it depends on the value of κ .

The momentum map for this case is given by

$$\begin{aligned}P_1 &= -\theta(X_1), & P_1(\rho, \phi, p_\rho, p_\phi) &= -p_\rho \cos \phi + p_\phi \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \sin \phi \\ P_2 &= -\theta(X_2), & P_2(\rho, \phi, p_\rho, p_\phi) &= -p_\rho \sin \phi - p_\phi \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \cos \phi \\ P_3 &= -\theta(X_3), & P_3(\rho, \phi, p_\rho, p_\phi) &= -p_\phi\end{aligned}$$

The Hamiltonian of the corresponding free particle is

$$H(\rho, \phi, p_\rho, p_\phi) = \frac{1}{2} p_\rho^2 + \frac{p_\phi^2}{2 S_\kappa^2(\rho)},$$

which can be expressed in terms of the components of the momentum map. From

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = - \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} p_\rho \\ p_\phi \frac{C_\kappa(\rho)}{S_\kappa(\rho)} \end{pmatrix}$$

we easily see that

$$P_1^2 + P_2^2 = p_\rho^2 + p_\phi^2 \frac{C_\kappa^2(\rho)}{S_\kappa^2(\rho)},$$

and consequently,

$$P_1^2 + P_2^2 + \kappa P_3^2 = p_\rho^2 + p_\phi^2 \frac{1}{S_\kappa^2(\rho)}.$$

This shows that the Hamiltonian can be written as

$$H = P_1^2 + P_2^2 + \kappa P_3^2.$$

Using Cartesian coordinates, $x = r \cos \phi$ and $y = r \sin \phi$, the Killing vector fields

are:

$$X_1(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x},$$

$$X_2(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y},$$

$$X_3(\lambda) = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

The conditions for a volume form $\text{vol} = \varrho(x, y) dx \wedge dy$ to be invariant under such vector fields are:

$$\sqrt{1 + \lambda r^2} \frac{\partial \varrho}{\partial x} dx \wedge dy + \varrho d(\sqrt{1 + \lambda r^2}) \wedge dy = \left(\sqrt{1 + \lambda r^2} \frac{\partial \varrho}{\partial x} + \frac{\lambda x \varrho}{\sqrt{1 + \lambda r^2}} \right) dx \wedge dy = 0,$$

$$\sqrt{1 + \lambda r^2} \frac{\partial \varrho}{\partial y} dx \wedge dy + \varrho dx \wedge d(\sqrt{1 + \lambda r^2}) = \left(\sqrt{1 + \lambda r^2} \frac{\partial \varrho}{\partial y} + \frac{\lambda y \varrho}{\sqrt{1 + \lambda r^2}} \right) dx \wedge dy = 0,$$

and therefore, y times the first equation minus x times the second one gives the invariance of the function ϱ under rotations $x \partial \varrho / \partial y - y \partial \varrho / \partial x = 0$, which implies:

$$\varrho(x, y) = f(r), \quad r^2 = x^2 + y^2,$$

and when using such expression in the previous equations we obtain (for $r \neq 0$):

$$\sqrt{1 + \lambda r^2} \frac{1}{r} \frac{df}{dr} + \frac{\lambda f}{\sqrt{1 + \lambda r^2}} = 0,$$

and therefore,

$$\frac{df}{f} = -\lambda \frac{r dr}{1 + \lambda r^2},$$

with general solution f proportional to

$$f(r) = \frac{1}{\sqrt{1 + \lambda r^2}}.$$

Natural mechanical systems involve a **potential term**. The two most studied situations are the **harmonic oscillator** and the **Kepler-Coulomb** system, that are the corresponding cases for constant curvature surface motion?

The harmonic oscillator on the unit sphere, on the Euclidean plane, or on the unit Lobachewski plane, arise as $V_\kappa(\rho) = \frac{1}{2} \alpha^2 T_\kappa^2(\rho)$, i.e.

$$V_1(\rho) = \frac{1}{2} \alpha^2 \tan^2 \rho, \quad V_0(\rho) = \frac{1}{2} \alpha^2 \rho^2, \quad V_{-1}(\rho) = \frac{1}{2} \alpha^2 \tanh^2 \rho.$$

Now if we consider the κ -dependent change $\rho \rightarrow r = S_\kappa(\rho)$ then the Lagrangian $L(\kappa)$ becomes

$$L(\kappa) = \frac{1}{2} \left(\frac{v_r^2}{1 - \kappa r^2} + r^2 v_\phi^2 \right) - \frac{1}{2} \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right)$$

and, if we change to Cartesian coordinates, we arrive to

$$L(\kappa) = \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left[v_x^2 + v_y^2 - \kappa (xv_y - yv_x)^2 \right] - \frac{1}{2} \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right), \quad r^2 = x^2 + y^2.$$

Similarly we can use this potential terms in the corresponding Hamiltonian

As far as the Kepler-Coulomb system in constant curvature surfaces is concerned one uses

$$V_K(\rho) = - \frac{k}{T_\kappa(\rho)}.$$

In Cartesian coordinate, using that $C_\kappa = \sqrt{1 - \kappa S_\kappa(\rho)}$

$$V_K(r) = - \frac{k}{r} \sqrt{1 - \kappa S_\kappa(\rho)}.$$

Both cases have received a lot of attention during the last years.

Quantisation of Noether momenta

In order to the differential operator be **selfadjoint** we need to consider a Hilbert space $\mathcal{L}^2(Q, d\mu)$ such that **the volume form be invariant under X_k** .

This shows that **the Hilbert space of the quantisation of constant curvature surfaces must be**

$$\mathcal{L}^2 \left(Q, \frac{1}{\sqrt{1 + \lambda r^2}} dx \wedge dy \right).$$

The Killing vector fields $X_1(\lambda)$ and $X_2(\lambda)$ are **skewsymmetric first order differential operators** generating ‘translations’, because as the functions are square integrable,

$$\int_{\mathbb{R}^2} \psi_1^*(x, y) \sqrt{1 + \lambda r^2} \frac{\partial \psi_2(x, y)}{\partial x} \frac{dx dy}{\sqrt{1 + \lambda r^2}} = - \int_{\mathbb{R}^2} \left[\sqrt{1 + \lambda r^2} \frac{\partial \psi_1^*(x, y)}{\partial x} \right] \psi_2(x) \frac{dx}{\sqrt{1 + \lambda r^2}}$$

and similarly when changing x by y , suggests us to quantise the momenta operators by associating the function P_k with $\hat{P}_k = -i X_k$.

$$\hat{P}_x = -i \hbar \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x}, \quad \hat{P}_y = -i \hbar \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y},$$

and then the quantum Hamiltonian for the free case is

$$\begin{aligned}\hat{H} &= -\frac{\hbar^2}{2}m \left((1 + \lambda r^2) \frac{\partial^2}{\partial x^2} + \lambda x \frac{\partial}{\partial x} \right) - \frac{\hbar^2}{2}m \left((1 + \lambda r^2) \frac{\partial^2}{\partial y^2} + \lambda y \frac{\partial}{\partial y} \right) \\ &+ \lambda \frac{\hbar^2}{2}m \left(x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)\end{aligned}$$

which can be written as

$$\hat{H} = \hat{H}_1 + \hat{H}_2 - \lambda \hat{J}^2,$$

with

$$\begin{aligned}\hat{H}_1 &= -\frac{\hbar^2}{2m} \left((1 + \lambda r^2) \frac{\partial^2}{\partial x^2} + \lambda x \frac{\partial}{\partial x} \right) \\ \hat{H}_2 &= -\frac{\hbar^2}{2m} \left((1 + \lambda r^2) \frac{\partial^2}{\partial y^2} + \lambda y \frac{\partial}{\partial y} \right) \\ \hat{J}^2 &= -\frac{\hbar^2}{2m} \left(x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)\end{aligned}$$

and each term commutes with the sum of the other two and therefore with \hat{H} . Therefore we can consider three different (complete) systems of compatible observables:

$$\{\hat{H}_1, \hat{H}_2 - \lambda J^2\}, \quad \{\hat{H}_1 - \lambda J^2, \hat{H}_2\}, \quad \{\hat{H}_1 + \hat{H}_2, J\}$$

The case of harmonic oscillator includes a term

$$\frac{1}{2} \alpha^2 \left(\frac{r^2}{1 - \kappa r^2} \right)$$

and it was exhaustively analysed in several papers, see e.g. J.F.C. + M.F. Rañada + M. Santander, A quantum exactly solvable nonlinear oscillator with quasi-harmonic behaviour, *Ann. Phys.* **322**, 434–459 (2007)

The case of Kepler-Coulomb problem was analysed in a recent paper: C. Quesne, Quantum oscillator and Kepler-Coulomb problems in curved spaces: deformed shape invariance, point canonical transformations, and rational extensions, *J. Math. Phys.* **57**, 102101 (2016) (also arXiv: 1604.04136)

Finally, let us remark that the computation of the Laplace–Beltrami operator corresponding to the given metric is very easy: the metric matrix is given by

$$(g_{ij}) = \frac{1}{1 + \lambda r^2} \begin{pmatrix} 1 + \lambda y^2 & -\lambda x y \\ -\lambda x y & 1 + \lambda x^2 \end{pmatrix}$$

with inverse

$$(g^{ij}) = \begin{pmatrix} 1 + \lambda x^2 & \lambda x y \\ \lambda x y & 1 + \lambda y^2 \end{pmatrix}$$

and therefore, taking into account that

$$g = \det(g_{ij}) = \frac{1}{1 + \lambda r^2},$$

the Laplace–Beltrami operator which is given by

$$\Delta\psi = |g|^{-1/2} \frac{\partial}{\partial x^i} \left(|g|^{1/2} g^{ij} \frac{\partial\psi}{\partial x^j} \right), \quad g = \det(g_{ij}).$$

turns out to be:

$$\Delta = (1 + \lambda x^2) \frac{\partial^2}{\partial x^2} + (1 + \lambda y^2) \frac{\partial^2}{\partial y^2} + 2\lambda \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + 2\lambda x y \frac{\partial^2}{\partial x \partial y}.$$

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THANKS FOR YOUR ATTENTION !!!