A way to build a conformal boundary of a spacetime based on light rays: the 3–dimensional case.

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60 years Alberto Ibort Fest

Classical and Quantum Physics: Geometry, Dynamics & Control

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4. L–extensions for dimension $m = 3$.

Introduction

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Aim

Construction and characterization of the conformal boundary suggested by R. Low, called L–boundary ([Low '06]), for 3–dimensional spacetimes.

Starting point

$(M, \mathcal{C}_{\mathbf{g}})$ conformal (Lorentz) manifold.

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- M m–dimensional Hausdorff differentiable manifold, $(m \geq 3)$.
- $\mathbf{g}$ Lorentz metric in M, $(- + + + \ldots +)$.
- $(M, \mathbf{g})$ time–oriented.
- $\mathcal{C}_{\mathbf{g}} = \left\{ \overline{\mathbf{g}} = e^{2\sigma}\mathbf{g} : \sigma \in \mathfrak{F}(M) \right\}$ conformal (Lorentz) structure in M.

Causality

Given $0 \neq v \in T_pM$, then v is said to be

- timelike $\iff \mathbf{g}(v, v) < 0$
- null or lightlike $\iff \mathbf{g}(v, v) = 0$
- spacelike $\iff \mathbf{g}(v, v) > 0$
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Causality is conformal.

Introduction

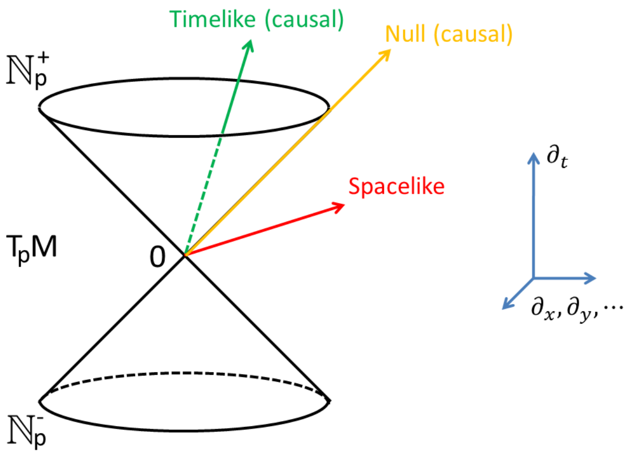

Figure: Causal character.

The space of light rays

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The space of light rays $\mathcal{N}$

Given $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g} \in \mathcal{C}_{\mathbf{g}}$, and a differentiable curve $\gamma : I \to M$, it is known that

γ is null $\mathbf{g}$–geodesic $\Longrightarrow$ γ is null $\overline{\mathbf{g}}$–pregeodesic. ([Kulkarni '88])

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- A light ray $\gamma \in \mathcal{N}$ can be seen as an unparametrized null geodesic.
- The definition of $\mathcal{N}$ is conformal.

Differentiable structure of $\mathcal{N}$

Strong causality, ([Minguzzi, Sánchez '08])

M is *strongly causal in* $x \in M$ if and only if there exists a neighbourhood basis at x of globally hyperbolic, causally convex (and convex normal) open sets.

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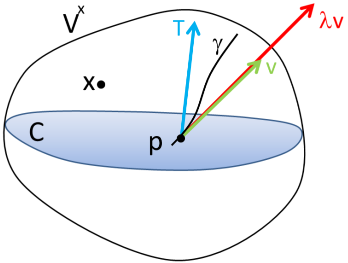

Figure: How can a light ray be defined by a basic neighbourhood?

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- $\mathbb{PN}(C) = \{ [v] = \mathrm{span}\{v\} : v \in \mathbb{N}^+(C) \}$. (Null directions at C).

The topology and the differentiable structure of $\mathcal{N}$ is inherited from $\mathbb{PN}(C)$ by the diffeomorphism

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Locally, $\mathcal{N}$ can be seen as a bundle of spheres: $\mathcal{N}_V \simeq C \times \mathbb{S}^{m-2}$.

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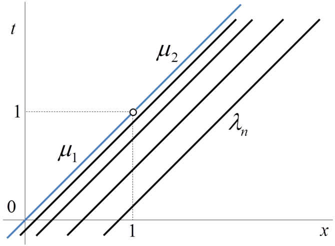

Figure: $\mathcal{N}$ is not Hausdorff.

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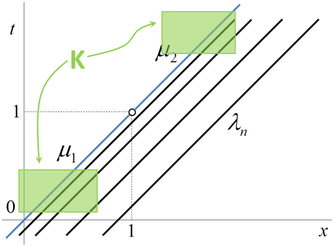

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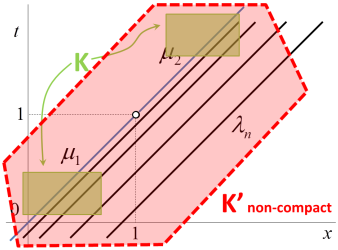

Figure: M is not null pseudo–convex.

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How can $T_\gamma \mathcal{N}$ be described with elements of M?

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Image of γ_0 in M

$\{\gamma_0(t) : t \in \mathrm{Dom}(\gamma_o)\}$

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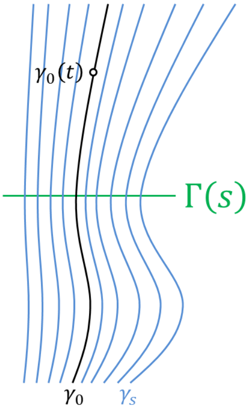
Curve of light rays in $\mathcal{N}$

$$\Gamma(s) = \gamma_s$$



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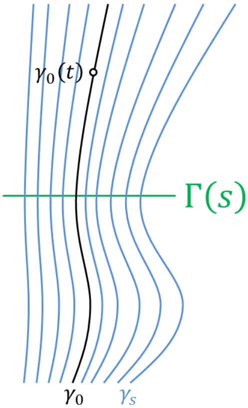
Variation in M

$x(s,t) = \gamma_s(t)$



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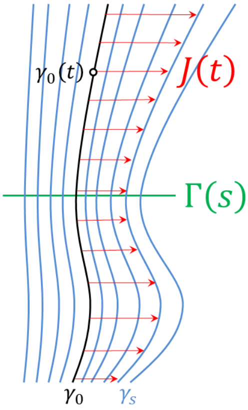
Tangent vector in $T_\gamma \mathcal{N}$

$\Gamma'(0) = \xi$



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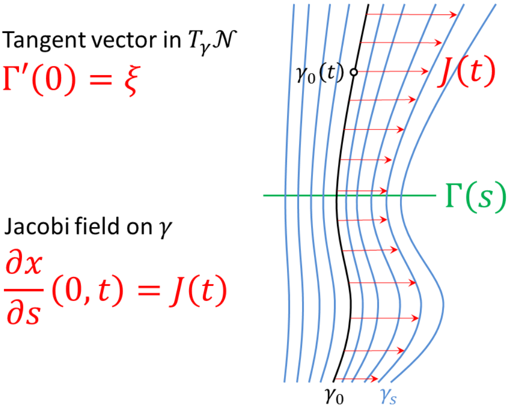

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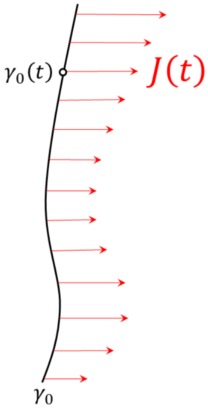
Equation of Jacobi fields

$$J'' + R(J, \gamma')\gamma' = 0$$



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- $\mathcal{J}_L\left(\gamma\right) = \left\{J \in \mathcal{J}\left(\gamma\right) : \mathbf{g}\left(J\left(t\right), \gamma'\left(t\right)\right) = \mathrm{const.}\right\} \simeq \mathbb{R}^{2m-1}$ Jacobi fields of null geodesic variation.
- $\mathcal{J}_0\left(\gamma\right) = \left\{J \in \mathcal{J}_L\left(\gamma\right) : J\left(t\right) = \left(\alpha t + \beta\right)\gamma'\left(t\right), \alpha, \beta \in \mathbb{R}\right\}$

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Proposition

$T_\gamma \mathcal{N}$ is isomorphic to $\mathcal{L}\left(\gamma\right) = \mathcal{J}_L\left(\gamma\right) / \mathcal{J}_0\left(\gamma\right) \simeq \mathbb{R}^{2m-3}$.
This means: $\xi \in T_\gamma \mathcal{N} \Longleftrightarrow J \left(\mathrm{mod} \gamma'\right) \in \mathcal{J}_L\left(\gamma\right) / \mathcal{J}_0\left(\gamma\right)$

The metric $\mathbf{g} \in \mathcal{C}$ and the parametrization of $\gamma \in \mathcal{N}$ are auxiliary elements.

Contact structure $\mathcal{H}$

Contact structure $\mathcal{H}$ in P where $\dim P = 2n+1$

- $\mathcal{H}$ is a differentiable distribution of hyperplanes in TP defined by a 1–form α verifying $\alpha \wedge (d\alpha)^n \neq 0$ (maximally non–integrable).
- If α is global then $\mathcal{H}$ is said to be *co–oriented*.

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Contact structure $\mathcal{H}$ in $\mathcal{N}$

$\mathcal{H}_\gamma = \{J \in T_\gamma\mathcal{N} : \alpha(J) \equiv \mathbf{g}(J, \gamma') = 0\}$.
Moreover, $\mathcal{H}$ is co–oriented.

$\mathcal{H}$ is **conformal**

The space of skies Σ

The sky of $x \in M$

The *sky* of x is defined by $X = S(x) = \{\gamma \in \mathcal{N} : x \in \gamma\} \simeq \mathbb{S}^{m-2}$

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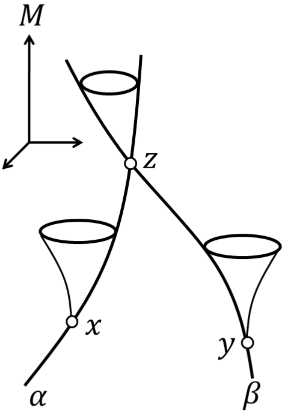

Figure: Skies at M.

The space of skies Σ

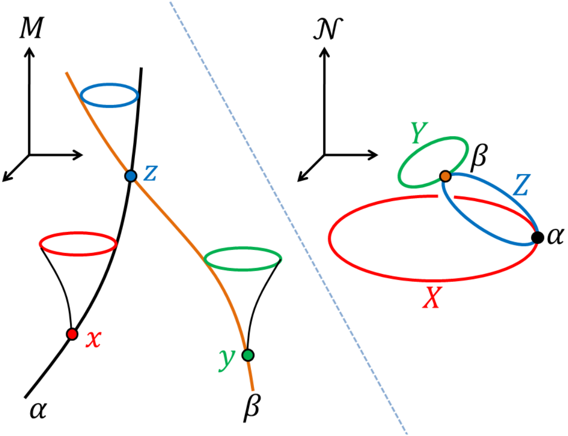

Figure: Skies at $\mathcal{N}$.

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For any $x \in M$ and $\gamma \in S(x)$, if $x = \gamma(t_0)$ then

Tangent space to a sky

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Light non-conjugation

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$\mathcal{H}$ is conformal

$$\mathcal{H}_\gamma = T_\gamma X \oplus T_\gamma Y \text{ for any } X, Y \text{ light non–conjugate.}$$

Topology in Σ

$X_0 \in \Sigma$ such that $X_0 \subset \mathcal{U} \subset \mathcal{N}$ where $\mathcal{U}$ is open.

Reconstructive or Low's topology in Σ

It is generated by $\Sigma(\mathcal{U}) = \{X \in \Sigma : X \subset \mathcal{U}\}$

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Theorem ([Kinlaw '11], [–, Ibort, Lafuente 14, 15])

The sky map $S : M \to \Sigma$ is an homeomorphism.

Differentiable structure in Σ

Theorem ([–, Ibort, Lafuente 14, 15])

Let $V \subset M$ be a relatively compact basic open set and $V_\Sigma = S(V) \subset \Sigma$. Then:

- $V_\Sigma \subset \Sigma$ is light non–conjugate.
- $\widehat{V} = \bigcup_{X \in V_\Sigma} \widehat{T}X \subset T\mathcal{N}$ is a **regular** submanifold of $\widehat{T}\mathcal{N}$.
- The distribution $\mathcal{D}$ in $T\mathcal{N}$ such that its leaves are $\widehat{T}X$ is regular.
- The map $V_\Sigma \to \widehat{V}/\mathcal{D}$ such that $X \mapsto \widehat{T}X$ is a diffeomorphism.

Theorem ([–, Ibort, Lafuente 14, 15])

The previous one is the unique differentiable structure of Σ compatible with the reconstructive topology and such that $S : M \to \Sigma$ is a diffeomorphism.

Reconstruction of $(M, \mathcal{C})$

Theorem of reconstruction ([–, Ibort, Lafuente, '14])

Let $(M, \mathcal{C})$, $(\overline{M}, \overline{\mathcal{C}})$ be two strongly causal spacetimes and $(\mathcal{N}, \Sigma)$, $(\overline{\mathcal{N}}, \overline{\Sigma})$ their corresponding pairs of spaces of light rays and skies. Let $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ be a diffeomorphism such that $\phi(\Sigma) \subset \overline{\Sigma}$ (i.e. *sky preserving*). Then the map

$$\varphi = \overline{S}^{-1} \circ \phi \circ S : M \to \overline{M}$$

is a conformal diffeomorphism onto its image, where $\overline{S} : \overline{M} \to \overline{\Sigma}$ is the sky map of $\overline{M}$.

The L–boundary for dimension $m = 3$

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Low's boundary

Idea

- $\gamma : (a, b) \to M$ maximal parametrized light ray.
- $\widetilde{\gamma} : (a, b) \to Gr^{m-2}(\mathcal{H}_\gamma)$ defined by $\widetilde{\gamma}(s) = T_\gamma S(\gamma(s))$ where $Gr^{m-2}(\mathcal{H}_\gamma)$ is the grassmannian manifold of $(m-2)$–dimensional subspaces of $\mathcal{H}_\gamma \subset T_\gamma \mathcal{N}$.
- $\begin{cases} \ominus_\gamma = \lim_{s \mapsto a^+} \widetilde{\gamma}(s) \in Gr^{m-2}(\mathcal{H}_\gamma) \\ \oplus_\gamma = \lim_{s \mapsto b^-} \widetilde{\gamma}(s) \in Gr^{m-2}(\mathcal{H}_\gamma) \end{cases}$ (distributions in $Gr^{m-2}(\mathcal{H})$).
- New future (past) causal boundary: integral manifolds of $\oplus$ ($\ominus$).

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Idea

- $\gamma : (a, b) \to M$ maximal parametrized light ray.
- $\widetilde{\gamma} : (a, b) \to Gr^{m-2}(\mathcal{H}_\gamma)$ defined by $\widetilde{\gamma}(s) = T_\gamma S(\gamma(s))$ where $Gr^{m-2}(\mathcal{H}_\gamma)$ is the grassmannian manifold of $(m-2)$–dimensional subspaces of $\mathcal{H}_\gamma \subset T_\gamma \mathcal{N}$.
- $\begin{cases} \ominus_\gamma = \lim_{s \mapsto a^+} \widetilde{\gamma}(s) \in Gr^{m-2}(\mathcal{H}_\gamma) \\ \oplus_\gamma = \lim_{s \mapsto b^-} \widetilde{\gamma}(s) \in Gr^{m-2}(\mathcal{H}_\gamma) \end{cases}$ (distributions in $Gr^{m-2}(\mathcal{H})$).
- New future (past) causal boundary: integral manifolds of $\oplus$ ($\ominus$).

Problem

- Do the limits $\ominus_\gamma$ and $\oplus_\gamma$ exist?

Low's boundary in $\dim M = 3$

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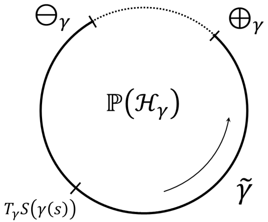

Figure: $\widetilde{\gamma} \subset \mathbb{P}(\mathcal{H}_\gamma)$.

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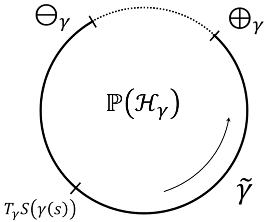

Figure: $\widetilde{\gamma} \subset \mathbb{P}(\mathcal{H}_\gamma)$.

If M is light non–conjugate $\Longrightarrow$ the limits $\ominus_\gamma, \oplus_\gamma$ exist.

Hypotheses

a. $\dim M = 3$.

b. $(M, \mathcal{C})$ is strongly causal, null–pseudo convex, light non–conjugate and sky–separating.

c. The distributions $\oplus, \ominus : \mathcal{N} \rightarrow \mathbb{P}(\mathcal{H})$ defined by $\oplus_\gamma = \lim_{s\mapsto b^-} T_\gamma S(\gamma(s))$ and $\ominus_\gamma = \lim_{s\mapsto a^+} T_\gamma S(\gamma(s))$ are differentiable and regular and such that $\oplus_\gamma \neq \ominus_\gamma$ for any maximally and future–directed parametrized light ray $\gamma : (a, b) \rightarrow M$.

The space $\widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$

Lemma

Let $\pi_M^{\mathbb{PN}} : \mathbb{PN} \to M$ be the canonical projection. Then the map

$$\begin{array}{rccl} \sigma : & \mathbb{PN} & \to & \mathbb{P}(\mathcal{H}) \\ & [u] & \mapsto & T_{\gamma_{[u]}} S\left(\pi_M^{\mathbb{PN}}([u])\right) \end{array} \tag{1}$$

is a diffeomorphism onto its image $\widetilde{\mathcal{N}} = \sigma(\mathbb{PN})$.

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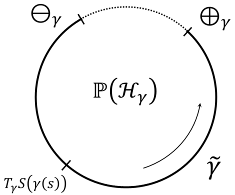
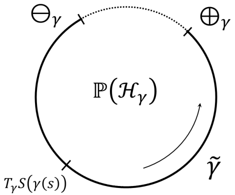
Corollary

$\widetilde{\mathcal{N}}$ is an open submanifold of $\mathbb{P}(\mathcal{H})$.



Every fibre $\mathbb{P}(\mathcal{H}_\gamma)$ is a projective line, so it is diffeomorphic to the circle $\mathbb{S}^1$.

The projective parameter



Every fibre $\mathbb{P}(\mathcal{H}_\gamma)$ is a projective line, so it is diffeomorphic to the circle $\mathbb{S}^1$.

We can construct a *"bunch"* of projective parametrizations

$$\begin{array}{rccl} \varepsilon : & \mathcal{N}_U \times \mathbb{R} & \rightarrow & \mathbb{P}(\mathcal{H}_U) - \widetilde{\infty} \\ & (\gamma, \mathbf{t}) & \mapsto & \widetilde{\gamma}(\mathbf{t}) = T_\gamma S(\gamma(\mathbf{t})) \end{array}$$

such that ε is a diffeomorphism and moreover

- $\mathbf{t} \in (-1,1) \Longleftrightarrow \widetilde{\gamma}(\mathbf{t}) \in \widetilde{\mathcal{N}}$.
- $\widetilde{\gamma}(\mathbf{1}) = \oplus_\gamma$, $\widetilde{\gamma}(-\mathbf{1}) = \ominus_\gamma$ and $\pi \circ \sigma^{-1}(\widetilde{\gamma}(\mathbf{0})) \in C$.

Distribution $\mathcal{D}^{\sim}$

$\sigma : \mathbb{PN} \to \widetilde{\mathcal{N}}$ diffeomorphism.

- $\mathcal{P}$ regular distribution in $\mathbb{PN}$ whose leaves are fibres $\mathbb{PN}_x$ of $\mathbb{PN} \to M$.
- Propagating $\mathcal{P}$ by σ we obtain a regular distribution $\mathcal{D}^{\sim}$ in $\widetilde{\mathcal{N}}$ whose leaves are $\sigma\left(\mathbb{PN}_x\right)$.

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The map $\mathbb{PN}/\mathcal{P} \to \widetilde{\mathcal{N}}/\mathcal{D}^{\sim}$ induced by σ is a diffeomorphism.
Since $\mathbb{PN}/\mathcal{P} \simeq M$ then

$$\widetilde{\mathcal{N}}/\mathcal{D}^{\sim} \simeq M$$

Distribution $\partial^+\mathcal{D}^\sim$

$\mathcal{N} \simeq \mathcal{N} \times \{1\} \stackrel{\varepsilon}{\to} \partial^+\widetilde{\mathcal{N}}$ is a diffeomorphism.

- $\oplus : \mathcal{N} \to \mathbb{P}(\mathcal{H})$ is a regular distribution in $\mathcal{N}$.
- Propagating the leaves of $\oplus$ by ε we obtain a regular distribution $\partial^+\mathcal{D}^\sim$ in $\partial^+\widetilde{\mathcal{N}}$ whose leaves are $\varepsilon(X^+)$ for X^+ leaf of $\oplus$.

Theorem, ([–, Ibort, Lafuente, '18])

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Then

$\overline{\widetilde{\mathcal{N}}}/\overline{\mathcal{D}^{\sim}}$ is a differentiable manifold and

$$\overline{\widetilde{\mathcal{N}}}/\overline{\mathcal{D}^{\sim}} = \widetilde{\mathcal{N}}/\mathcal{D}^{\sim} \cup \partial^{+}\widetilde{\mathcal{N}}/\partial^{+}\mathcal{D}^{\sim} \simeq M \cup \partial^{+}\widetilde{\mathcal{N}}/\partial^{+}\mathcal{D}^{\sim} = \overline{M}$$

$\partial M = \partial^{+}\widetilde{\mathcal{N}}/\partial^{+}\mathcal{D}^{\sim}$ is the L–boundary.

L–extensions for dimension $m = 3$.

1. Introduction.
2. The space of light rays.
3. The L–boundary for dimension $m = 3$.
4. L–extensions for dimension $m = 3$.

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3. **projective** if $\gamma : (a, b) \to M$ is a regular parametrization and $\widetilde{\gamma}(s) \in \mathbb{P}(\mathcal{H}_\gamma)$ defines a projectivity in the fibre $\mathbb{P}(\mathcal{H}_\gamma)$, and

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1. **continuous** if $\gamma : (a, b) \to M$ is a continuous map,
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4. **admissible** if there exists a diffeomorphism $h : (c, d] \to (a, b]$ such that $h' (t) > 0$ for all $t \in (c, d]$ and $\gamma \circ h : (c, d) \to M$ is a projective parametrization.

Definition

A *future L–extension* of $(M, \mathcal{C})$ is a Hausdorff smooth manifold $\overline{M} = M \cup \partial^+ M$ where $\partial^+ M = \overline{M} - M$ is a closed hypersurface of $\overline{M}$ called the *future L–boundary* such that:

1. $\lim_{s \mapsto b^-} \gamma(s) = \infty^+_\gamma \in \partial^+ M$ for any continuous parametrization of $\gamma \in \mathcal{N}$.
2. The map $\infty^+ : \mathcal{N} \to \partial^+ M$ defined by $\infty^+(\gamma) = \infty^+_\gamma$ is a surjective submersion.
3. For every $\gamma_0 \in \mathcal{N}$ there exists a neighbourhood $\mathcal{U} \subset \mathcal{N}$ and a differentiable map $\Psi_{\mathcal{U}} : \mathcal{U} \times (a, b] \to \overline{M}$, where $\gamma(s) = \Psi_{\mathcal{U}}(\gamma, s)$ is an admissible parametrization of $\gamma \in \mathcal{U}$ for $s \in (a, b)$ and such that $\frac{\partial \Psi_{\mathcal{U}}}{\partial s}(\gamma, b) \notin T_{\infty^+(\gamma)} \partial^+ M$.

Analogously, we can define a *past L–extension* $\overline{M} = M \cup \partial^- M$.

L–extensions

Since $\infty^+ : \mathcal{N} \to \partial^+ M$ is a surjective submersion then every

$$S(p) = \left(\infty^+\right)^{-1}(p) = \{\gamma \in \mathcal{N} : p = \infty^+(\gamma)\} \subset \mathcal{N}$$

defines a leaf of a regular distribution $\boxplus : \mathcal{N} \to \mathbb{P}(T\mathcal{N})$ given by $\boxplus(\gamma) = T_\gamma S(\infty^+(\gamma))$, and the map

$$\begin{array}{rccc} S: & \partial^+ M & \to & \mathcal{N}/\boxplus \\ & p & \mapsto & S(p) \end{array}$$

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Theorem, ([–, Ibort, Lafuente, '18])

The extension constructed by the L–boundary is a L–extension.

L–extensions

Theorem, ([–, Ibort, Lafuente, '18])

Let $\overline{M}_1 = M \cup \partial^+ M_1$ and $\overline{M}_2 = M \cup \partial^+ M_2$ be future L–extension of $(M, \mathcal{C})$, then the identity map $\mathrm{id} : M \to M$ can be extended as a diffeomorphism $\overline{\mathrm{id}} : \overline{M}_1 \to \overline{M}_2$.

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Conversely,

Theorem, ([–, Ibort, Lafuente, '18])

Under the hypotheses:

a. $\dim M = 3$.

b. $(M, \mathcal{C})$ is strongly causal, null–pseudo convex, light non–conjugate and sky–separating.

c. there is a future L–extension of $(M, \mathcal{C})$

then $\boxplus = \oplus$, and therefore $\oplus$ is differentiable and regular.

Present and future

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  All results try to indicate, when possible, a way to afford the construction for the general higher dimensional case.

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What's next?

- Follow the suggested way (or find another one) to obtain analogue results for m–dimensional spacetimes with $m \geq 3$.

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Thanks to all for your attention...

... and congratulations to Alberto for his birthday.

Happy ~~sixty~~ sixteen!!!