The Gauss Law: A Tale

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This talk covers work done with many colleagues over the years.

Recent work involves Parameswaran Nair, Arshad Momen, Amilcar Queiroz and Sachin Vaidya.

We will analyze the Gauss Law and see its impact on

- Superselection Rules
- Edge States
- Mixed Vector states in a) QCD & b) Non-linear σ models

QED

Let us begin our discussion with the standard Gauss law

$$(\partial_i E^i + J_0)|\cdot\rangle = 0 \tag{1}$$

with E_i being the electric field which is conjugate to the potential A_j at equal times.

But E_i are (operator-valued) distributions.

So we must smear (1) with test functions and transfer the action of the derivatives onto the test functions.

So if $\Lambda \in C_0^\infty$ so that it has compact support and is infinitely differentiable , one instead works with

$$G_E(\Lambda)|\cdot\rangle = 0, G_E(\Lambda) = \int d^3x \; (-E^i \; \partial_i \Lambda + \Lambda J_0) \tag{2}$$

Now the charge Q is defined is a similar manner:

$$Q_E(\mu) \equiv Q(\mu) = \int d^3x \; (-E^i \; \partial_i \mu + \mu J_0)$$

But μ is not required to vanish at ∞ like Λ, $\mu \in C^\infty$.

So we have two groups, namely:

- Group generated by $G_E(\Lambda) : \mathcal{G}_E^\infty \to 1$ acting on $|\cdot\rangle$.
- Group generated by $Q(\mu) : \mathcal{G}_E$.

Note: Local observables commute with $G(\Lambda), Q(\mu) \Rightarrow$ (electric) superselection rule, which is fixed by elements of $\mathcal{G}_E / \mathcal{G}_E^\infty$.

These depend only on the $\mu|_{S^2_\infty}$, with S^2_∞ being the sphere at ∞.

If μ is constant, then these elements end up being the standard $U(1)$ electric charges.

When $\mu|_{S^2_\infty}$ are dependent on the angles (θ, ϕ), one gets the electric part of the Sky group of Balachandran and Vaidya.

Magnetic Constraints

Starting with the "magnetic Gauss law"

$$\partial_i B^i |\cdot\rangle = 0,$$

one can proceed (as before) and get

$$G_M(\Lambda)|.\rangle, \qquad G_M(\Lambda) = -\int d^3x \ B^i \partial_i \Lambda$$

with $\Lambda \in C_0^\infty$. They generate the group $\mathcal{G}_M^\infty$.
For $\nu \in C^\infty$, where ν need not be zero at infinity, one has $\mathcal{G}_M$ generated by

$$G_M(\nu) \equiv M(\nu) = -\int d^3x \ B^i \partial_i \nu$$

$\mathcal{G}_M / \mathcal{G}_M^\infty$ is also superselected, depends only on $\nu|_{S^2_\infty}$.

Remark

If we allow

$$\mu(\mathbf{x}) \rightarrow_{r\rightarrow\infty} \mu_\infty(\hat{\mathbf{x}}) \neq 0; \quad \nu(\mathbf{x}) \rightarrow_{r\rightarrow\infty} \nu_\infty(\hat{\mathbf{x}})$$

the resulting $Q(\mu), M(\nu)$ will generate the Sky group of Balachandran and Vaidya.

The Sky group

A natural question to ask:

$$[Q(\mu, M(\nu)] =?$$

In QED, the above commutator is zero, as shown by using

$$[B_i(\mathbf{x}, t), E_j(\mathbf{y}, t)] = -i\varepsilon_{ijk}\partial_k\delta(\mathbf{x} - \mathbf{y})$$

The full Sky group for QED is thus $(\mathcal{G}_E/\mathcal{G}_E^\infty) \times (\mathcal{G}_M/\mathcal{G}_M^\infty)$.
These are isomorphic to maps $S^2_\infty \to U(1)\times$ maps $S^2_\infty \to U(1)$.
Different charge sectors have different $Q(\mu)$.
Even without charges or monopoles, $Q(\mu), M(\nu)$ can differ from zero due to the distribution of **E** and **B** at infinity.

The Sky group(contd.)

Since $\partial_i B^i = 0$ is nothing but the Bianchi identity, $M(\nu)$ would measure the flux associated with the Bianchi identity and its moments - which would be like "magnetic charges" and their moments. Note: $\mathbf{B} = \nabla \wedge \mathbf{A}$ is only on vectors $|\cdot\rangle$ fulfilling

$$G_M(\Lambda)|\cdot\rangle = 0.$$

We proceed as before for non-abelian systems but take test functions $\Lambda, \mu, \nu \cdots$ which are valued in the Lie algebra:

$$\Lambda \equiv \Lambda^\alpha \lambda_\alpha, \qquad \Lambda^\alpha \in C_0^\infty(\mathbb{R}).$$

λ_α form a basis of the Lie algebra of the group G.

The Gauss law

$$G_E(\Lambda)|\cdot\rangle = 0, \qquad G_E(\Lambda) \equiv \int d^3x \;\; \mathrm{Tr}\; (D_i\Lambda(\mathbf{x})E^i(\mathbf{x})$$

generates the group $\mathcal{G}^\infty$.

As before, $\mu = \mu^{\alpha}\lambda_{\alpha}, \mu^{\alpha} \in C^{\infty}(\mathbb{R}^3)$,

$$Q_E(\mu) \equiv Q(\mu) = \int d^3x \ \ \mathrm{Tr}\ (D_i\mu(\mathbf{x})E^i(\mathbf{x})$$

with

$$[Q(\mu_1), Q(\mu_2)] = iQ([\mu_1, \mu_2])$$

which represents the group $\mathcal{G}_E$.

The (electric) superselection group is thus $\mathcal{G}_E/\mathcal{G}_E^{\infty}$.

It depends only on $\mu|_{S^2_{\infty}}$.

If $\mu|_{S^2_{\infty}} =$ constant, we get the global group G.

QCD and Superselection rules

For QCD , we will consider only constant maps of S^2_∞ to $SU(3)$ as the superselection group.

We can also consider $SU(3)_{\mathrm{Sky}}$ where maps vary over the points of S^2_∞.

For simplicity. c onsider only the former. It is superselected.

In a a superselection sector, we can diagonalize a complete commuting set: via two $SU(3)$ Casimirs C_i, one $SU(2)$ Casimir $\mathbf{I}^2$, and I_3, Y:

$$|\Psi\rangle = |C_2, C_3, \mathbf{I}^2, I_3, Y, \cdots\rangle$$

No observable changes the sector.

However, generic $SU(3)$ does that $\Rightarrow$ it is not observable!

Also if $g \in SU(3)$ and $|\chi\rangle = g|\Psi\rangle = |C_2, C_3, \mathbf{I}'^2, I_3', Y', \cdots\rangle$ and

$$\rho(\lambda) = \lambda|\Psi\rangle\langle\Psi| + (1-\lambda)|\chi\rangle\langle\chi|,$$

then for any observable a,

$$\mathrm{Tr}\ (\rho(\lambda)a) = \ \mathrm{Tr}\ (\rho(0)a) \qquad \lambda \in [0,1]$$

But von Neumann entropy depends on λ. So the question is:

Are these states mixed?

If states are regarded as density matrices, they are mixed.

The mixture is not even unique ! $\Rightarrow$ von Neumann entropy also not unique.

But abstractly, a state ω is determined by the numbers $\omega(a)$.

In this sense, we should identify the above ρ's as definiting the same state.

But different ρ's give different entropies $\Rightarrow$ we cannot uniquely associate an entropy with such a state.

Note: If observables are enlarged to include $\mathbf{I}^2, I_3, Y$, then $\rho(\lambda)$ becomes impure for $\lambda \neq 0, 1$ and pure if $\lambda = 0, 1$.

Is $SU(3)_c$ of color spontaneously broken?

An operator which changes superselection sector is , by definition, spontaneously broken. So $su(3)_c$, the group algebra of $SU(3)_c$, is spontaneously broken to the algebra generated by $C_i, \mathbf{I}^2, I_3, Y$.

So where are the Goldstone modes?

We note that $SU(3)_c$ and Sky group create edge excitations at infinity. They come from Gauss law and connection field A_μ which has spin 1.

Superselected (Magnetic) Group in Non-Abelian Gauge Theories

We have Bianchi identity or "Magnetic" Gauss law

$$D_i B^i = 0$$

which we rewrite as

$$G_M(\Lambda)|\cdot\rangle = 0, \text{ with } G_M(\Lambda) = -\int d^3x \ \mathrm{Tr} \ (D_i\Lambda \ B^i)$$

and $\Lambda \in C_0^\infty(\mathbb{R}^3)$. Then the magnetic fields on S^2_∞ or Bianchi flux is identified as

$$Q_M(\nu) = M(\nu) = -\int d^3x \ \mathrm{Tr} \ (D_i\nu \ B^i), \quad \nu \in C^\infty(\mathbb{R}^3)$$

It does not commute with $Q(\mu)$:

$$[Q(\mu), M(\nu)] = iM([\mu,\nu])$$

Thus we have the full have the full Sky group

$$
\begin{aligned}
&[Q(\mu_1), Q(\mu_2)] = iQ([\mu_1, \mu_2]),\\
&[Q(\mu), M(\nu)] = iM([\mu, \nu]),\\
&[M(\nu), M(\nu')] = 0,
\end{aligned}
$$

which has a semi-direct product structure.

Now that we have established the existence of these edge excitations in QCD, a natural question to ask -

Can there be edge excitations from (scalar) fields?

The answer seems yes.

We can establish this observation in σ-models.

Non-linear Models and Edge Excitations

Consider a model for Goldstone modes with gauge group G which is spontaneously broken to $H \subset G$.
Then the model describes Goldstone modes with target space G/H.
If the model can be described as a gauge theory, then we can apply the previous discussion. This can be done as follows(*Balachandran, Stern and Trahern* - **Phys Rev D 19***(1979)2416*).
We fix an orthonormal basis of the Lie algebra of G :

$$T(\alpha), \quad \alpha = 1, 2, \cdots |H|$$

$$S(i) \qquad \text{remaining generators of G}$$

Then under an action of $h \in H$,

$$hT(\alpha)h^{-1} = T(\beta)h_{\beta\alpha}$$
$$hS(i)h^{-1} = S(j)D_{ji}(h)$$

Set

$$A_\mu(g) = T(\alpha)\ \mathrm{Tr}\ T(\alpha)g^{-1}(x)\partial_\mu g(x)$$
$$B_\mu(g) = S(i)\ \mathrm{Tr}\ S(i)g^{-1}(x)\partial_\mu g(x)$$

Then under the right action of H ,

$$A_\mu(gh) = h^{-1}A_\mu(g)h + h^{-1}\partial_\mu h$$
$$B_\mu(gh) = h^{-1}B_\mu(g)h$$

i.e. A_μ is a connection while B_μ is a tensor field.

For gauge group $\mathcal{H} \ni h : \mathbb{R}^d \to H$, we can write Lagrangian densities like

$$\mathcal{L}_1 = -\lambda \ \mathrm{Tr} \ B_\mu(g) B^\mu(g) \tag{3}$$

or

$$\mathcal{L}_2 = -\lambda \ \mathrm{Tr} \ F_{\mu\nu}(A) F^{\mu\nu}(A) \tag{4}$$

They reduce to standard σ-model Lagrangians, e.g. with $G = SU(2), H = U(1) \Rightarrow G/H = S^2$. Explicitly writing:

$$g(x)\sigma_3 g^{-1}(x) = \sigma_\alpha \varphi_\alpha(x) \Rightarrow \varphi_\alpha(x)\varphi_\alpha(x) = \mathbb{I},$$

we get

$$\mathcal{L}_1 \sim -\lambda(\partial_\mu \varphi_\alpha)(\partial^\mu \varphi_\alpha)$$

$$\mathcal{L}_2 \sim -\varepsilon^{\alpha\beta\gamma}\lambda(\varphi_\alpha \partial^\mu \varphi_\beta \partial^\nu \varphi_\gamma)^2$$

But (non-local) observables need be invariant only under

$$\mathcal{H}_\infty = \{h \in \mathcal{H} | h_\infty(\hat{\mathbf{x}}) = \lim_{r \to \infty} h(r\hat{\mathbf{x}}) = \mathbb{I}\}.$$

Can we find such observables invariant only under $\mathcal{H}_\infty$ and not under $\mathcal{H}$?
Consider the Wilson line

$$W(g, x, e) = \exp \int_\infty^x d\lambda \; e^\mu A_\mu(g(x + \lambda e))$$

where e^μ is a spacelike unit vector.
Under gauge transformation by $h \in \mathcal{H}$,

$$W(g, x, e) \to h_\infty(\hat{\mathbf{x}}) W(g, x, e) h^{-1}(x).$$

Hence

$$\tilde{B}_\mu(g,x,e) \equiv W(g,x,e)B_\mu(x)[W(g,x,e)]^{-1}$$

is invariant by small , but not by large gauge transformations.

$\tilde{B}_\mu(g,x,e)$ is *not* a local field .

$\tilde{B}_\mu(g,x,e)|x\rangle$ is a state with edge excitations.

How do we see them?

Perhaps through instantons. (New work with Nair).

Thus we have the θ-vacuum term

$$\frac{\theta}{32\pi^2}\int \mathrm{Tr}\, F(A)\wedge F(A)$$

that we can add to the action.

There are also instanton solutions of

$$F = *F$$

(For certain groups, the ADHM method works!)

But this topological term cannot be reduced to an integral of standard G/H -model fields.

It violates CP and can induce electric dipole moment.

Present limit

$$\theta \leq 10^{-10}.$$

Some of these ideas extend to self-dual gravity as well.

- The deceptively simple Gauss law leads to many significant physical results, in particular to non-abelian superselection rules. The latter are poorly studied.
- Gauge systems, despite being well-studied over the years, still have important results yet to be explored.

Thank you!

Finally

WE CELEBRATE ALBERTO'S MANY TALENTS

I WISH HIM A GLORIOUS FUTURE.