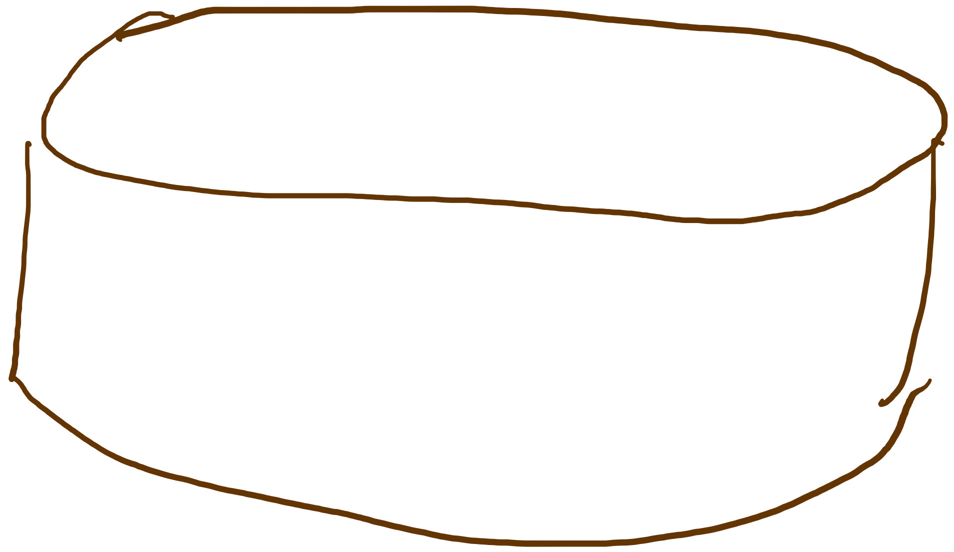
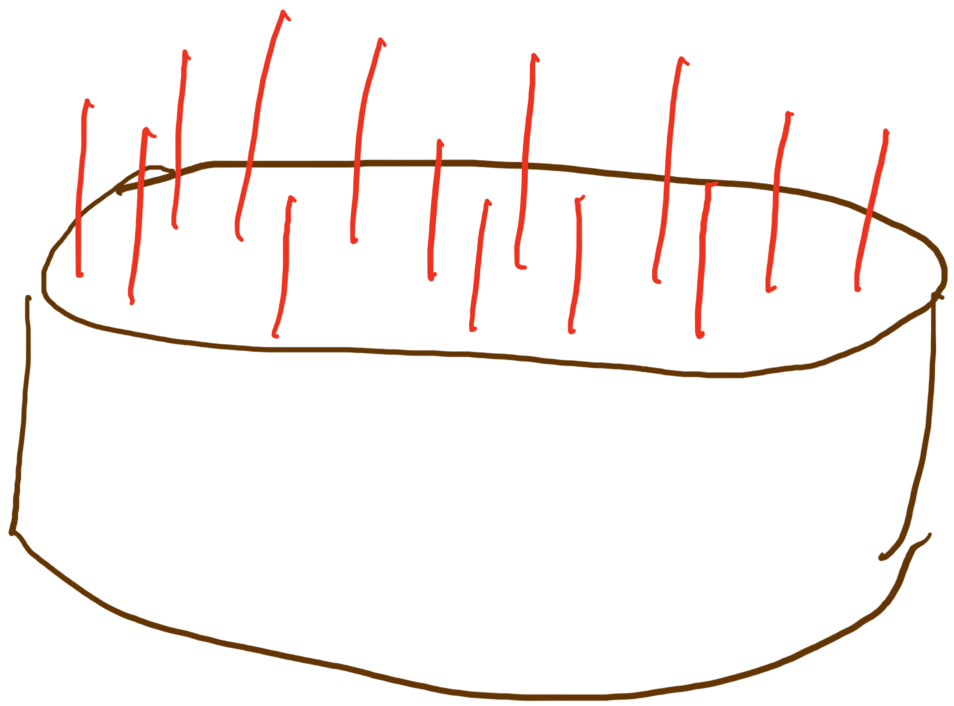
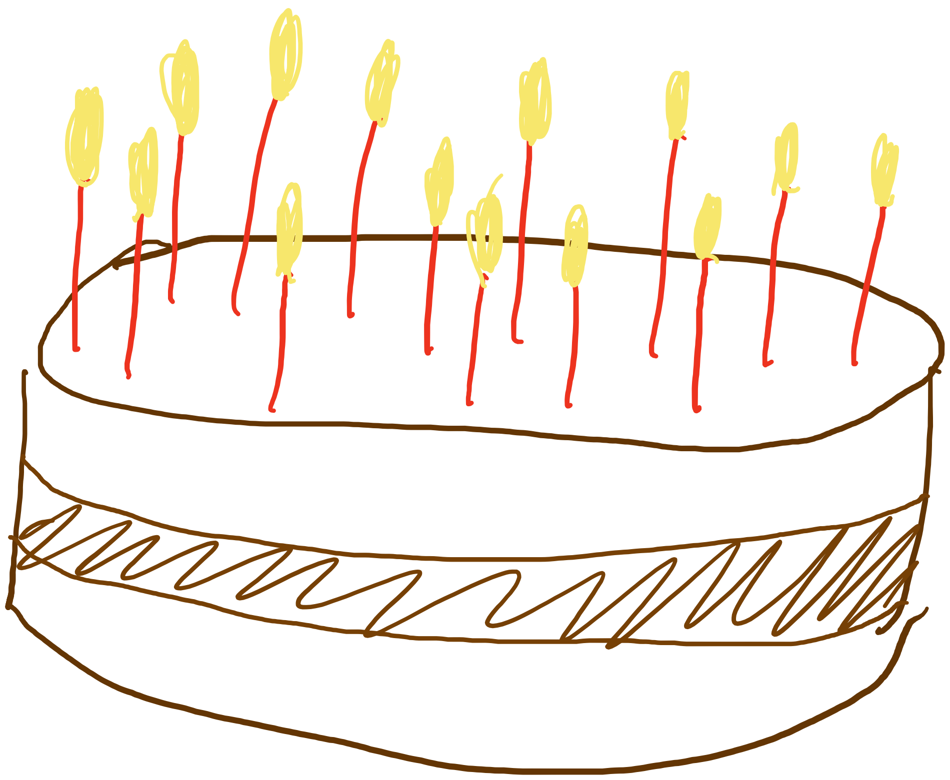
Hybrid Large deformation diffeomorphic metric mapping

July 2017

Laurent Younes







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Growth form and self-organisation

- Overview
- Participants
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Current Programmes

Future Programmes

Past Programmes

Programmes by Year

Programme
22nd August 2017 to 20th December 2017
Organisers: Arezki Boudaoud, Robert Dudley, Andrzej Herczynski, Darryl Holm, Eric Lauga, Jean-Luc Thiffeault

Programme Theme

This programme, planned on the 100th anniversary of On Growth and Form by D'Arcy Thompson, will endeavour to bring various contemporary strands of research related to shape selection, deformations, and self-organisation into focus. The aim is to stimulate new collaborations between mathematicians, biologists, physicists, and other scholars or creators whose work concerns or employs evolution of

"LDDMM"

→ The "large deformation diffeomorphic metric mapping" method is a family of algorithms designed for shape registration.

→ They provide a (local) representation shape space in the diffeomorphism group

Applied in Computational Anatomy

Study of organ shape variation

from me

Basic Principles of LDDMM

Denote by Diff_0^p the space of Diffeomorphisms φ in $\mathbb{R}^d$ who:

→ are C^p

→ such that $\varphi - id$ and derivatives of order p or less tend to 0 at infinity.

→ $\mathrm{Diff}_0^p - id$ open subset of $\left(C_0^p(\mathbb{R}^d, \mathbb{R}^d), \| \, \|_{p,\infty}\right)$

→ Let V be a Hilbert space continuously included in $C_0^p(\mathbb{R}^d, \mathbb{R}^d)$ $(p \geq 1)$.

→ Consider the distribution

$$\varphi \longmapsto V_\varphi = V\circ\varphi = \{ v\circ\varphi \ , \ v \in V \}$$

→ Define the sub-riemannian structure

$$\| v\circ\varphi \|_\varphi = \| v \|_V$$

→ Diff_V : group of attainable diffeomorphisms through finite energy paths

$$\dot\varphi \in V_\varphi \ , \qquad \int_0^1 \| \dot\varphi \|_\varphi^2 \, dt < \infty$$

→ Basic example :

⊗ $K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow M_d(\mathbb{R})$

positive kernel : $K(x,y) = K(y,x)^T$ and

$$\sum_{i,j=1}^{N} a_i^T K(x_i, x_j) a_j \geqslant 0 \quad , \forall x_1, \dots, x_N, a_1, \dots, a_N \in \mathbb{R}^d$$

⊗ V = associated RKHS

⊗ $V_\varphi = V \circ \varphi$

Optimal Control Problem (version 1)

Minimize $\displaystyle\int_0^1 \|v(t)\|_V^2 \, dt \quad + \quad U(\varphi(1))$

subject to $\quad \varphi(0) = \mathrm{id}$

and $\quad \dot{\varphi} = v \circ \varphi$

Optimal Control Problem (version 2)

→ Assume that Diff_0^θ acts on a "shape space" Q

→ Minimize

$$\int_0^1 \|v(t)\|_V^2 \, dt \quad + \quad D(q(1), q_{targ})$$

subject to $\quad q(0) = q_{template}$

and $\quad \partial_t q = v \cdot q$

Infinitesimal action

Interpretation

- LDDMM deforms the whole space in order to move the template in a position consistent with the target (up to invariance).
- The deformation cost treats $\mathbb{R}^d$ as a homogeneous medium.
- In particular, this cost does not depend on deformed objects.

This is good because...

- The shape space geometries derives from a right-invariant metric on Diff through a Riemanian submersion.
- Geodesic equations are well known (EP_DIFF) and have important conservation laws.
- Numerical procedures are well explored.
- Dependences on shape can be brought in the data attachement term.

However...

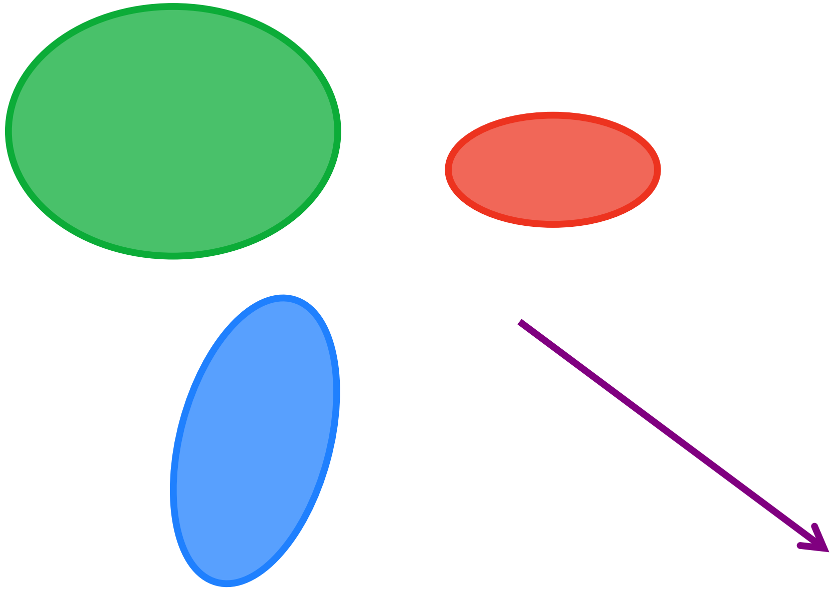
Including object information to drive the deformation process can be beneficial in some important cases, like:

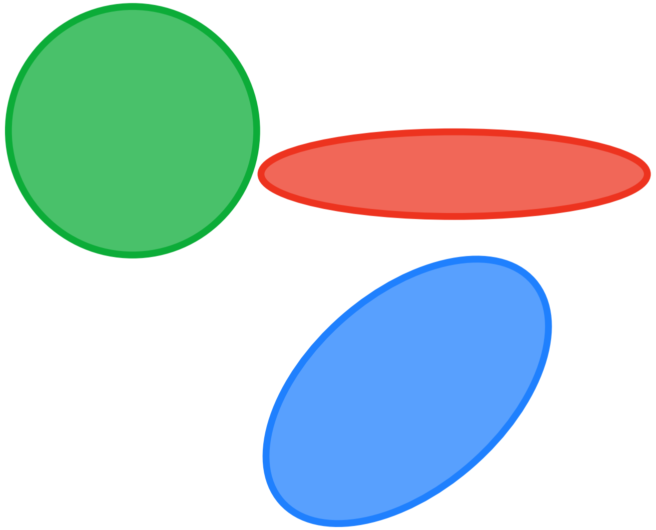
→ Shape complexes (Multi-shape).

→ Articulation.

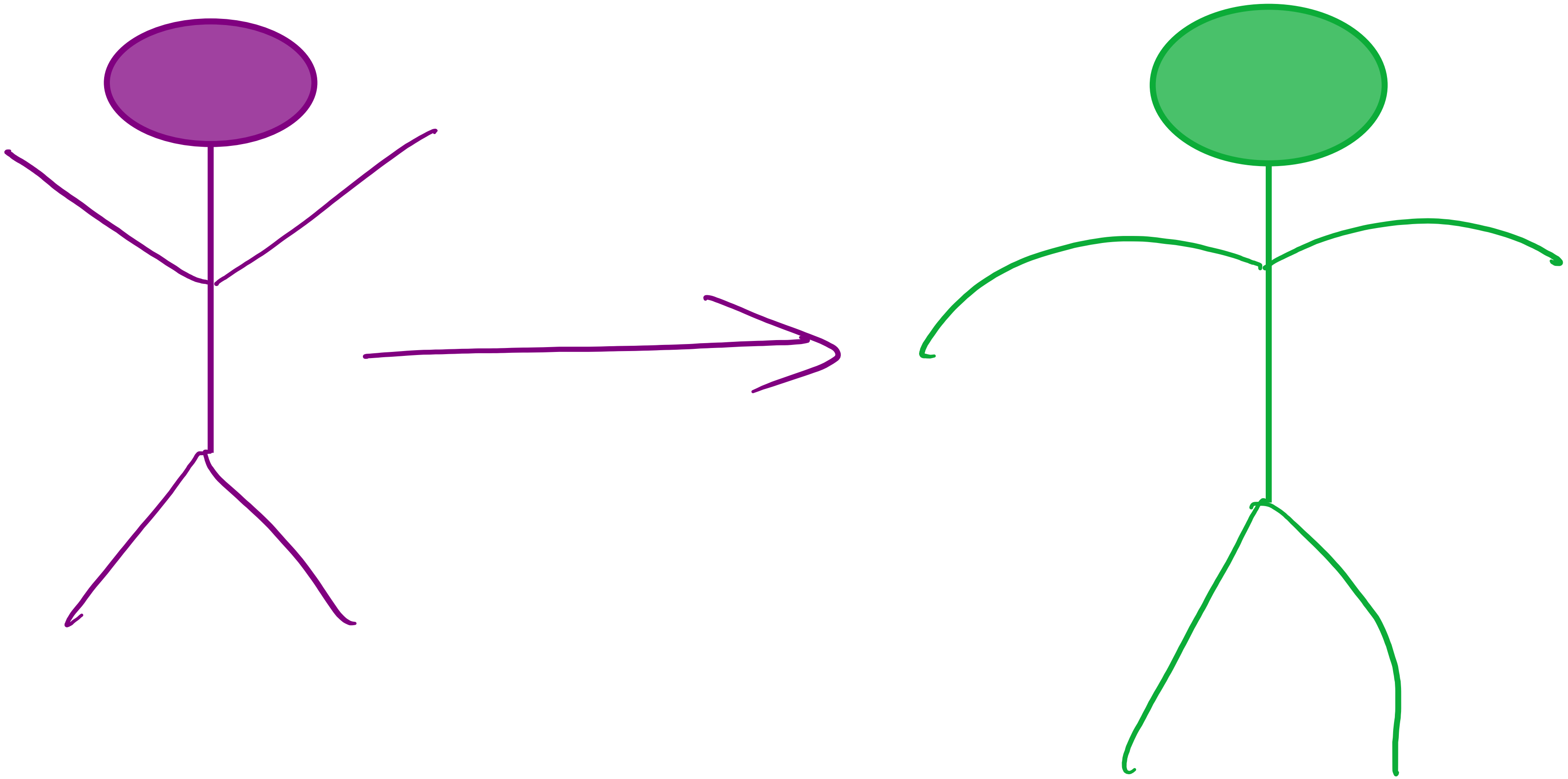
→ non-change in topology.

Multi-Shape

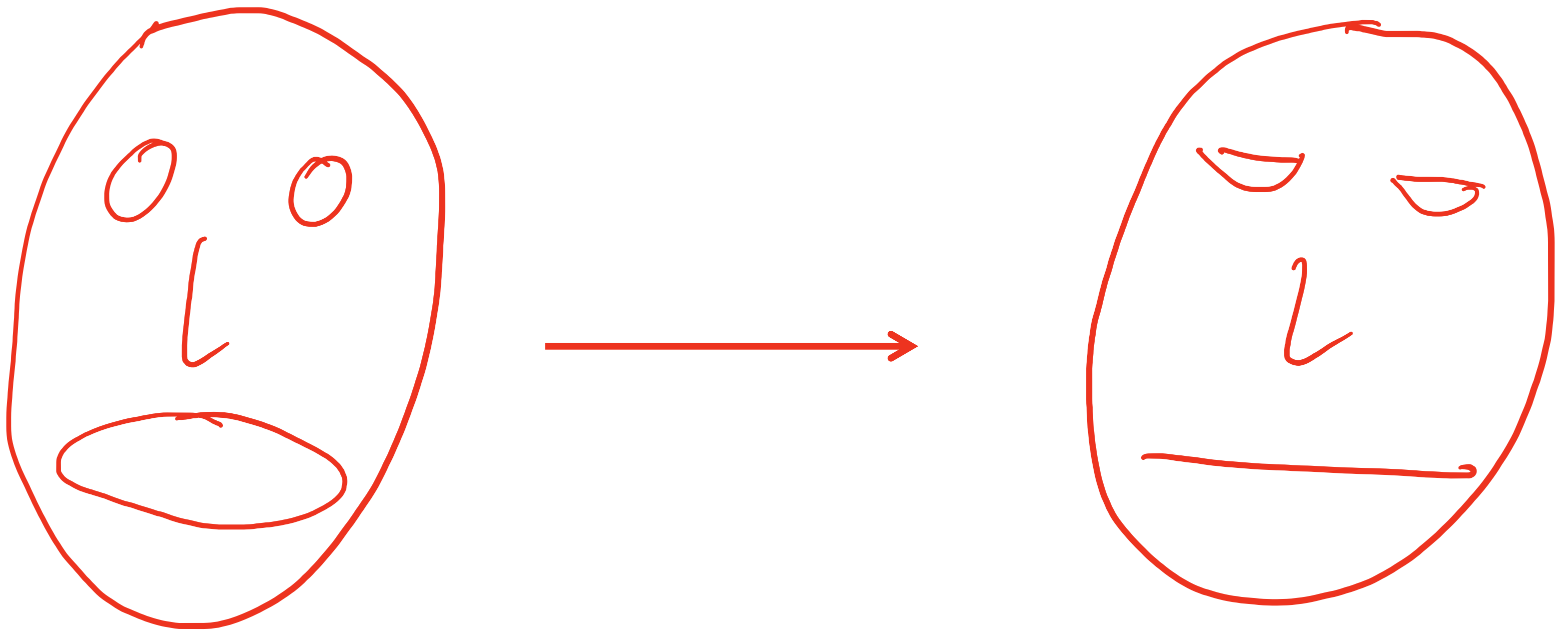




Shapes move easily relative to each other while deforming slightly.



Articulation



Topological changes

(Sub) Riemannian submersion

Notation and Setting

→ V: Hilbert space $\hookrightarrow C_0^p(\mathbb{R}^d, \mathbb{R}^d)$, $p \geq 1$

→ $\forall \varphi \in \mathrm{Diff}_0^p$: $\| \ \|_{V,\varphi}$ is a Hilbert norm on V satisfying $\|v\|_{V,\varphi} \geq c\|v\|_V$ for some $c > 0$

→ Denote $V_\varphi = \{v \circ \varphi,\ v \in V\}$; $\|v \circ \varphi\|_\varphi = \|v\|_{V,\varphi}$

→ Diff_0 : group of attainable diffeomorphisms, end point of paths $\varphi(t,\cdot)$ such that

$$\int_0^1 \|\partial_t \varphi(t,\cdot)\|^2 \, dt < \infty$$

$\rightarrow$ M : shape space. Diff_γ acting on M

$$(\varphi, q) \longmapsto \varphi \cdot q = \pi_q(\varphi)$$

$$(v, q) \longmapsto v \cdot q = \xi_q v = d\pi_q(id).v$$

(inf. action)

Assume that M is open in Q, a Banach space

→ Fix $q_0 \in \mathcal{M}$: the "template"

Let $\mathcal{M}_0 = \{ \pi_{q_0}(\varphi) \, , \, \varphi \in \mathrm{Diff}_0 \}$

→ $\varphi \in \mathrm{Diff}_0$: $H_\varphi = \mathrm{Null}(\xi_q)^{\perp_{V,\varphi}}$, $q = \pi_{q_0}(\varphi)$

$$\boxed{v \in H_\varphi \iff \left(\xi_q w = 0 \Rightarrow \langle v, w \rangle_{V,\varphi} = 0 \right)}$$

→ If $\pi_{q_0}(\varphi) = \pi_{q_0}(\psi) = q$, the condition

$$\boxed{\xi_q(\mathcal{J}(v)) = \xi_q v}$$

uniquely defines an isomorphism $\mathcal{J} : H_\varphi \to H_\psi$

($\mathcal{J}(v)$ is the $\perp$ projection of 0 on the space $\{\xi_q w = \xi_q v\}$ for the $\langle , \rangle_{r,\psi}$ dot product)

→ Assumption : $\mathcal{J}$ is an isometry between H_φ & H_ψ

→ Define

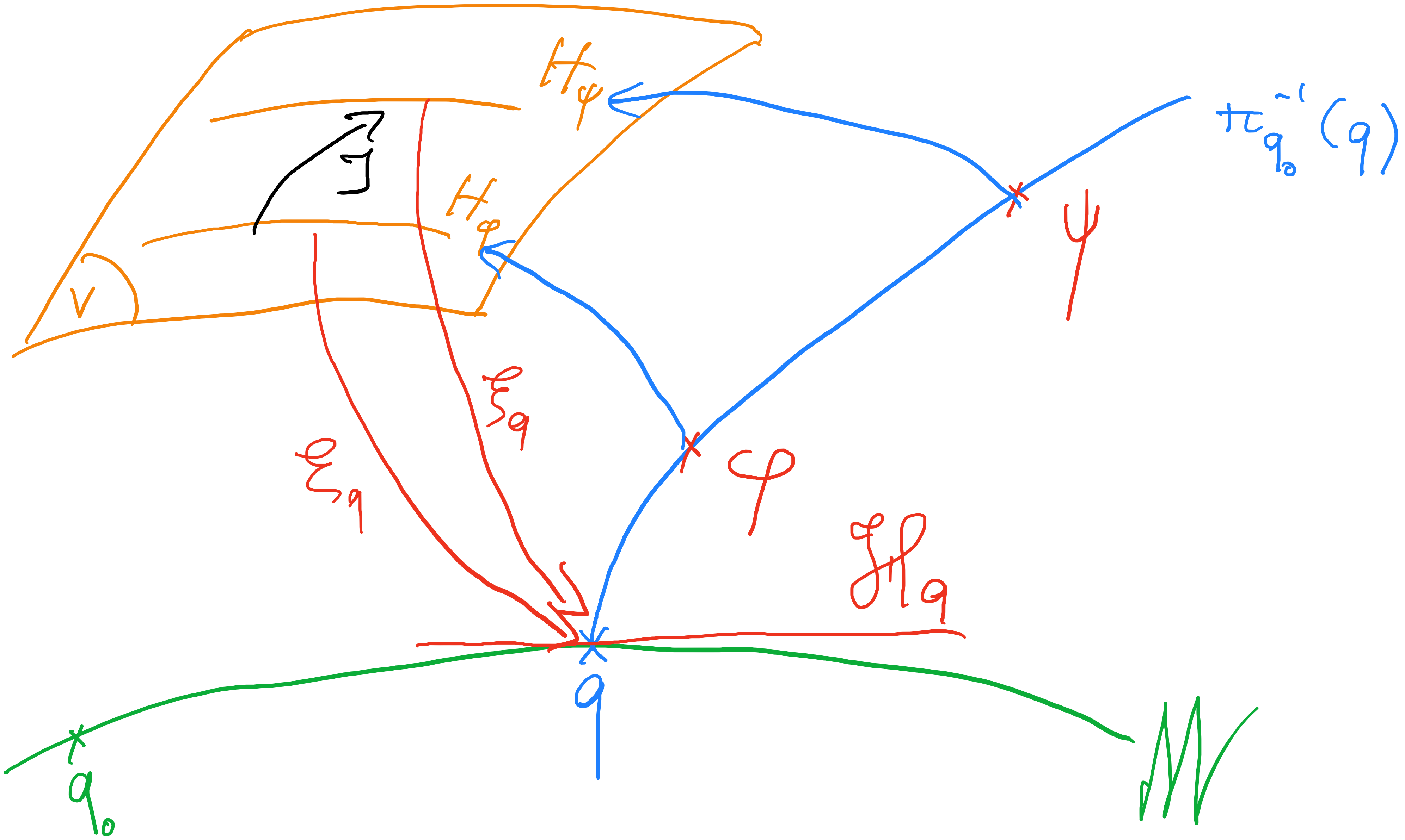
$$\mathcal{H}_q = \xi_q H_\varphi \quad \text{for} \quad \pi_{q_0}(\varphi) = q$$

$$= \{\xi_q v : v \in H_\varphi\}$$

$$\|\xi_q v\|_q = \|v\|_{V,\varphi} \;, \quad v \in H_\varphi$$

(independent of $\varphi \in \pi_{q_0}^{-1}(q)$ by assumption)

Diff



This provide a sub-Riemannian metric on the shape space $\mathcal{M}$ with distribution $\mathcal{H}_q$:

$$\|h\|_q = \|\xi_q v\|_q$$

with v such that $\xi_q v = h$, $v \in H_\varphi$, $\varphi \cdot q_0 = q$

Special case : LDDMM

$$\|\ \|_{r,\varphi} = \|\ \|_V$$

$$H_\varphi = H_V \quad , \quad \mathcal{J} = id$$

$$\|\xi_q v\|_q = \|v\|_V \quad , v \in H_\varphi$$

$$\varphi \cdot q_0 = q$$

Slightly less trivial

$$\| \quad \|_{V,\varphi} = \| \quad \|_{V,\psi} \qquad \varphi \cdot q_0 = \psi \cdot q_0$$

then $\quad H_\varphi = H_\psi \quad , \quad \underline{J = \mathrm{id}}$

All our examples today in this category

Running construction

$(q, h) \longmapsto G_q(h, h)$ pseudo-Riemannian metric on $\mathcal{M}$.

Let
$$\boxed{\begin{gathered} \|v\|^2_{V,\varphi} = \lambda \|v\|^2_V + G_q(\xi_q v, \xi_q v) \\ q = \pi_{q_0}(\varphi) \end{gathered}}$$

Denote also $\|v\|^2_q$

Two interpretations

(1) Enrich the LDDMM norm with shape-dependent (geometric) information

(2) Modify shape-space pseudo-norms to force geodesic to correspond to diffeomorphic transformations.

Invariance associated with the pseudo norm are partially inherited by the combined norm.

Important note

Easy to apply to products of shape spaces: → replace $\mathcal{M}$ by $\mathcal{M}^n$ with the product pseudo-Riemmanian metric

→ use action $\varphi \cdot (q_1, \dots, q_n) = (\varphi \cdot q_1, \dots, \varphi \cdot q_n)$

Applications to spaces of curves

→ A lot of pseudo-Riemannian metrics have be described and studied in the literature, notably by P. Michor's group in Vienna, or by Srivastava, Klassen, Mumford, Shah, etc.

→ Work with parametrized curves, represented as embeddings

$$m : \begin{array}{c}[0,1] \\ \text{or} \\ S^1\end{array} \longrightarrow \mathbb{R}^2$$

→ The shape spaces of interest are curves modulo parametrization.
Invariance is achieved by using parametrization-blind cost functions

Scientific Articles

[146] Martin Bauer, Martins Bruveris, Philipp Harms, Peter W. Michor: **Soliton solutions for the elastic metric on spaces of curves.** 22 pages. arxiv:1702.04344 (pdf).

[145] Martins Bruveris, Peter W. Michor: **Geometry of the Fisher-Rao metric on the space of smooth densities.** 13 pages. arxiv:1607.04550 (pdf).

[144] Martins Bruveris, Peter W. Michor, Adam Parusinski, Armin Rainer: **Moser's theorem on manifolds with corners.** 9 pages. arxiv:1604.07787 (pdf).

[143] Martin Bauer, Peter W. Michor, Olaf Müller: **Riemannian geometry of the space of volume preserving immersions.** *Differential Geometry and its Applications* **49** (December 2016), 23â€“42. doi:10.1016/j.difgeo.2016.07.002. arxiv:1603.05916. (pdf).

[142] Peter W. Michor: **Manifolds of mappings and shapes.** In the book: *The legacy of Bernhard Riemann after one hundred and fifty years*. Editors: Lizhen Ji, Frans Oort, Shing-Tung Yau. Series: Advanced Lectures of Mathematics 35, pp. 459â€“486. Higher Education Press and International Press Beijingâ€“Boston 2016. arXiv:1505.02359. (pdf).

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[137] Andreas Kriegl, Peter W. Michor, Armin Rainer: **An exotic zoo of diffeomorphism groups on $\mathbb{R}^n$.** *Ann. Glob. Anal. Geom.* **47**, 2 (2015), 179-222. doi:10.1007/s10455-014-9442-0. arXiv:1404.7033. (pdf).

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[134] Martin Bauer, Martins Bruveris, Peter W. Michor: **Overview of the Geometries of Shape Spaces and Diffeomorphism Groups.** *Journal of Mathematical Imaging and Vision*, **50**, 1-2, 60-97, 2014. doi:10.1007/s10851-013-0490-z. arXiv:1305.1150. (pdf).

[133] Giuseppe Marmo, Peter W. Michor, Yuri Neretin: **The Lagrangian Radon Transform and the Weil representation.** *Journal of Fourier Analysis and Applications* **20**, 2 (2014), 321-361. doi:10.1007/s00041-013-9315-0. arXiv:1212.4610. (pdf).

[132] Martin Bauer, Martins Bruveris, Peter W. Michor: **Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. II.** *Ann. Glob. Anal. Geom.* **44**, 4 (2013), 361-368. doi:10.1007/s10455-013-9370-4. arXiv:1211.7254. (pdf).

[131] Peter W. Michor and David Mumford: **A zoo of diffeomorphism groups on $\mathbb{R}^n$.** *Ann. Glob. Anal. Geom.* **44**, 4 (2013), 529-540. doi:10.1007/s10455-013-9380-2. arXiv:1211.5704. (pdf).

Hybrid LDDMM problem

Minimize $$\boxed{\int_0^1 \|v(t)\|^2_{q(t)}\,dt + D\left(q(1), q_{targ}\right)}$$

subject to

$$(0) = q_{templ} \qquad \partial_t q(t) = v(t) \cdot q(t)$$

Maximum Principle : assumptions

- $\mathcal{M} = \{ C^r$ embeddings from S^1 (or $[0,1]$) to $\mathbb{R}^2 \}$
- $V \hookrightarrow C^p_0(\mathbb{R}^d, \mathbb{R}^d)$, $p > r$
- $G_q(h,h) \leq C(q) \|h\|^2_{p,\infty}$, C continuous
- $q \longmapsto G_q(h,h)$ C^1 (from $\mathcal{M}$ to $\mathbb{R}$)
- $q \longmapsto D(q, q_{\text{targ}})$ $\underline{\hspace{6em}}$
- $Q = C^r(S_1, \mathbb{R}^2)$

Maximum Principle

With these assumptions, Pontryagin's maximum principle is true:

Let $\left[H_v(p,q) = (p \mid v \circ q) - \frac{1}{2}\|v\|_q^2 \right]$

Then, along optimal solution, there exists $p : [0,1] \to Q^*$ with

$$\begin{cases} \partial_r q = \partial_p H_v(p,q) \\ \partial_r p = -\partial_q H_v(p,q) \\ v = \operatorname{argmax} H_v(p,q) \end{cases}$$

Reduction

→ The PMP implies that $v = K\xi_q^* \alpha$ for some $\alpha \in Q^*$

→ Use α to reparametrize the problem.

→ Minimize $\frac{1}{2}\int_0^1 \|\alpha(t)\|^2_{q(t)}\,dt + D(q(1), q_1)$

with $\|\alpha\|_q^2 = \lambda(\alpha \mid K_q\alpha) + G_q(K_q\alpha, K_q\alpha)$

$\partial_t q = K_q \alpha \quad , \quad K_q = \xi_q K \xi_q^*$

H^1 norms for experiments

- h vector field along the curve.
- H^1 norm:

$$G_q(h,h) = \int_0^{\ell(q)} |\partial_s h|^2 \, ds$$

$$\ell(q) = \text{length}(q)$$

→ Rescaled H':

$$G_q(h,h) = \frac{1}{\ell(q)} \int_0^{\ell(q)} |\partial_s h|^2 \, ds$$

→ rotation corrected H'

$$G_q(h,h) = \int_0^{\ell(q)} |\partial_s h|^2 \, ds - \frac{1}{\ell(q)} \left(\int_0^{\ell(q)} (\partial_s h)^T N_q \, ds \right)^2$$

N_q : unit normal to q.

→ Rotation and scale corrected rescaled H^1

$$\frac{1}{\ell(q)}\int_0^{\ell(q)} |\partial_s h|^2\, ds - \left(\frac{1}{\ell(q)}\int_0^{\ell(q)} (\partial_s h)^T T_q\, ds\right)^2$$

$$- \left(\frac{1}{\ell(q)}\int_0^{\ell(q)} (\partial_s h)^T N_q\, ds\right)^2$$

T_q : unit tangent.

Choosing V and its norm

→ Start with positive definite kernel K:

$$K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow M_d(\mathbb{R})$$

→ Let V be the Hilbert completion of

$$\sum_{i=1}^{n} K(\cdot, x_i)\alpha_i \quad , \quad x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^d$$

→ Gives a space of smooth functions under some assumptions on K

Examples

→ Gaussian kernel :

$$K(x,y) = e^{-\frac{|x-y|^2}{2a^2}} \, \mathrm{Id}$$

identity matrix

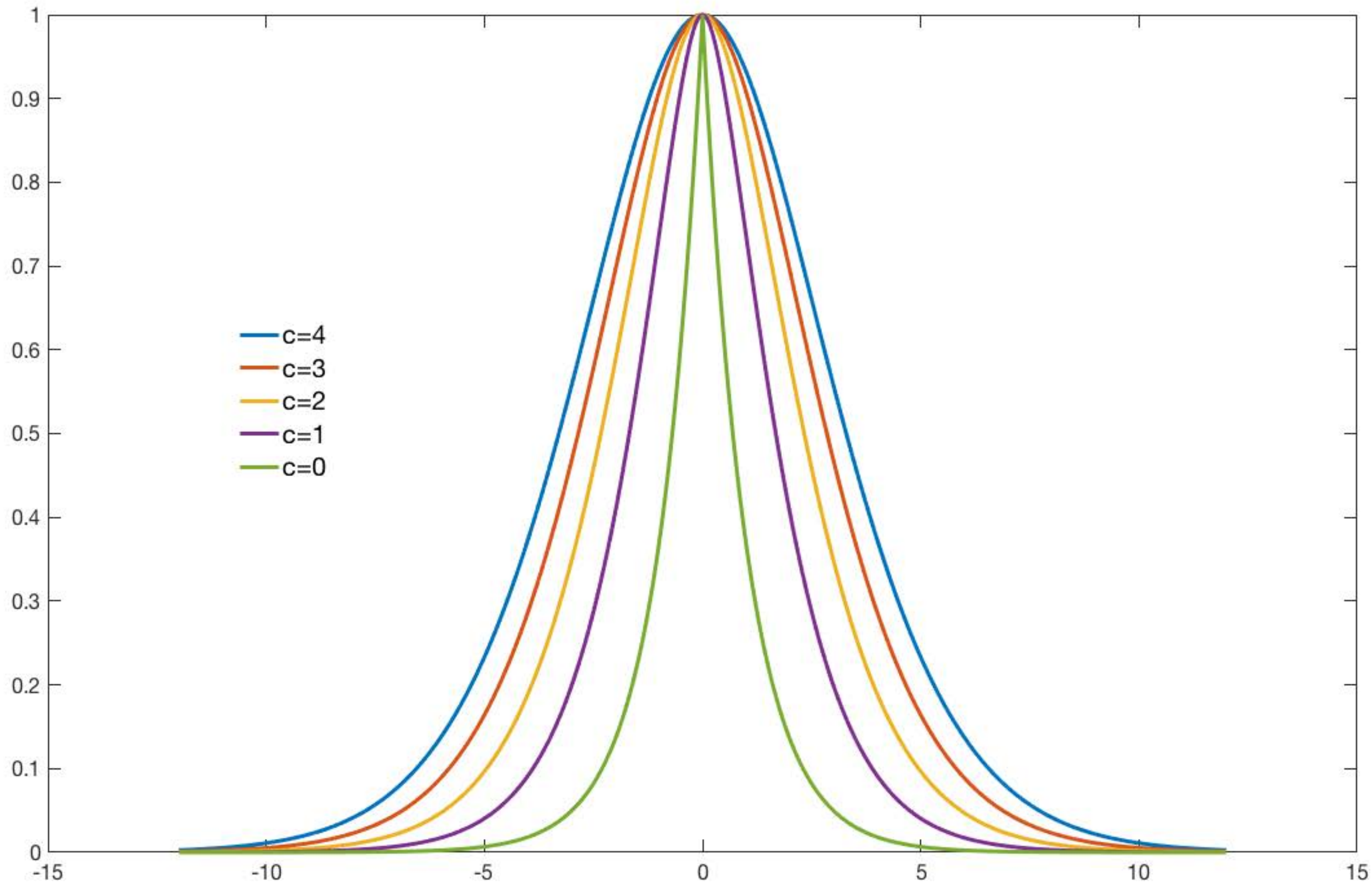
→ Laplacian / Abel kernel

$$K(x,y) = P_c\left(\frac{|x|}{a}\right) e^{-\frac{|x|}{a}}$$

where P_c is a reverse Bessel polynomial degree c.

Equivalent to Sobolev $H^{\frac{d+1}{2}+r}$

in odd dimension



c=4
c=3
c=2
c=1
c=0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-15
-10
-5
0
5
10
15

Cost function

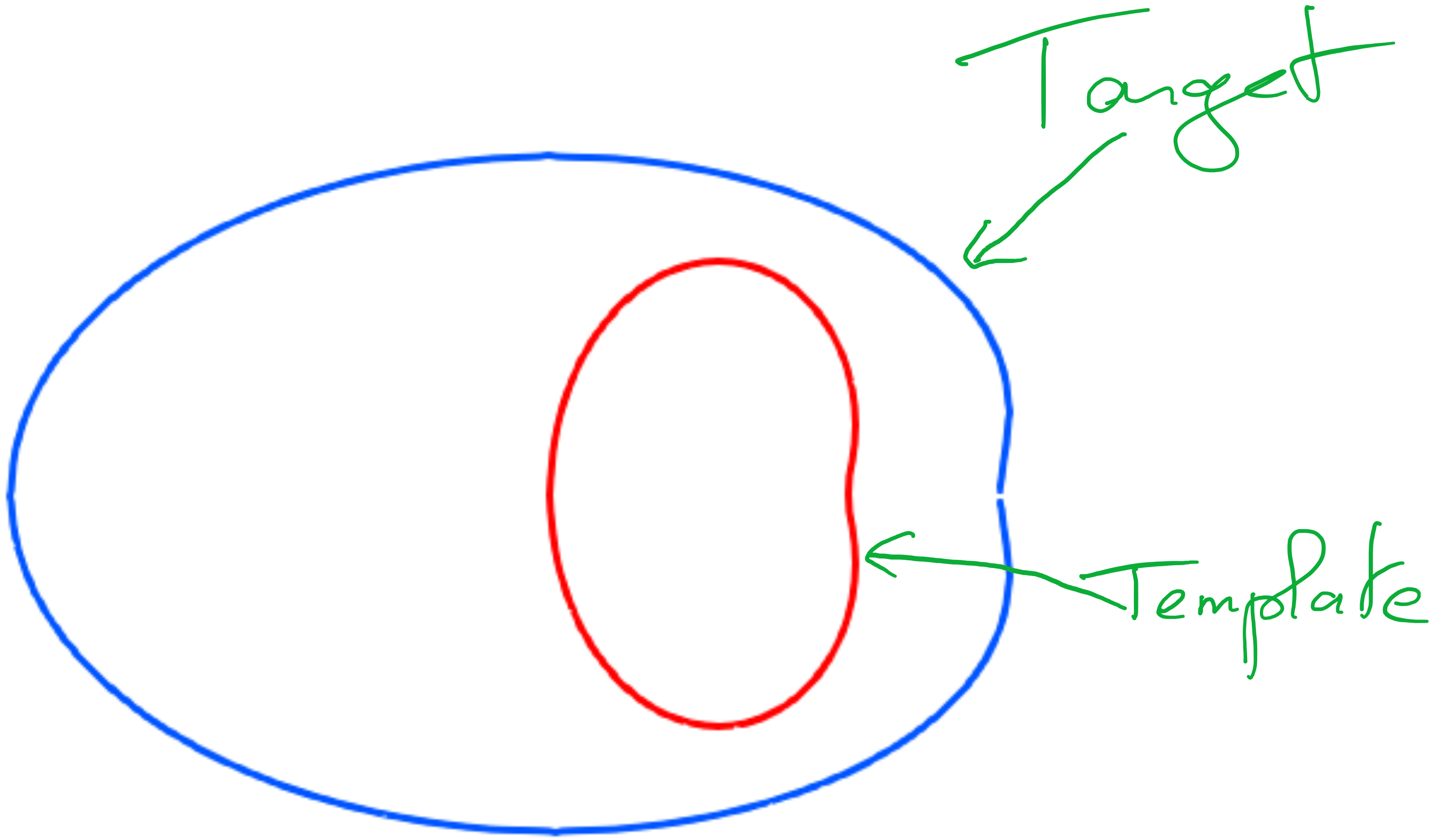
→ We used a version of the varifold norm introduced by Trouvé and Charon:

$$D(q,q_1) = \|q\|_X^2 - 2\langle q, q_1\rangle_X + \|q_1\|_X^2$$

with

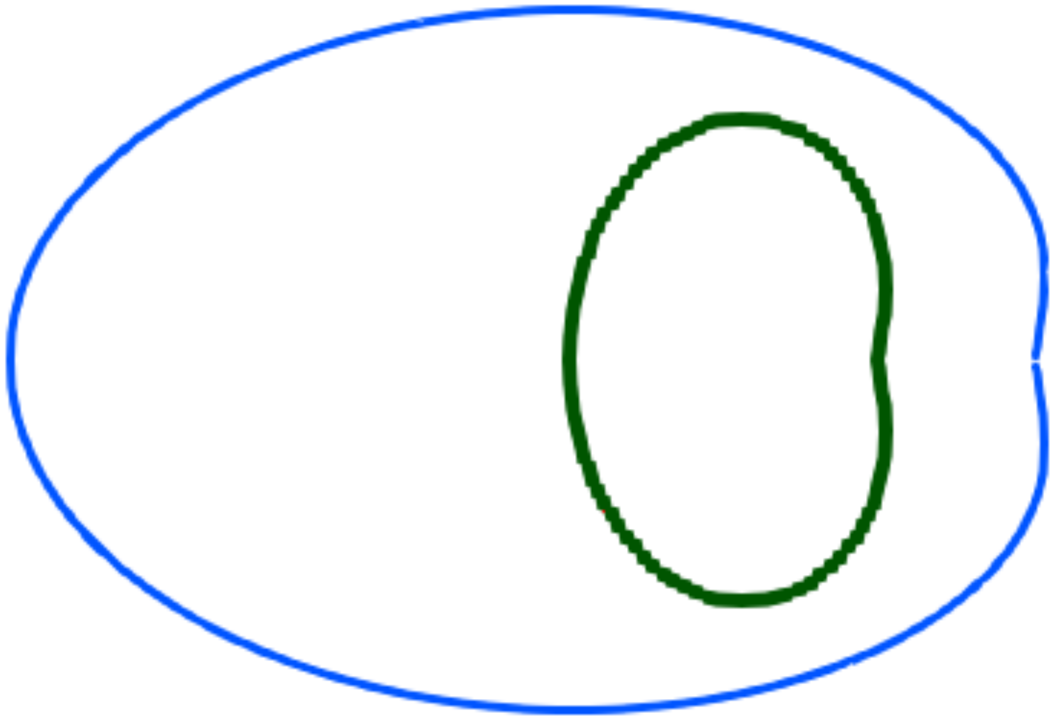
$$\langle q, q_1\rangle_X = \int_{S^1}\int_{S^1} X(q(u), q_1(u_1)) \left(1 + c\left(N(u)^T N_1(u_1)\right)^2\right) |q'(u)|\, |q_1'(u_1)|\, du\, du_1$$

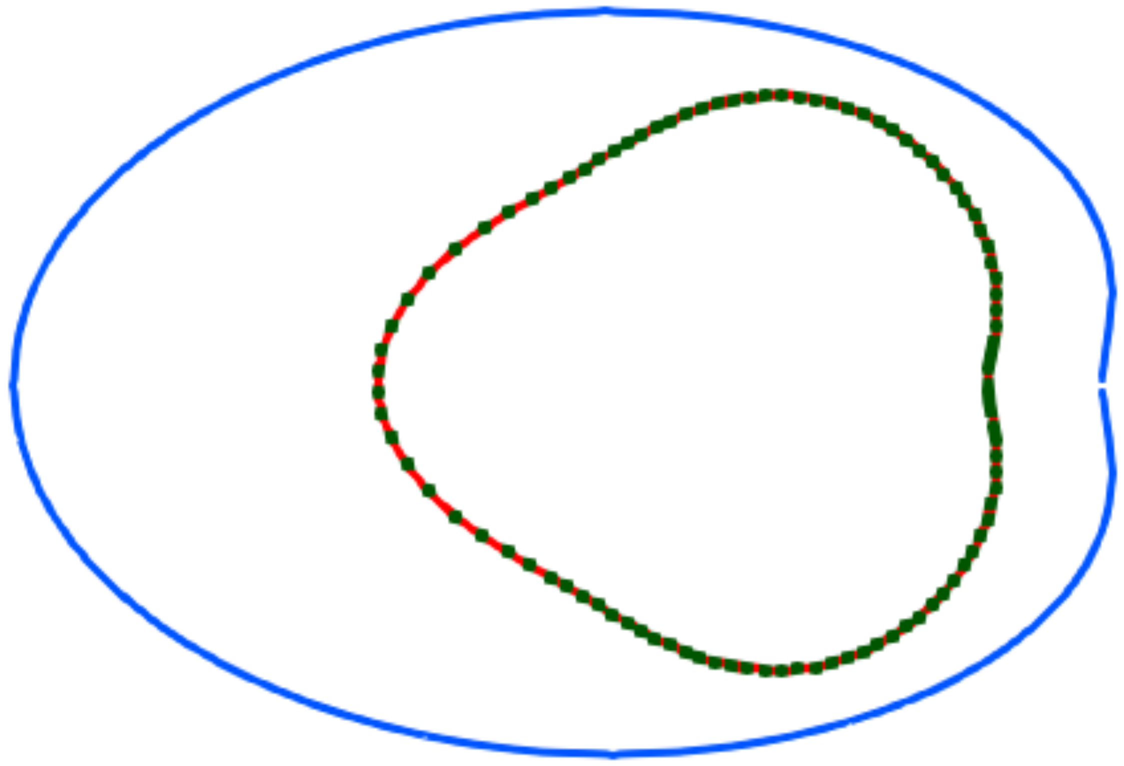
EXAMPLES

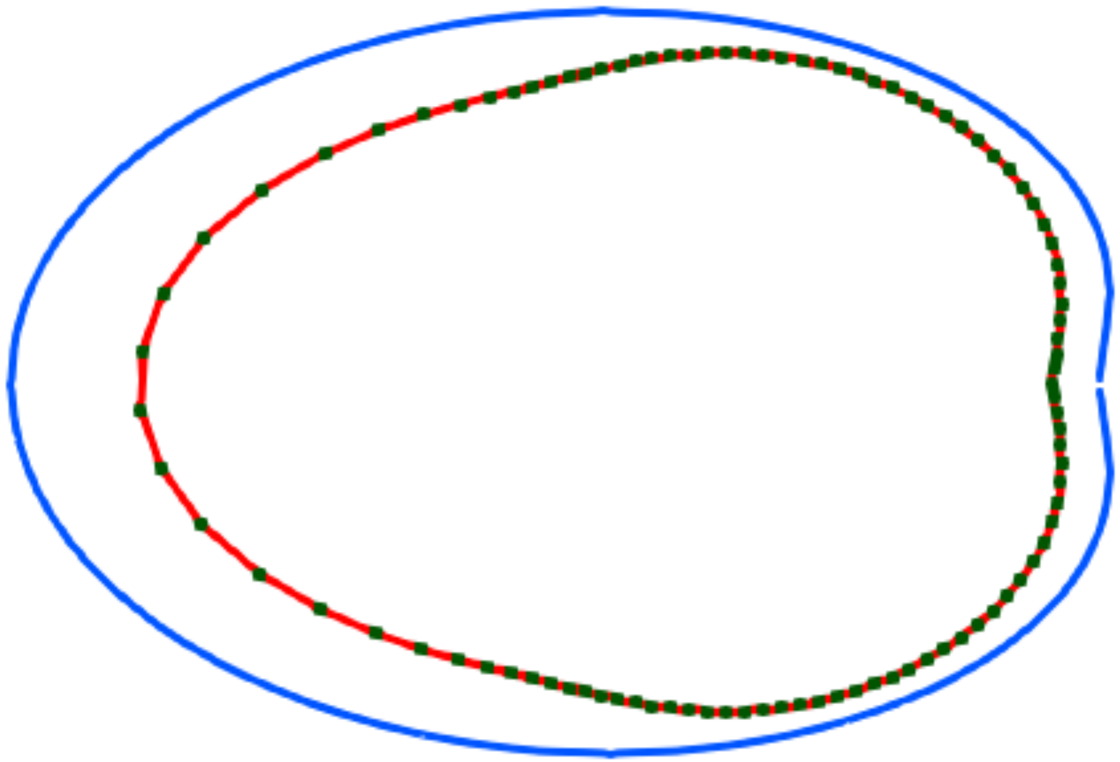


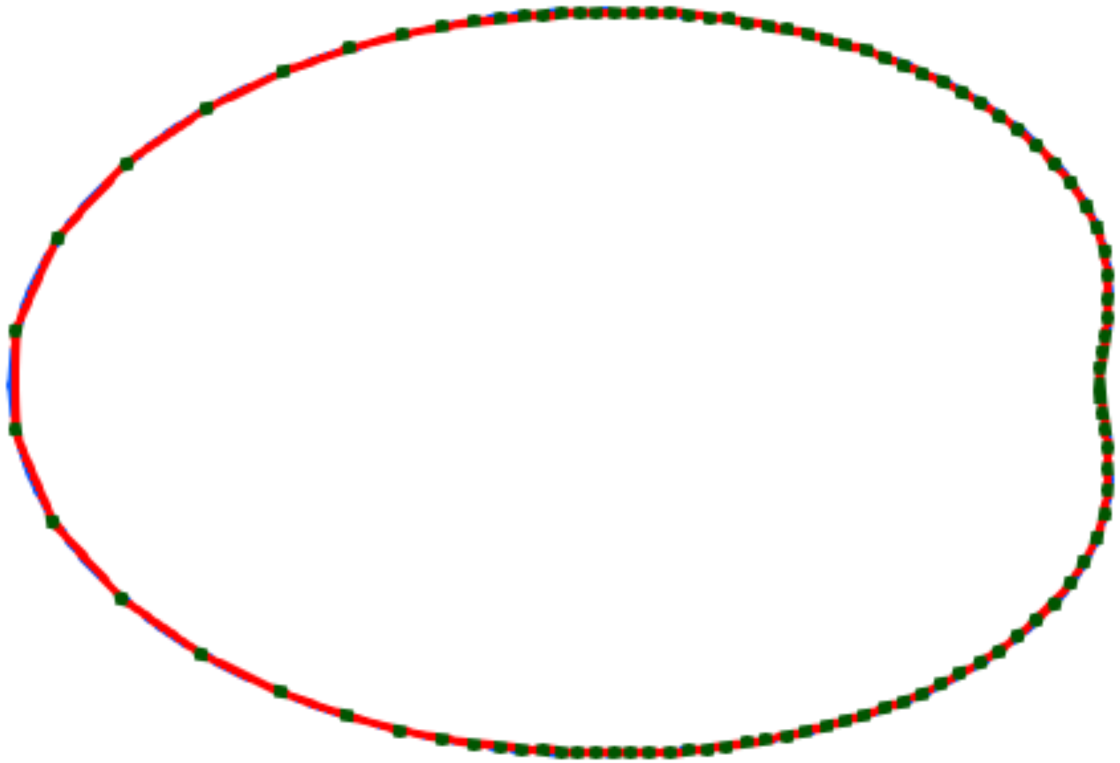
Target
Template

LDDMM

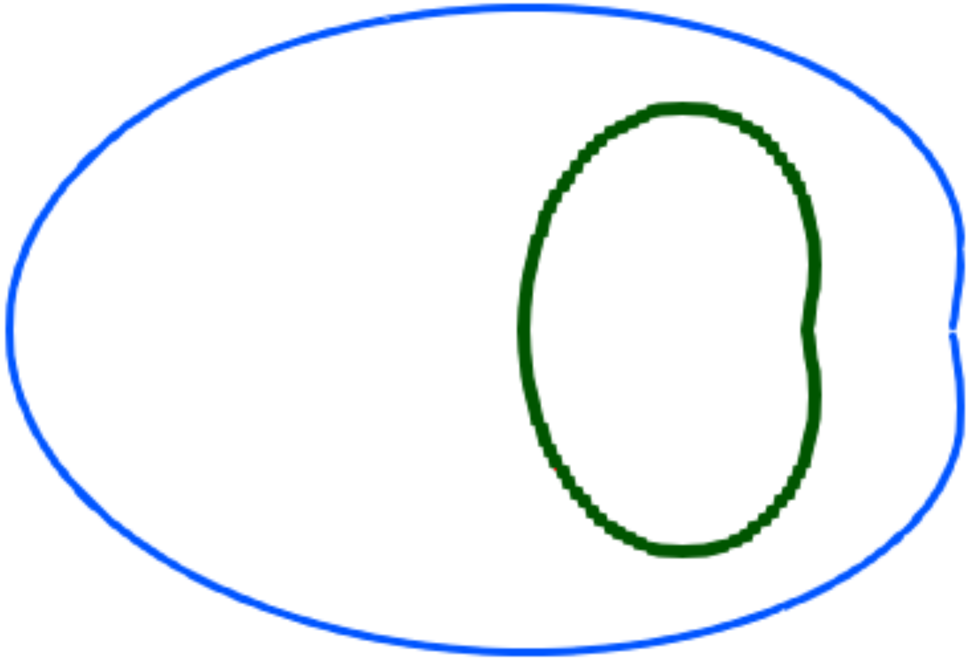


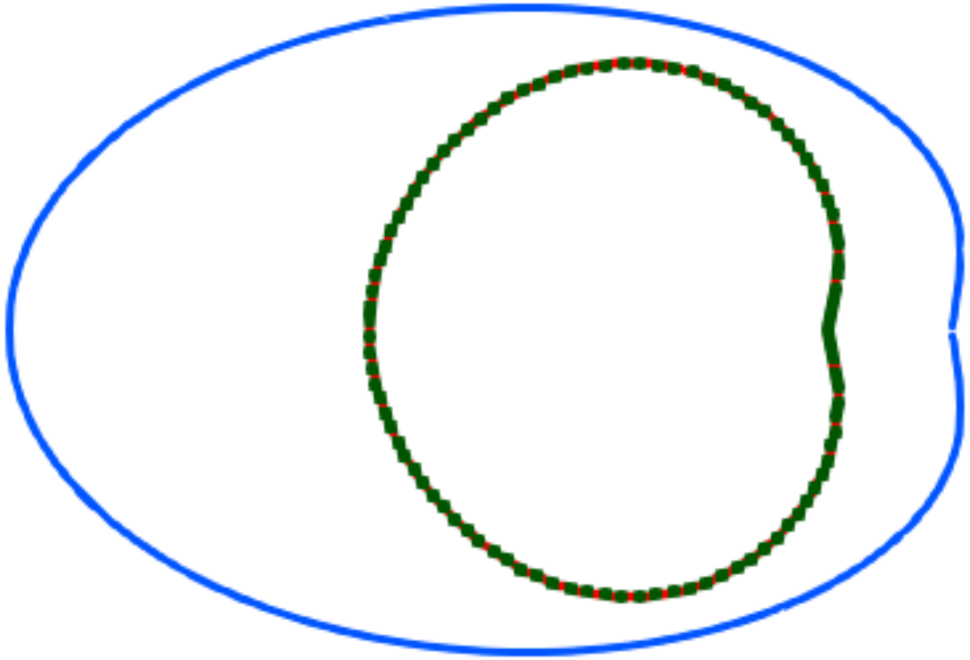


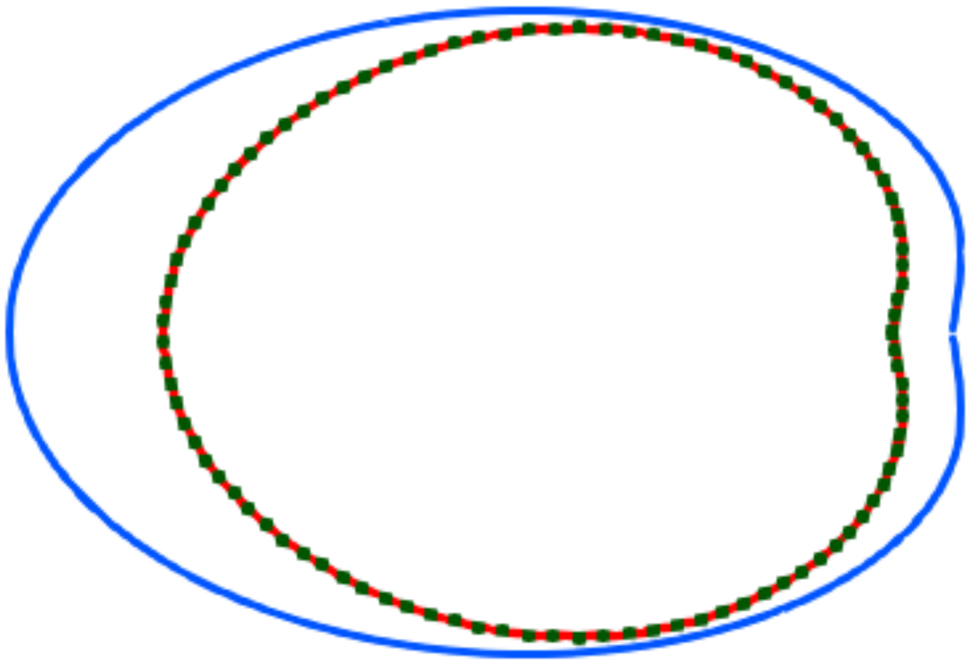


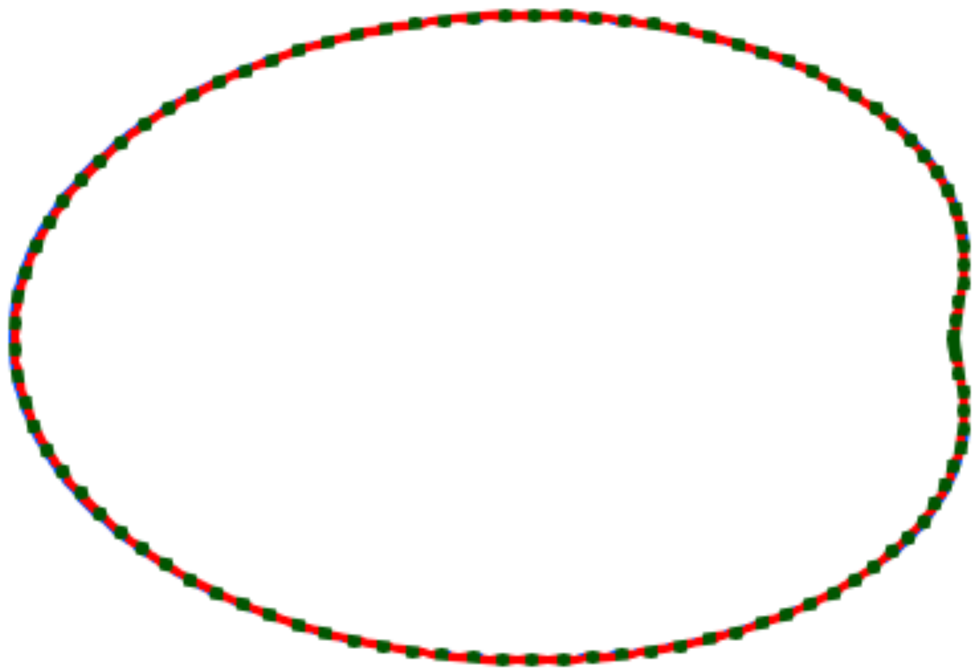


H_LDDMM with H'_invariant norm

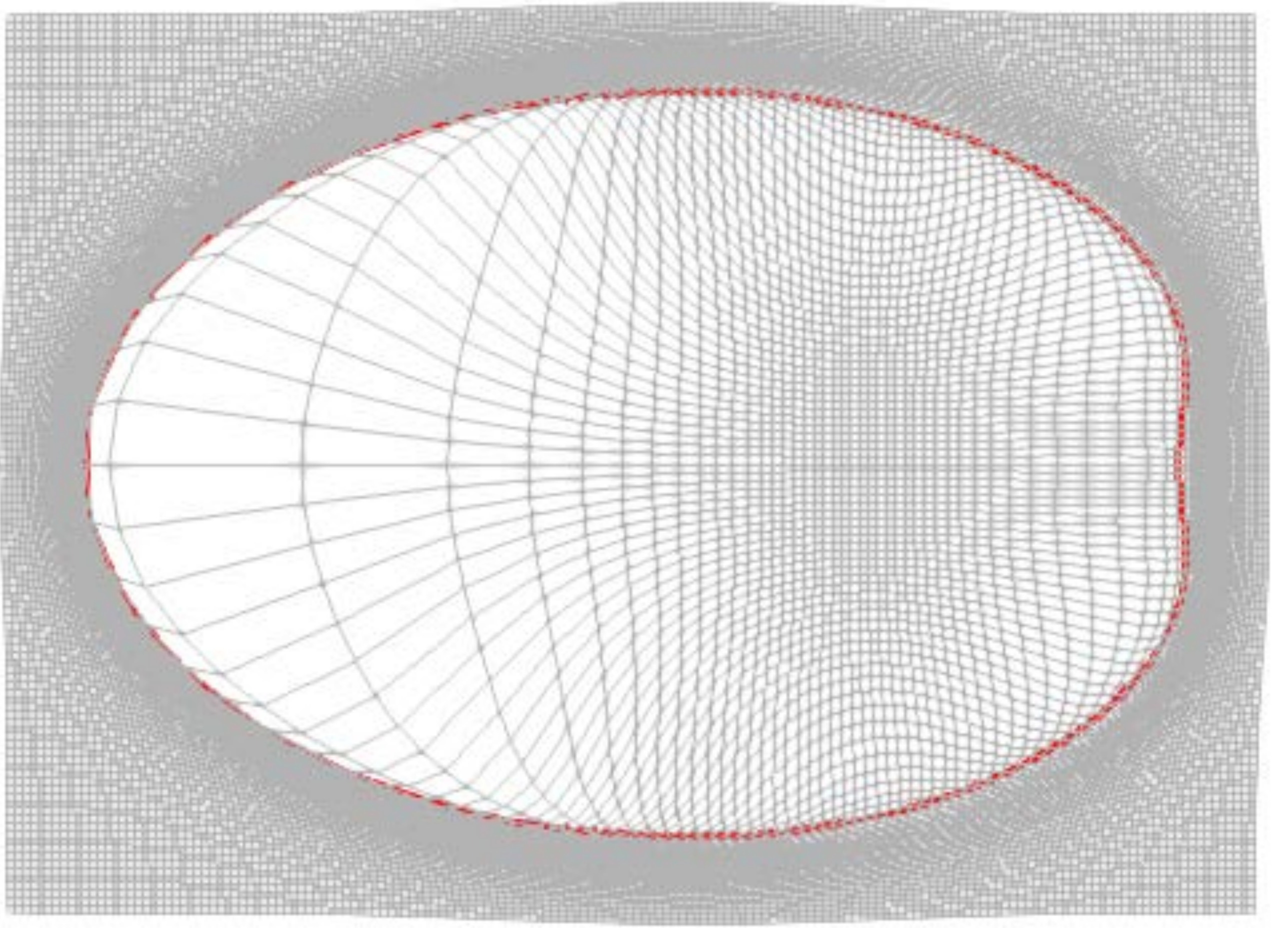




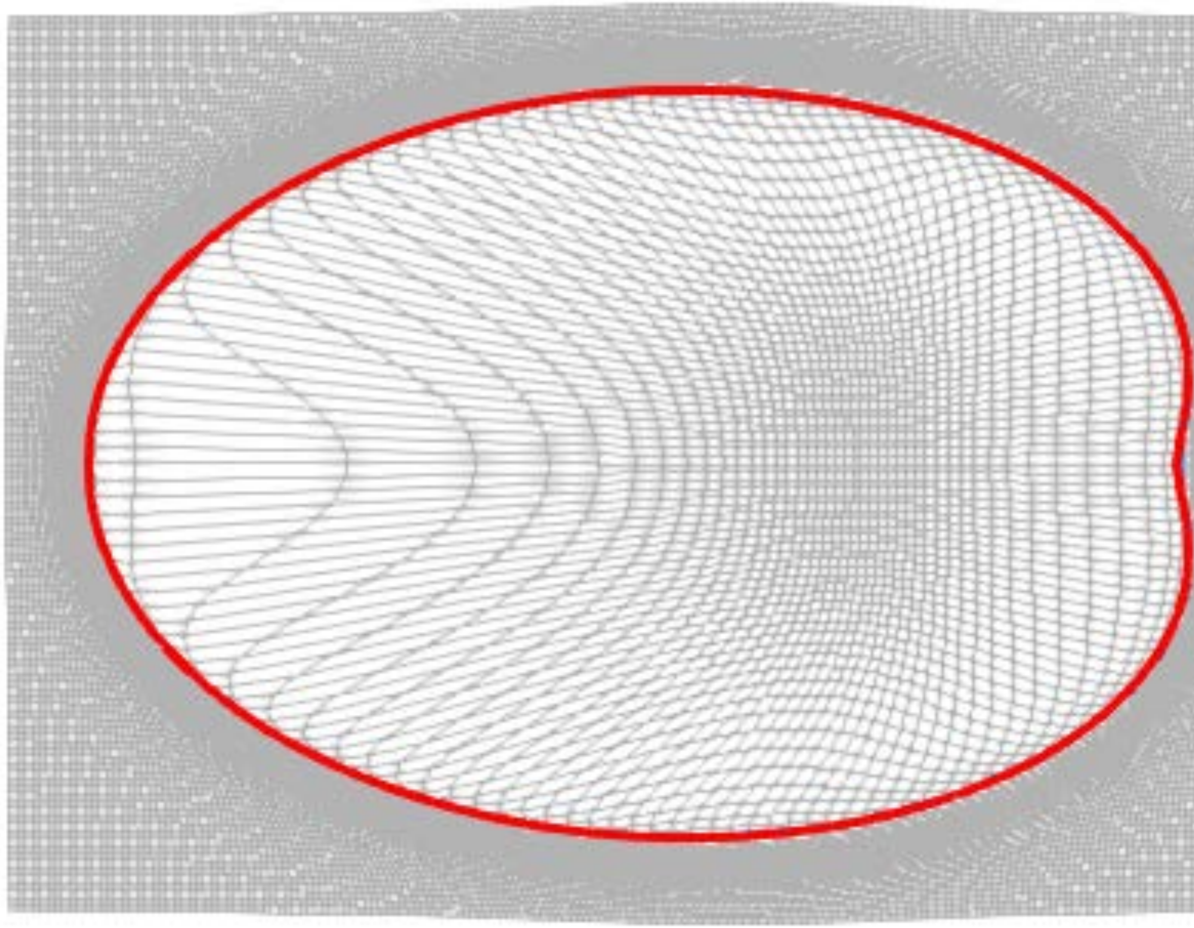




Estimated transformations

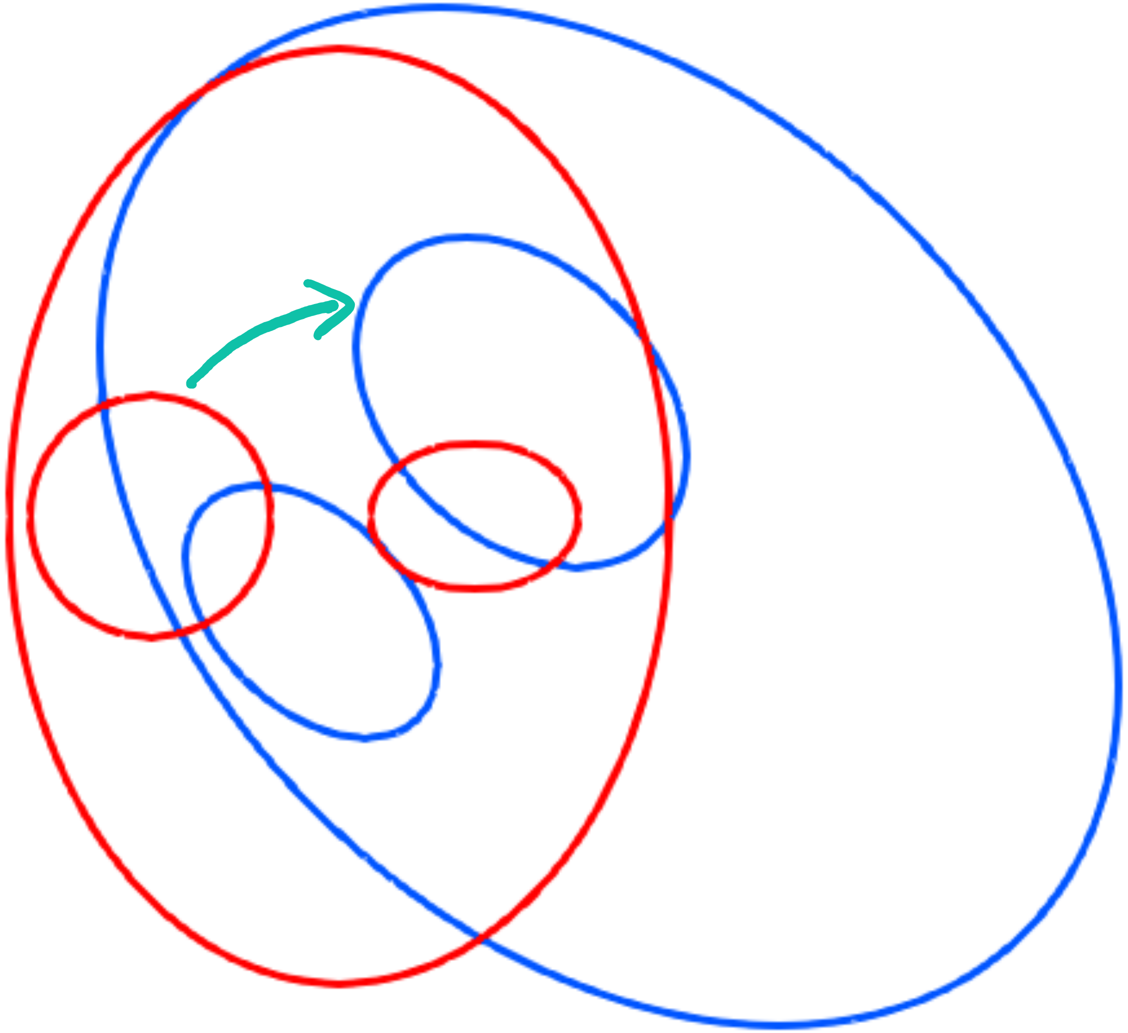


LDDMM



H-LDDMM

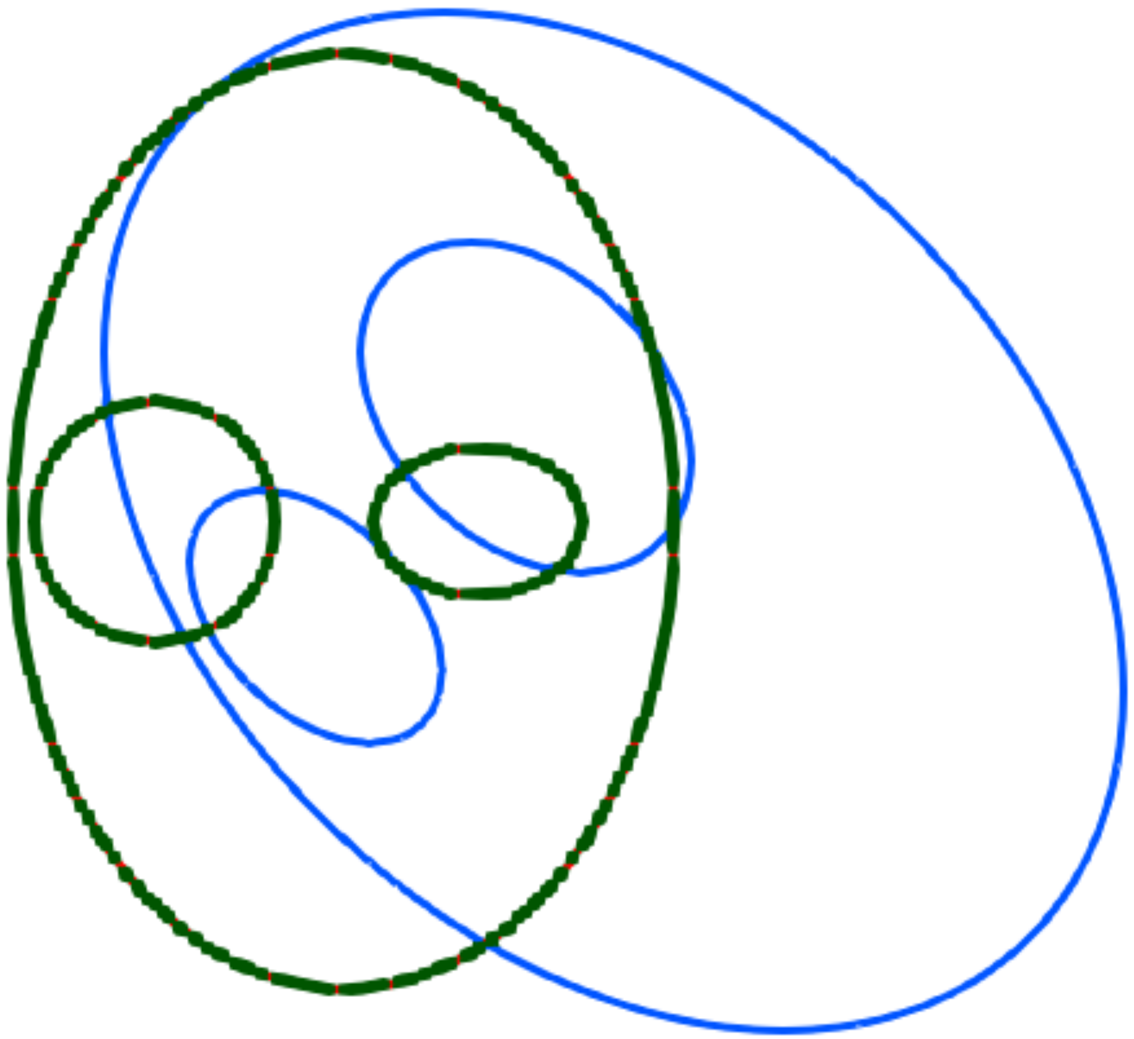
ELLIPSES...

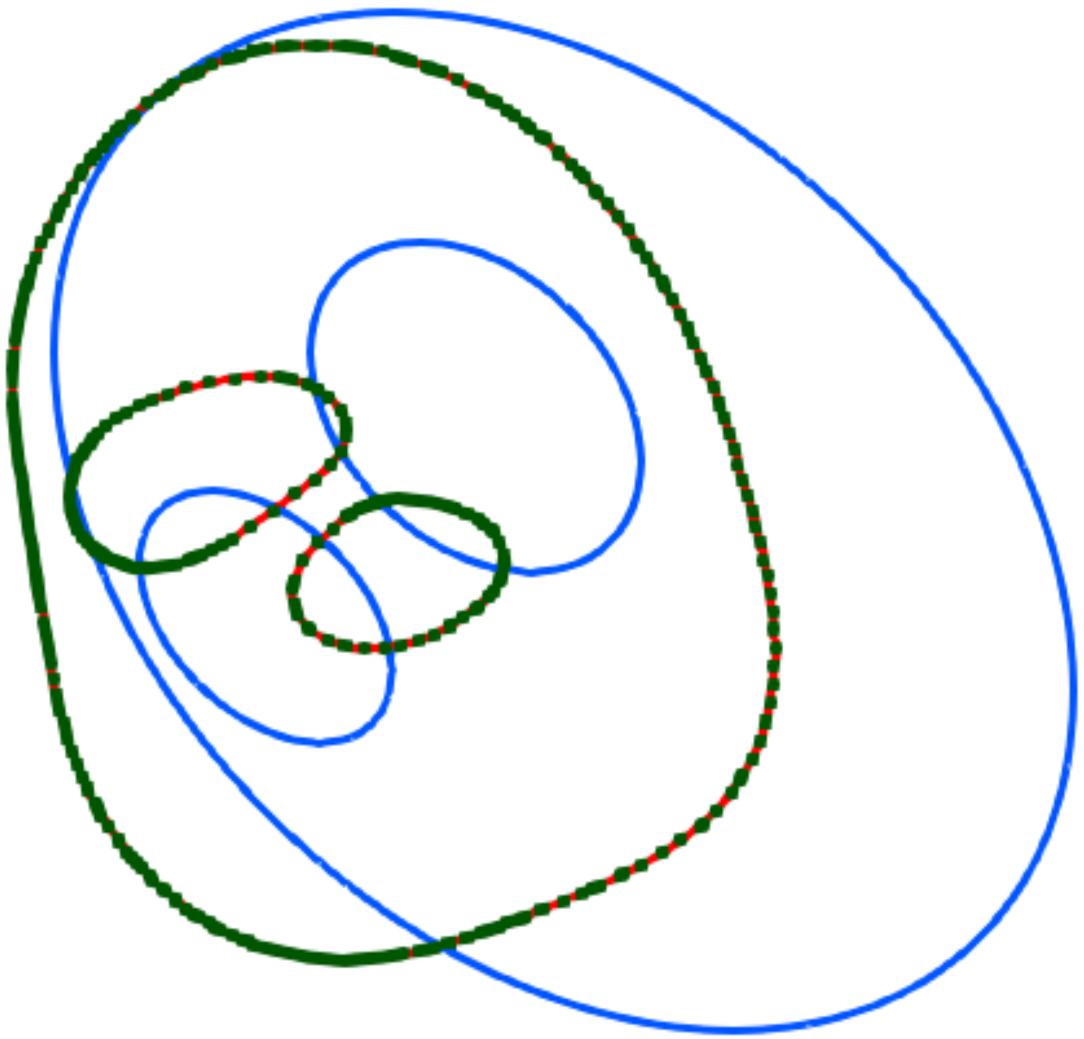


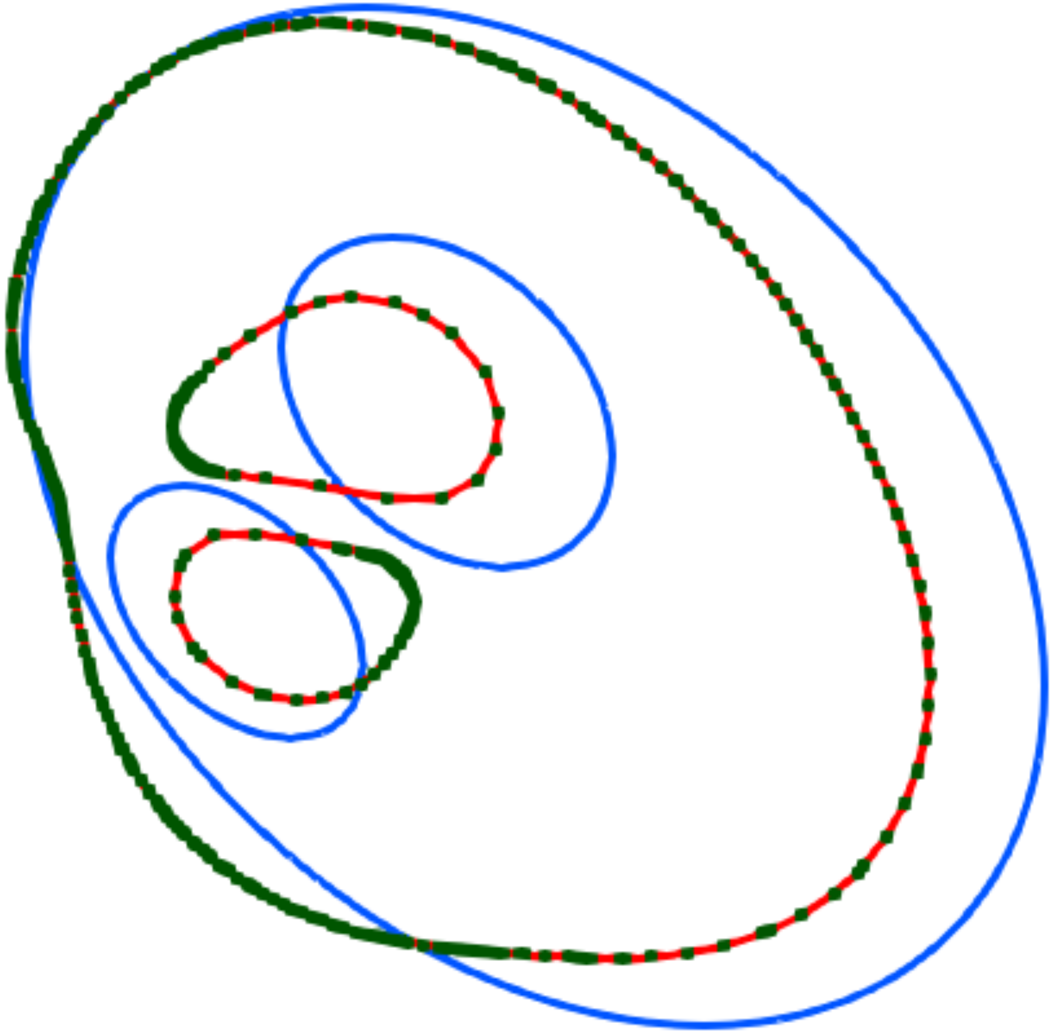
<u>Remark</u>: For this multiple-curve example only: the cost function ensures that homologous curves are mapped on each other (i.e., the curves are labeled).

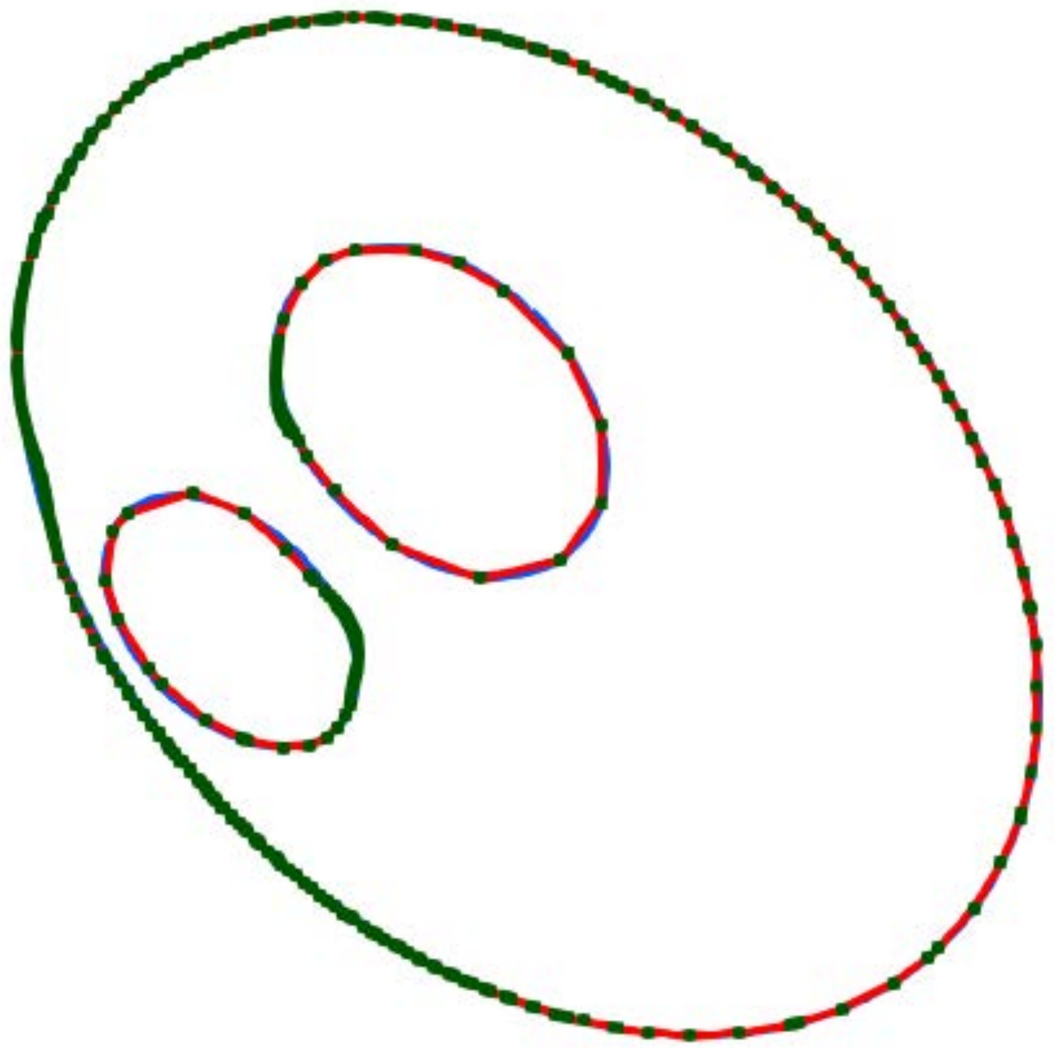
In all subsequent examples, the cost function is blind to curve relabelling.

LDDMM



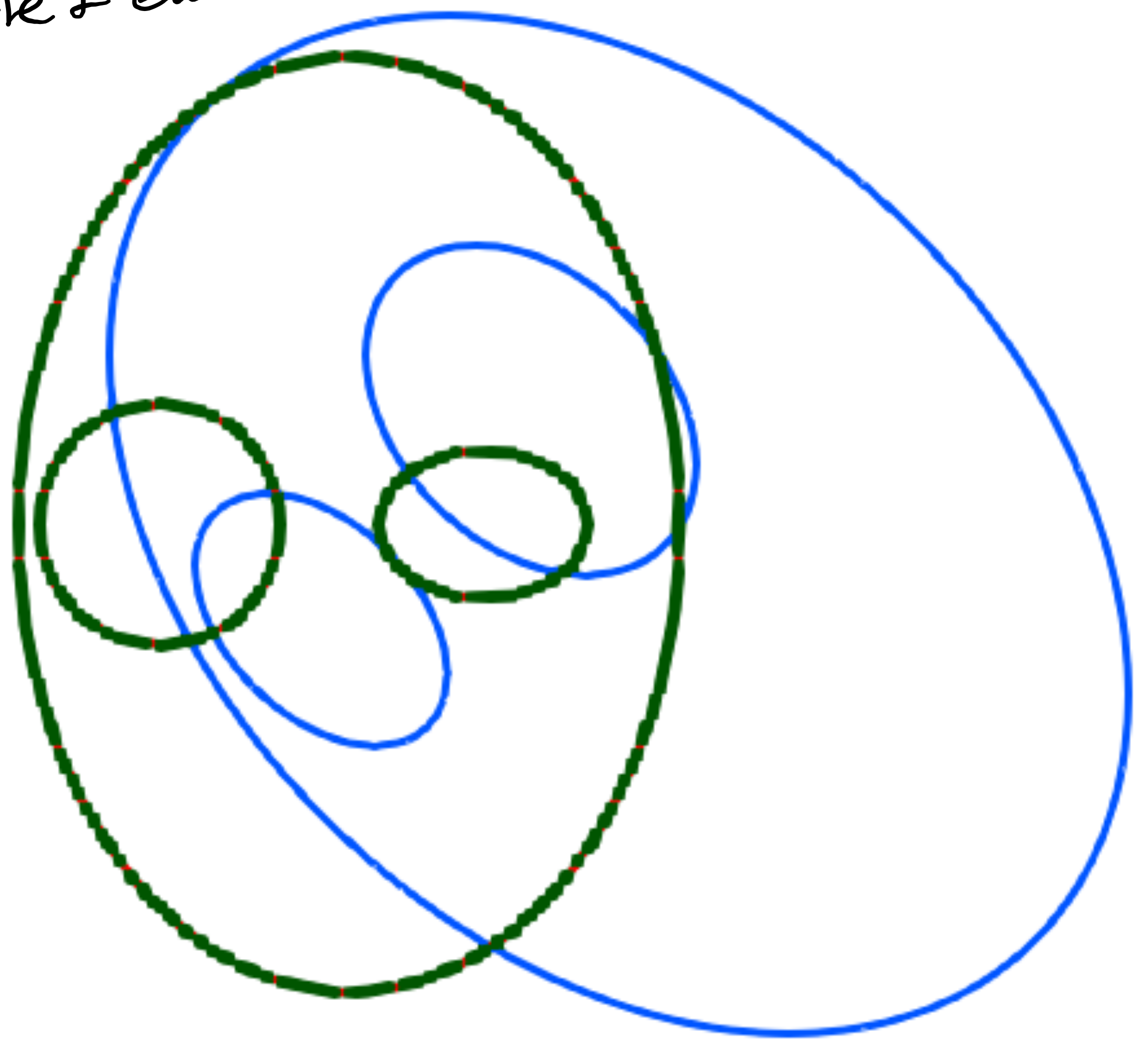


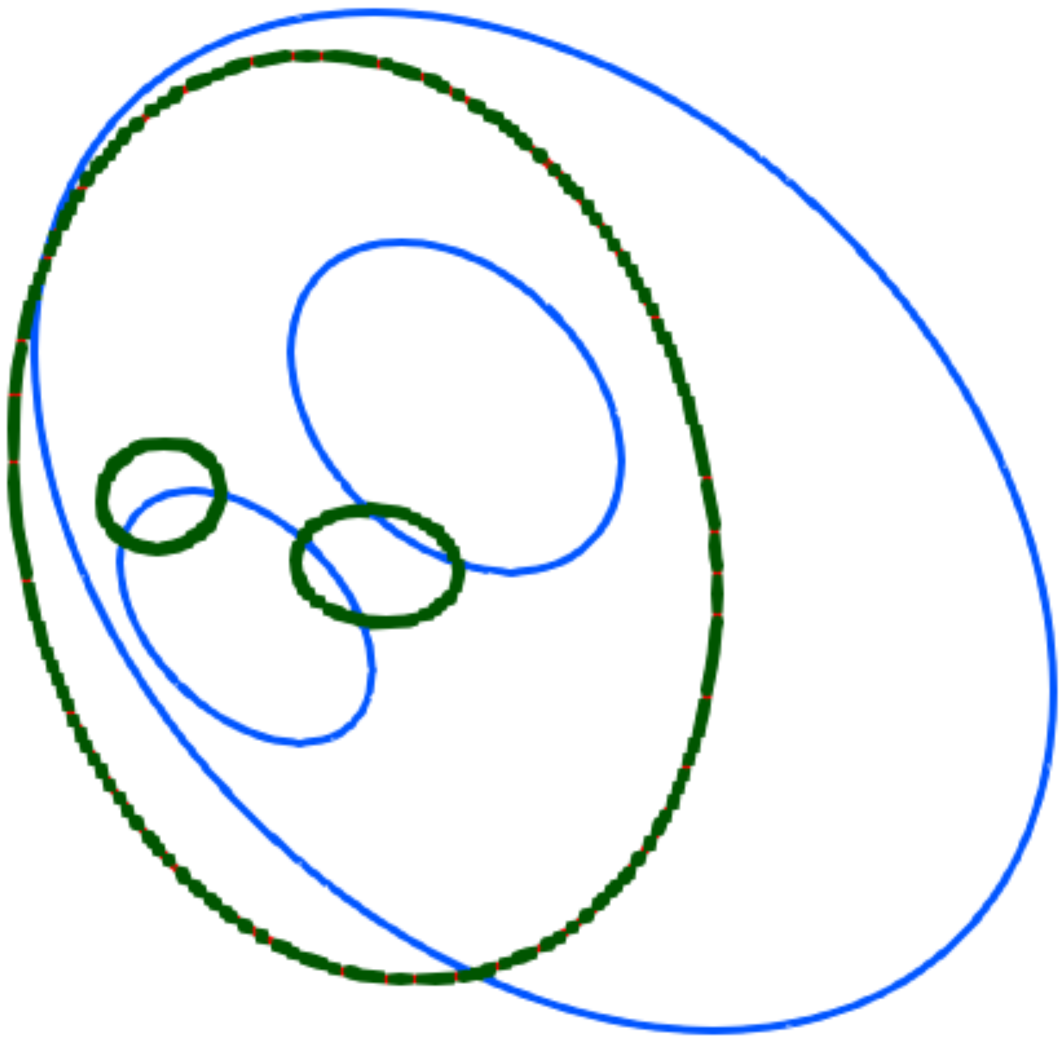


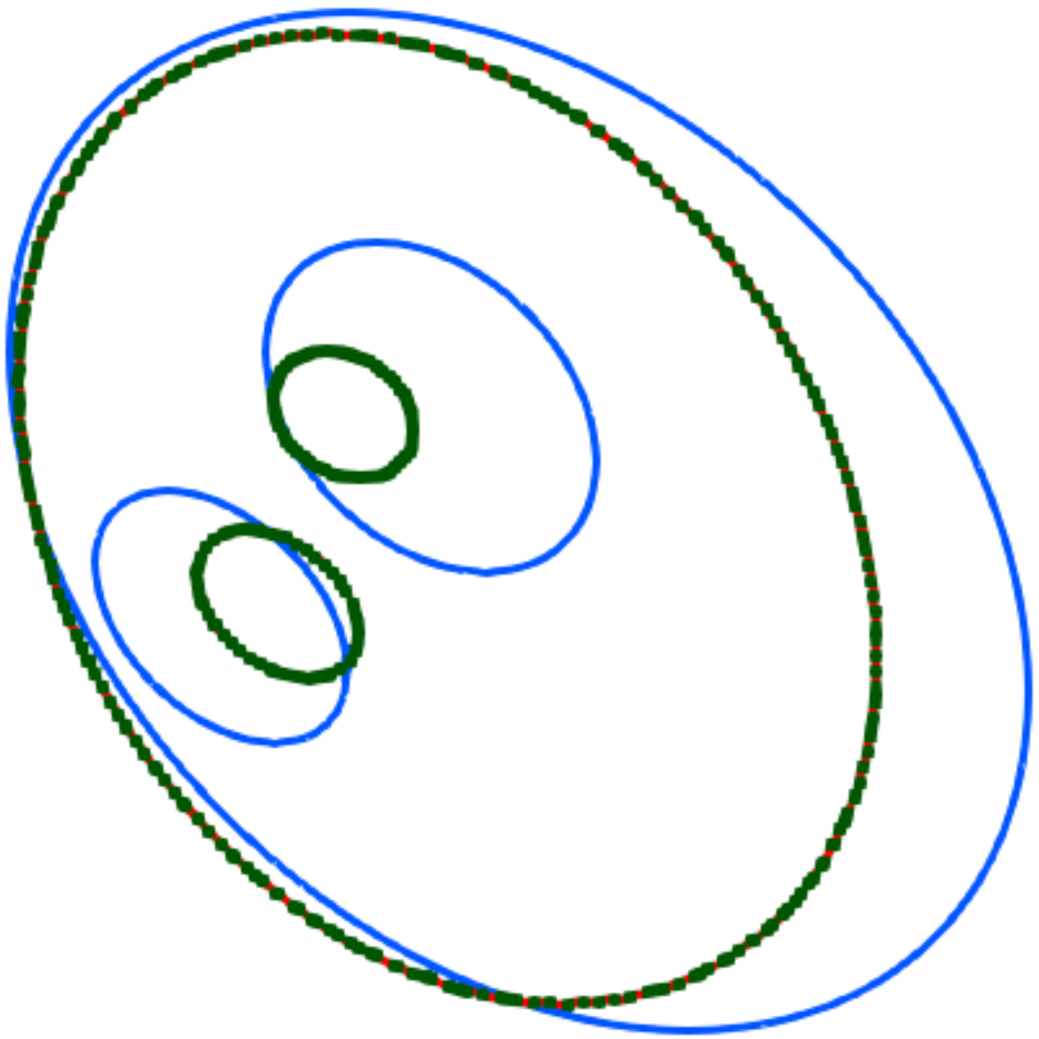


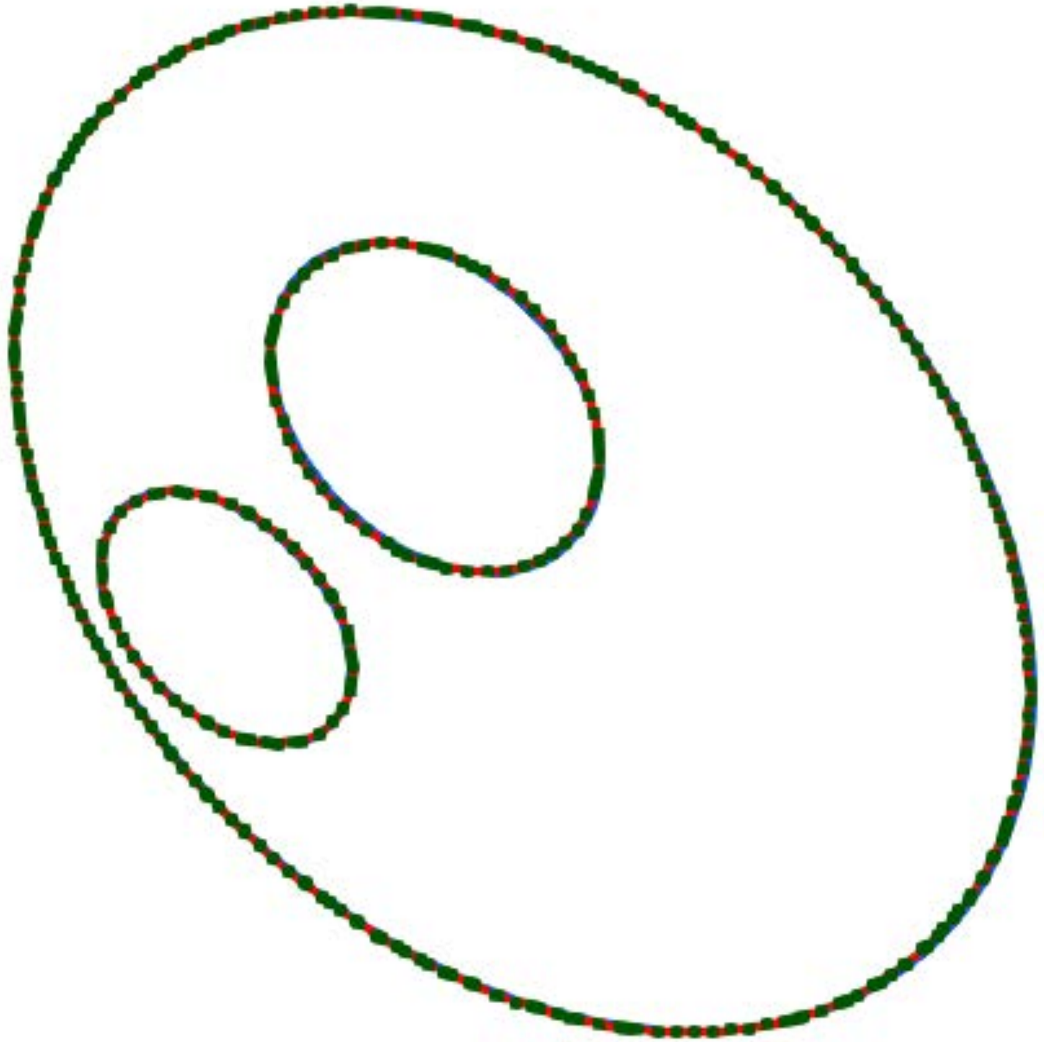
H_ LDDMM

H'_ invariant (scale + euclidean)



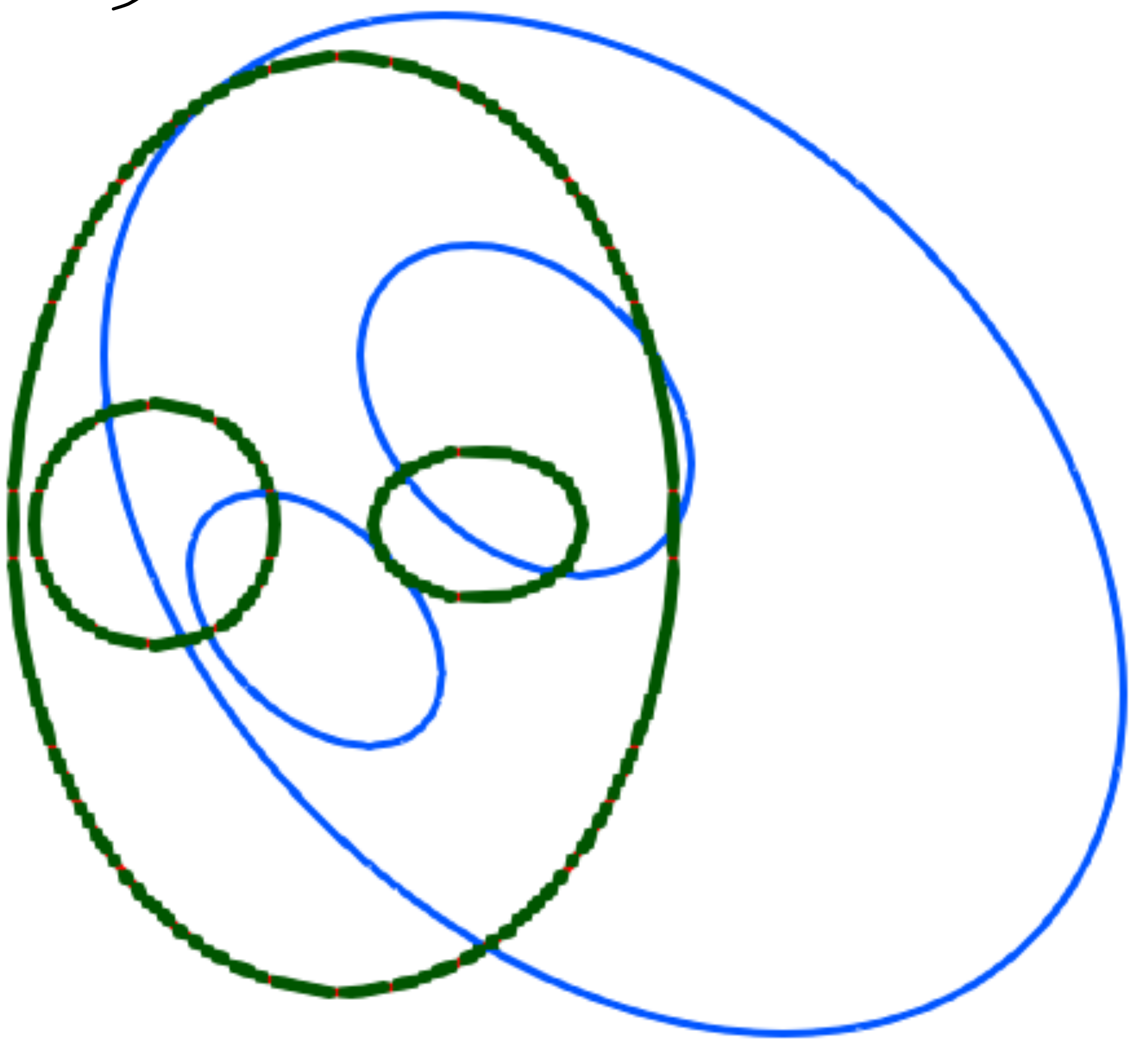


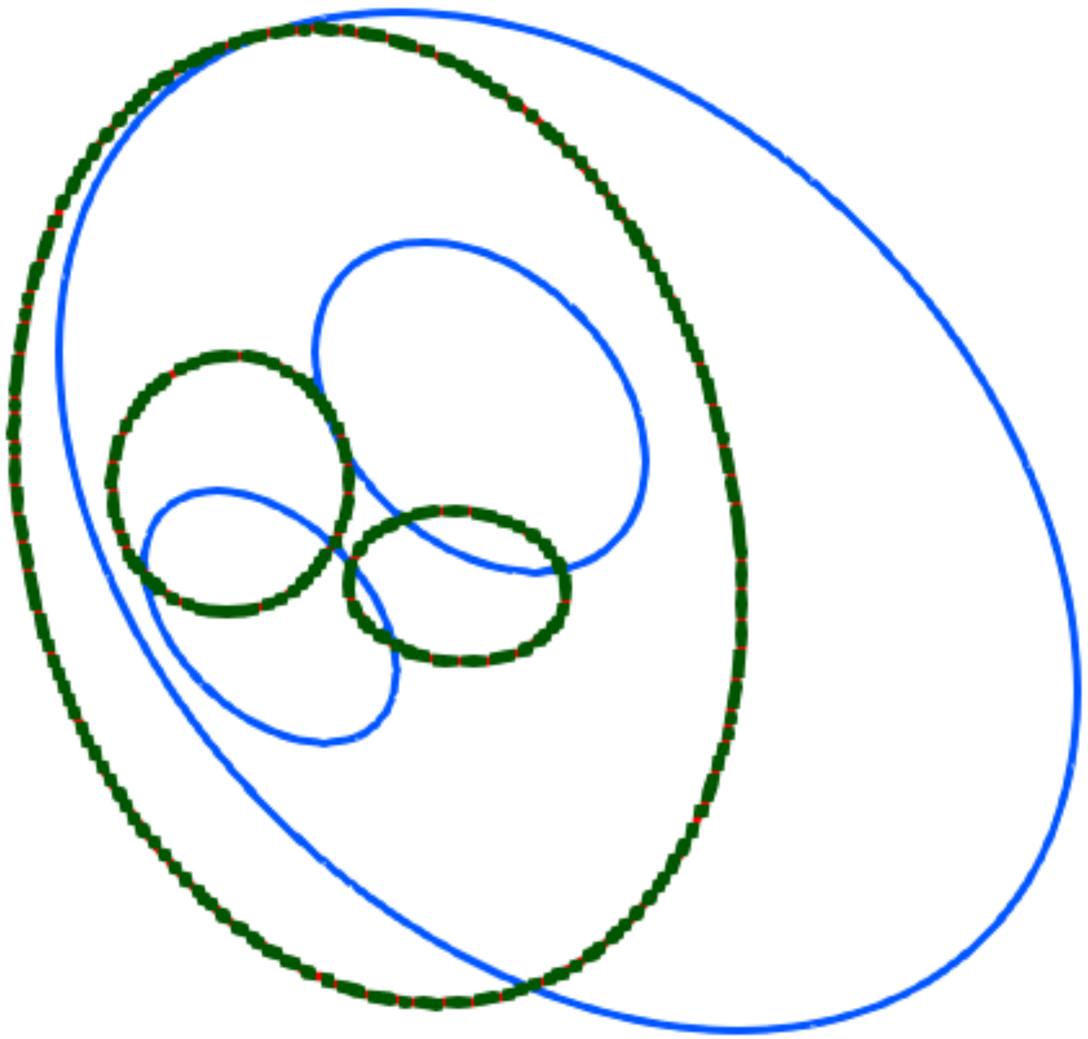


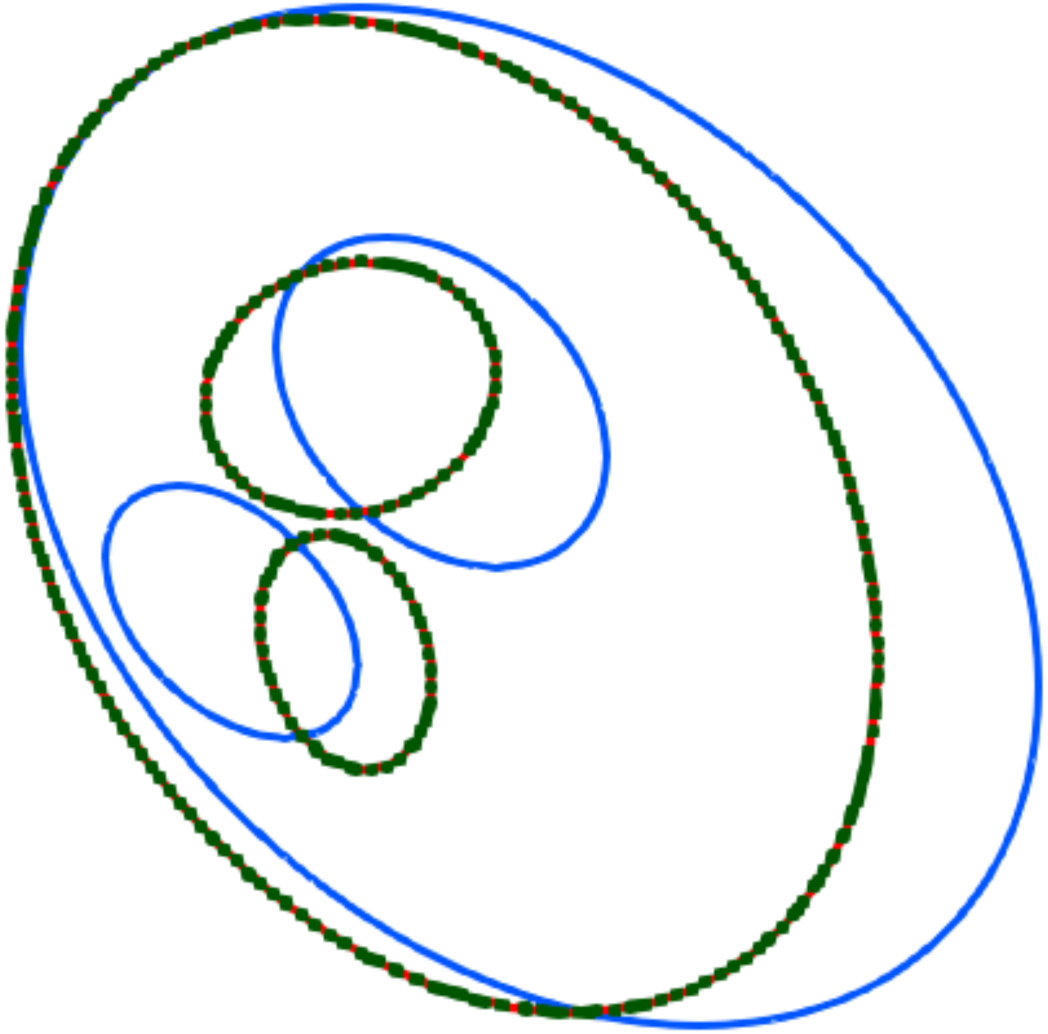


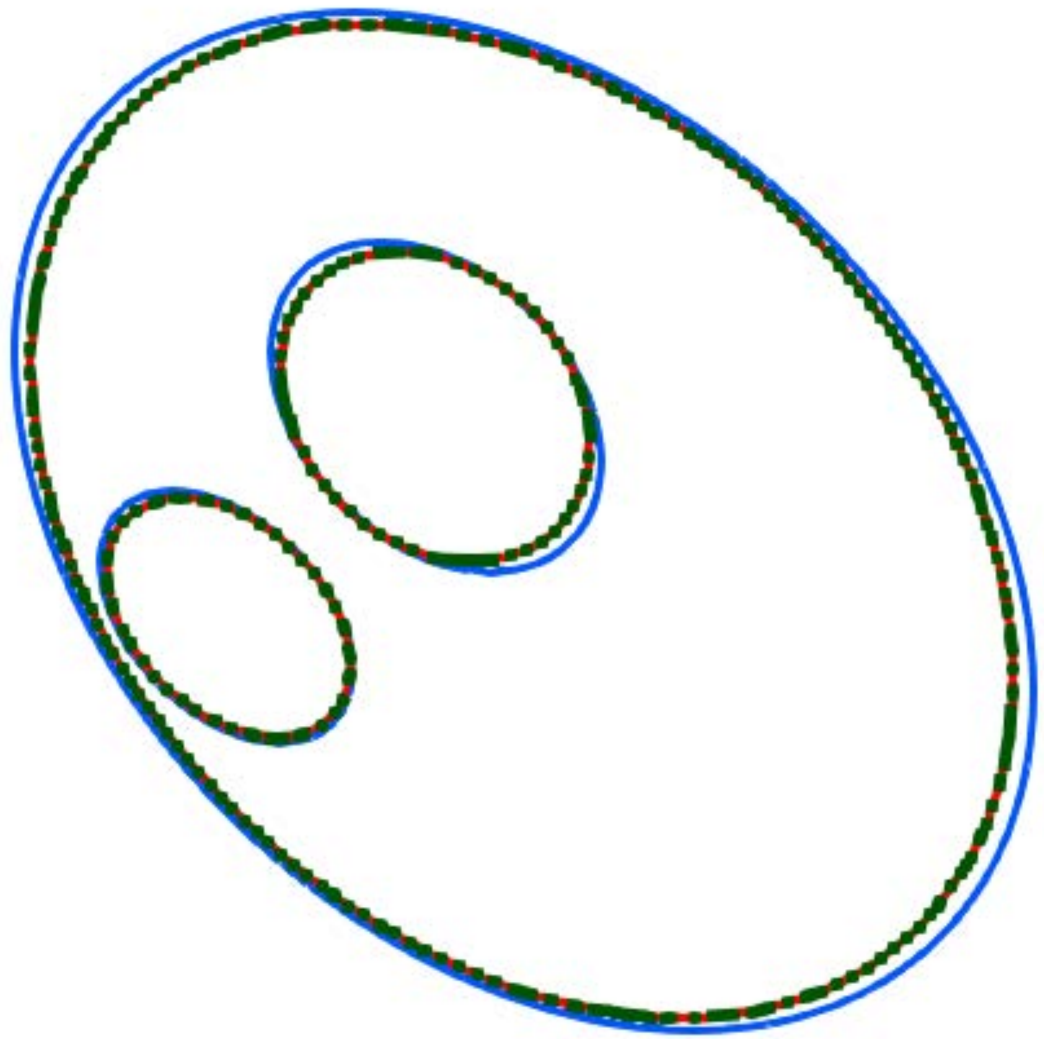
H_LDDMM

H'_invariant (euclidean)

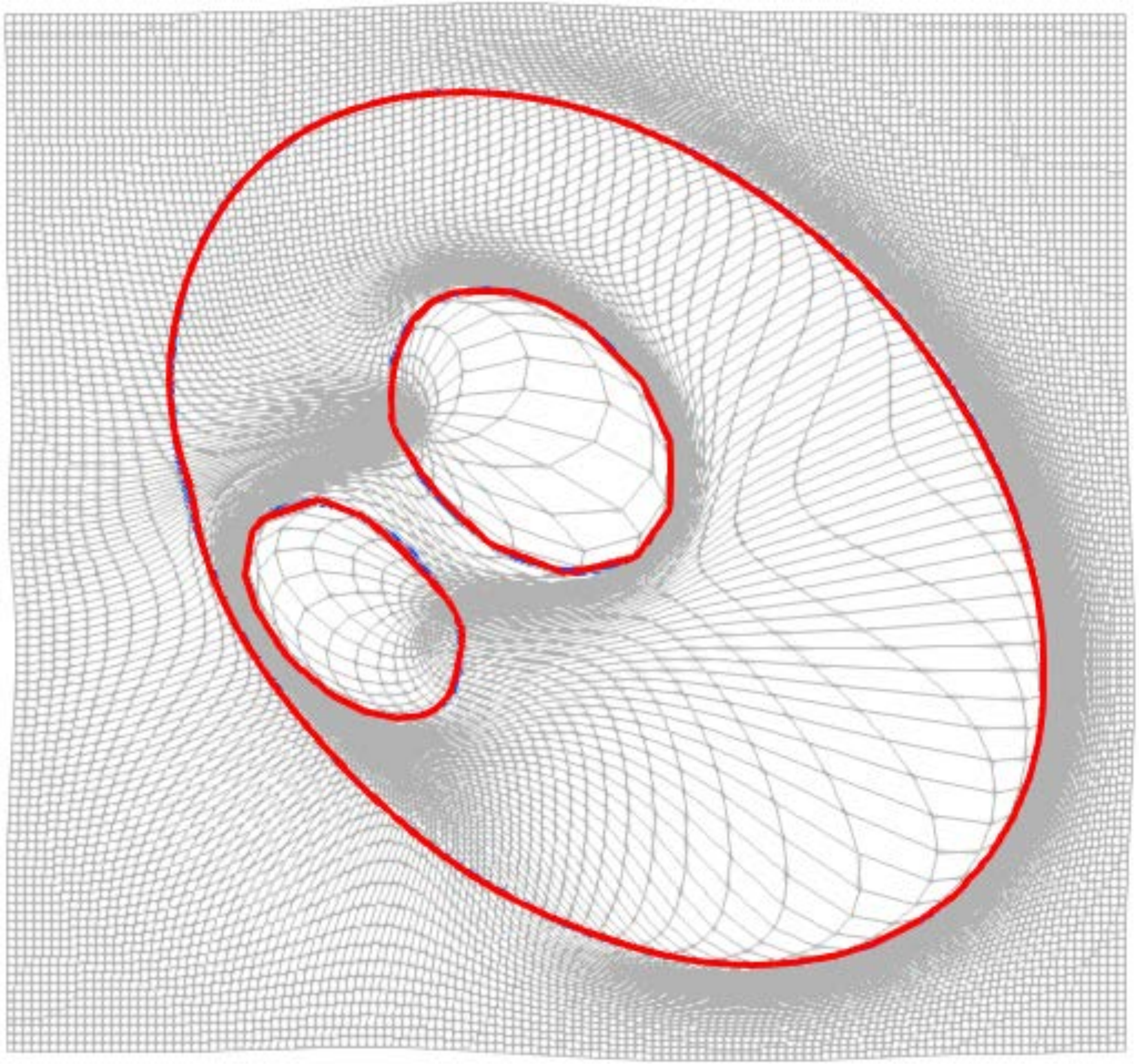








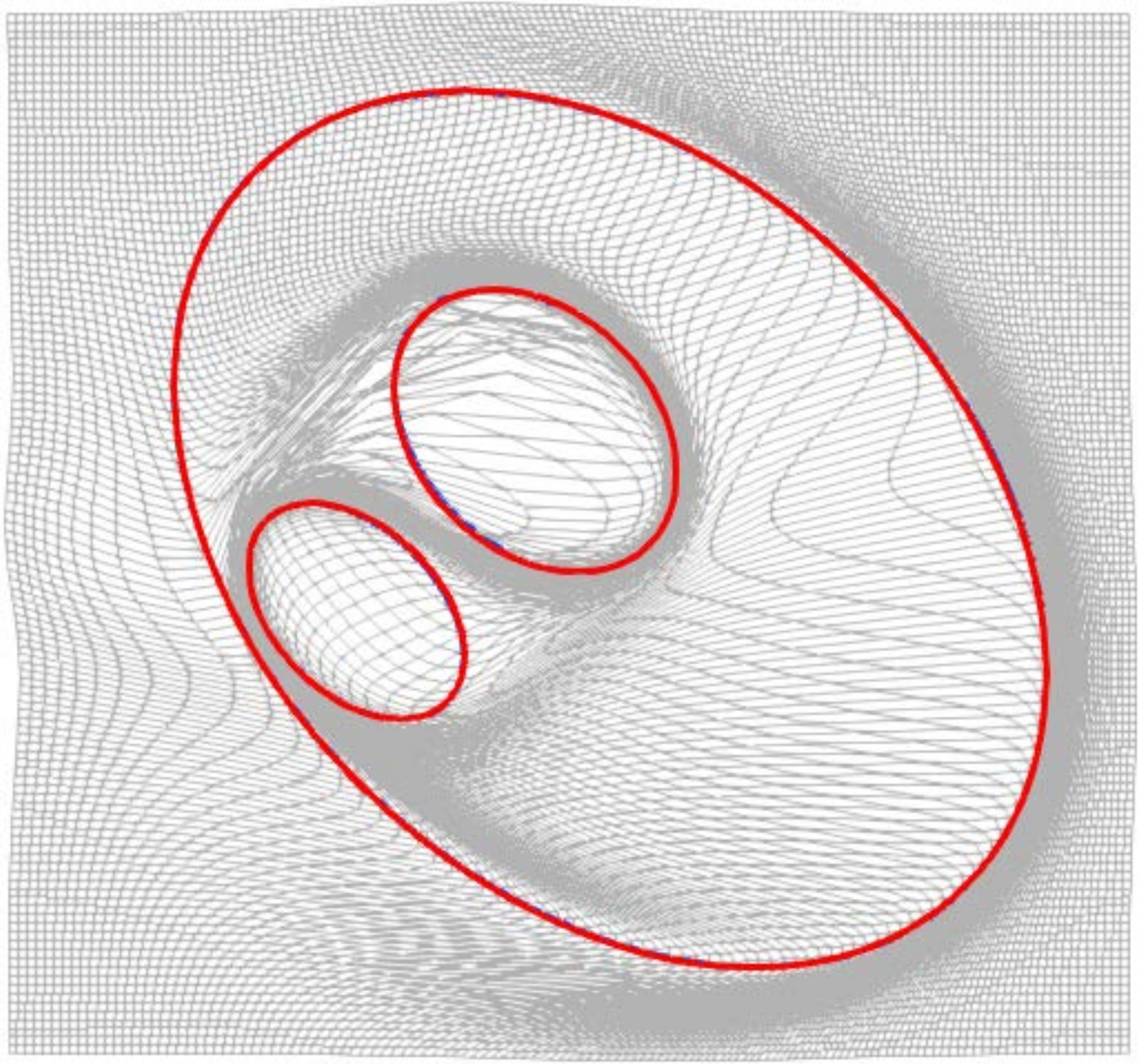
LDDMM



H_LDDMM

H'_invariant

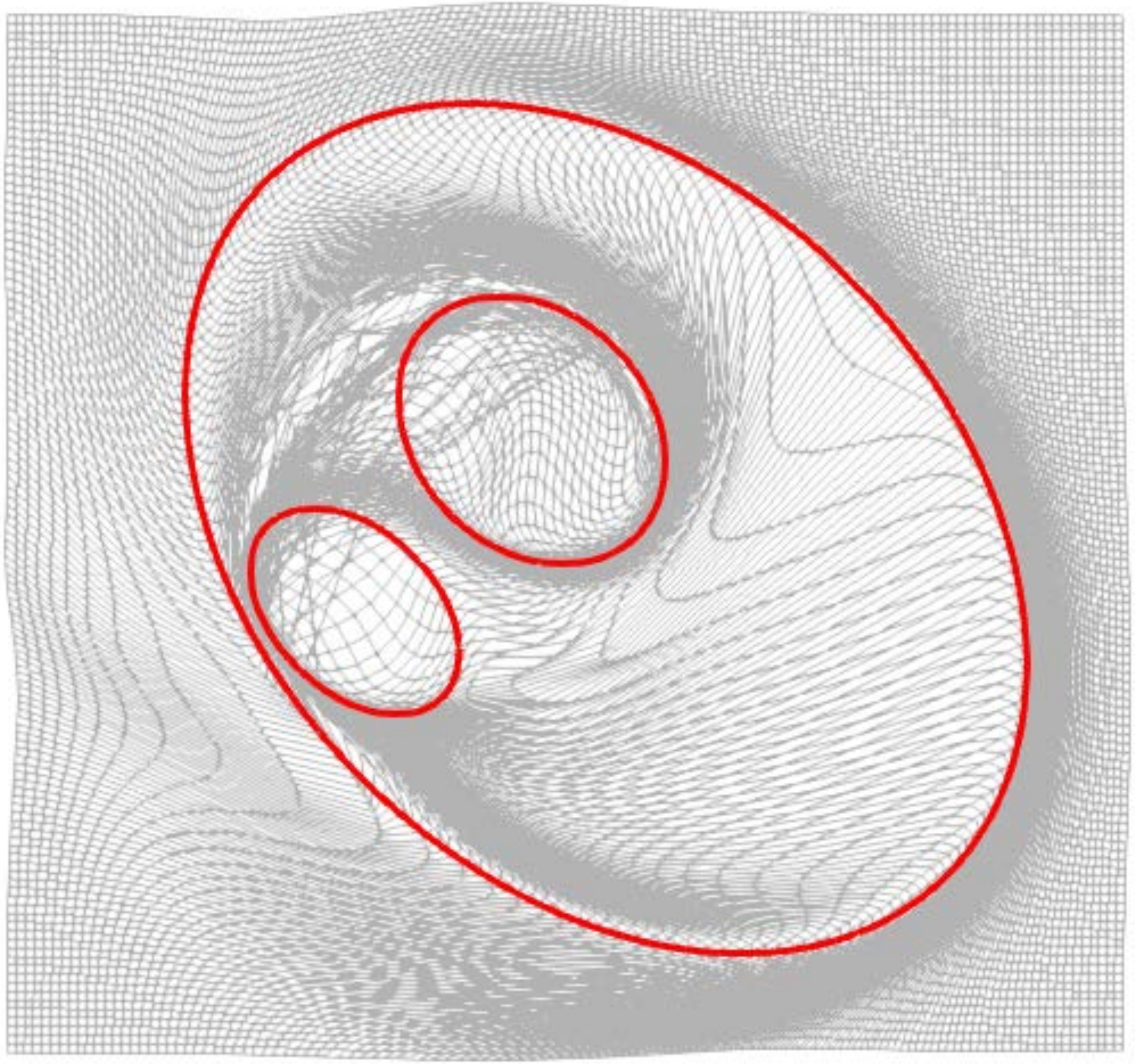
(scale + euclidean)

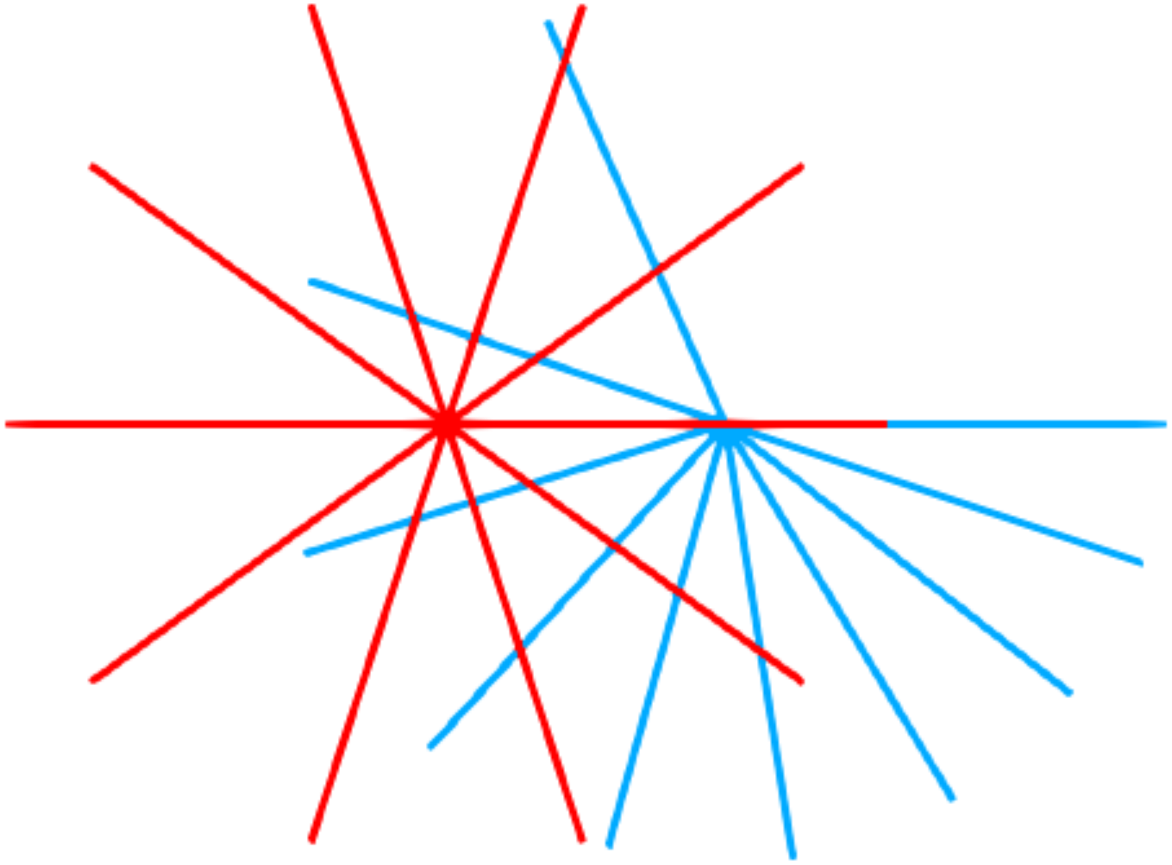


H_LDDMM

H'_invariant

(euclidean)





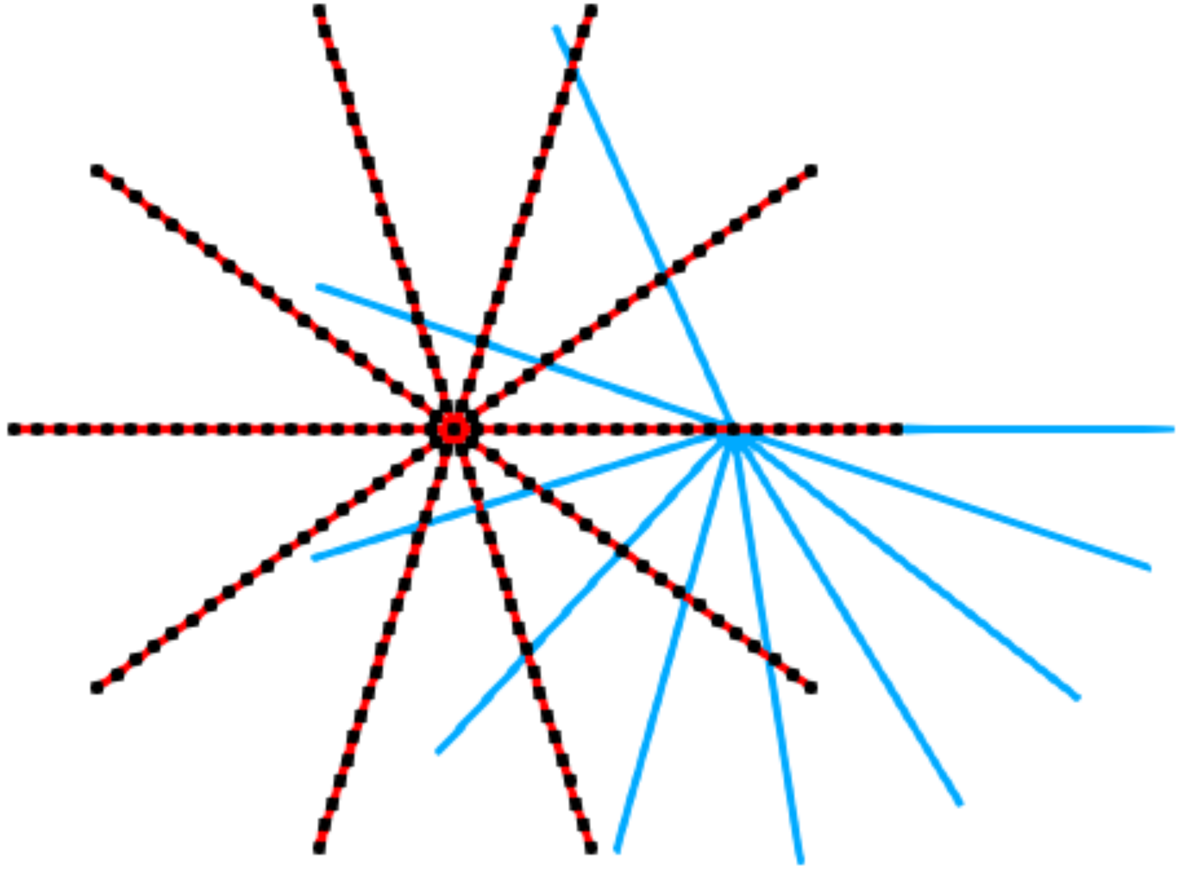
→ No explicit ray labelling

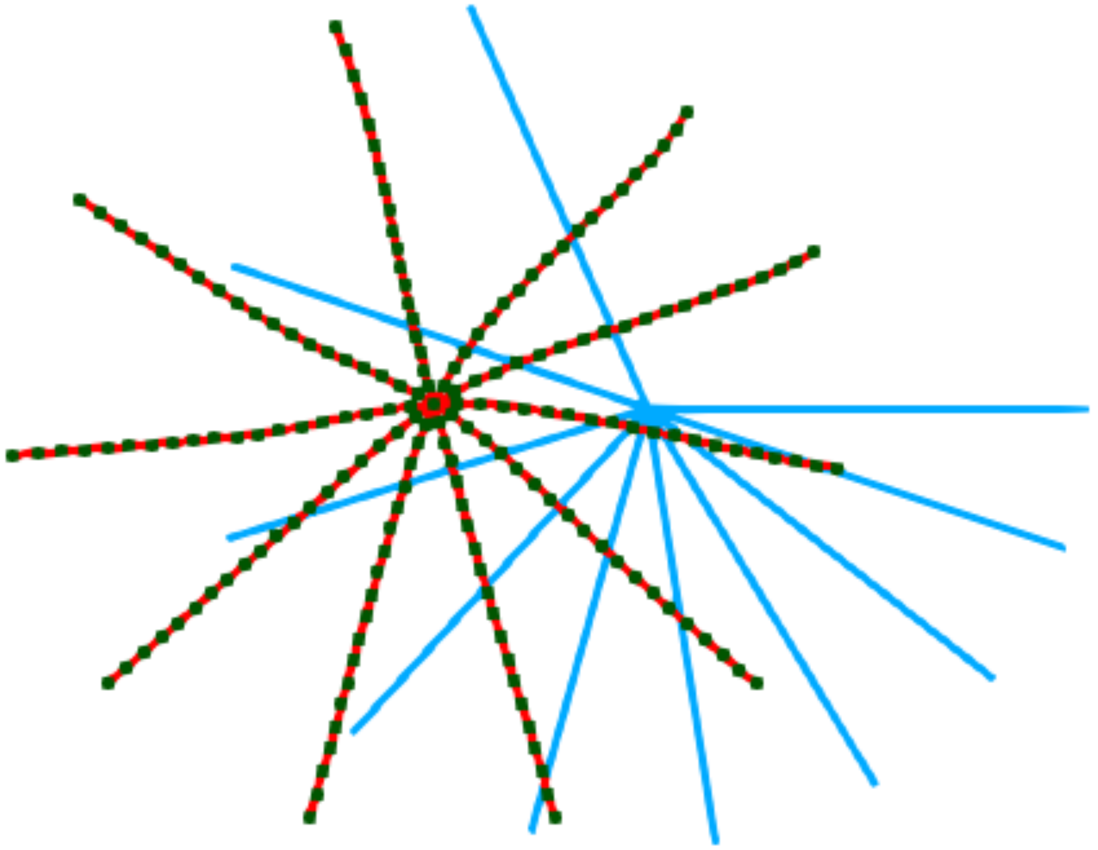
→ Mapping challenging ⟷ many possible local minima

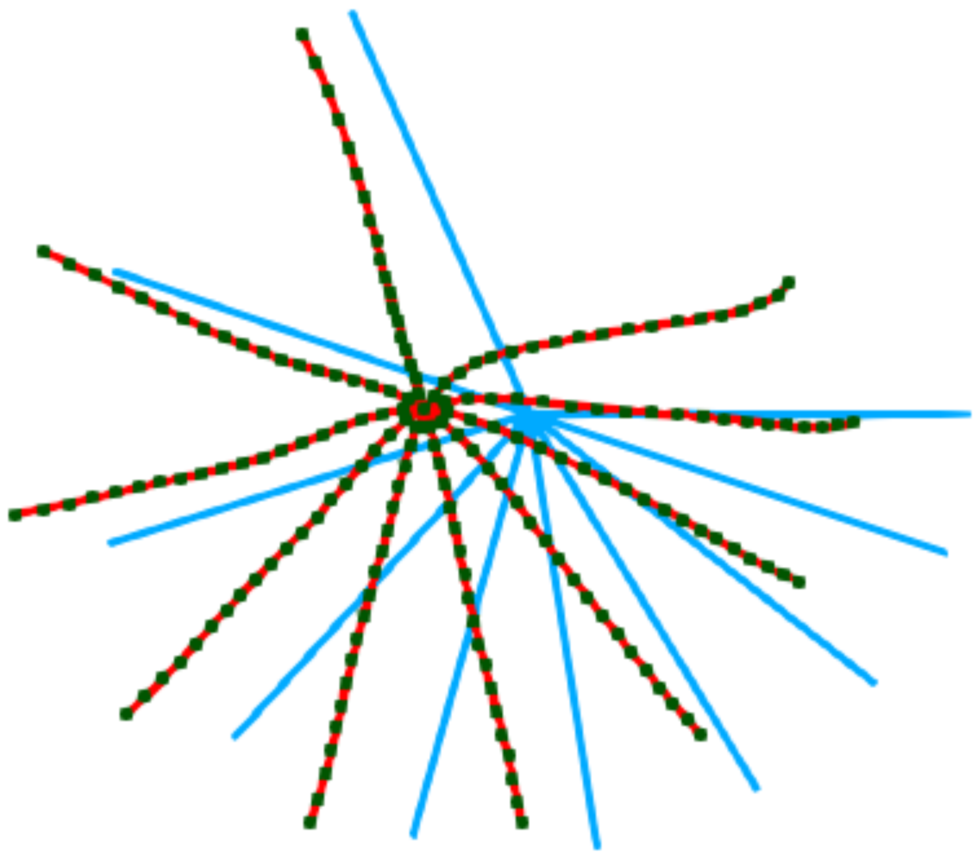
→ LDDMM with small kernel fails

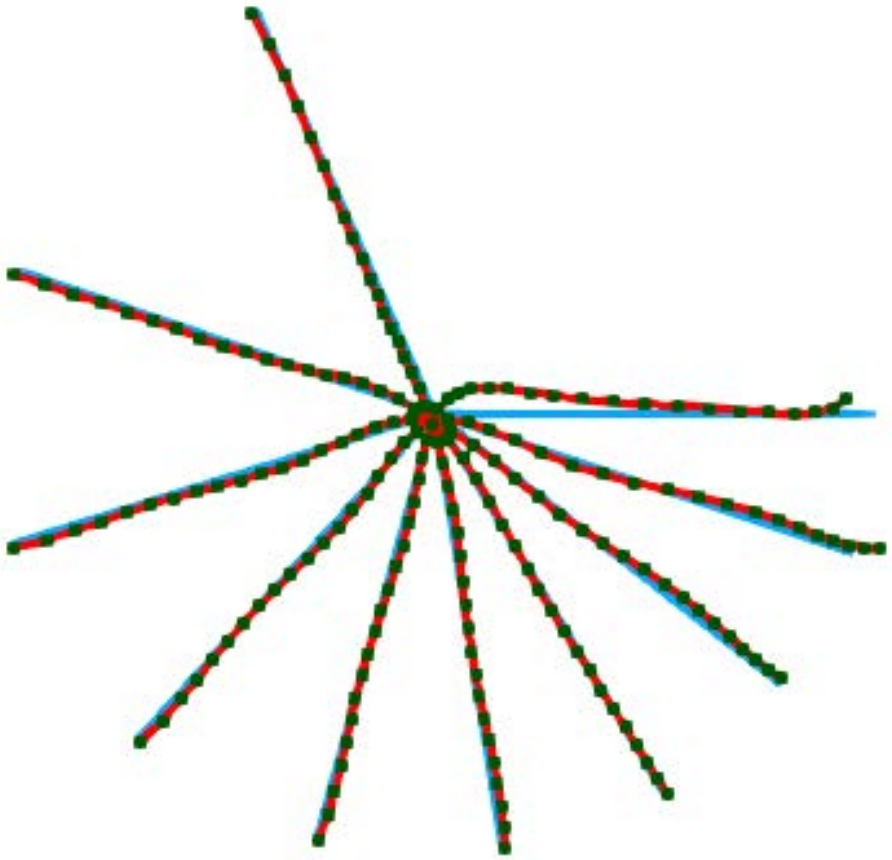
→ example use "smoother" kernel.

LDDMM



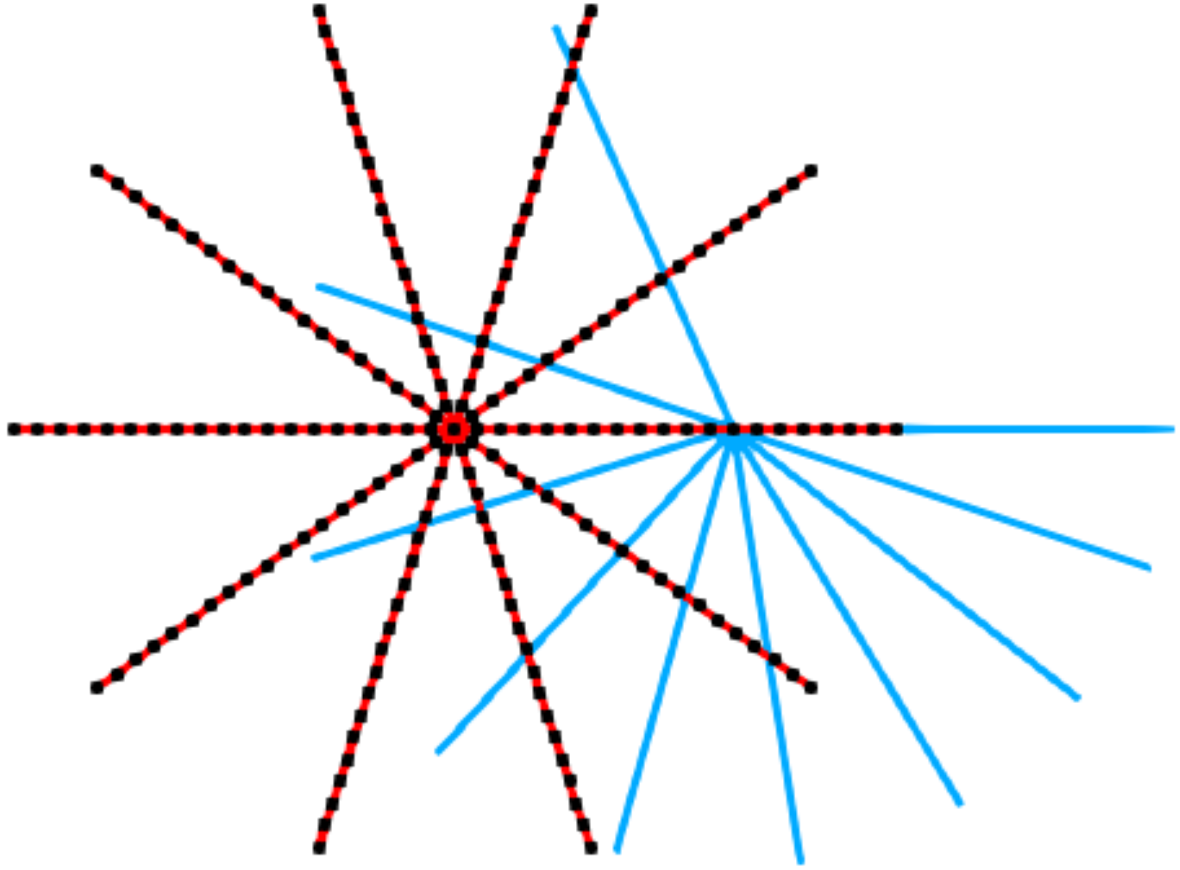


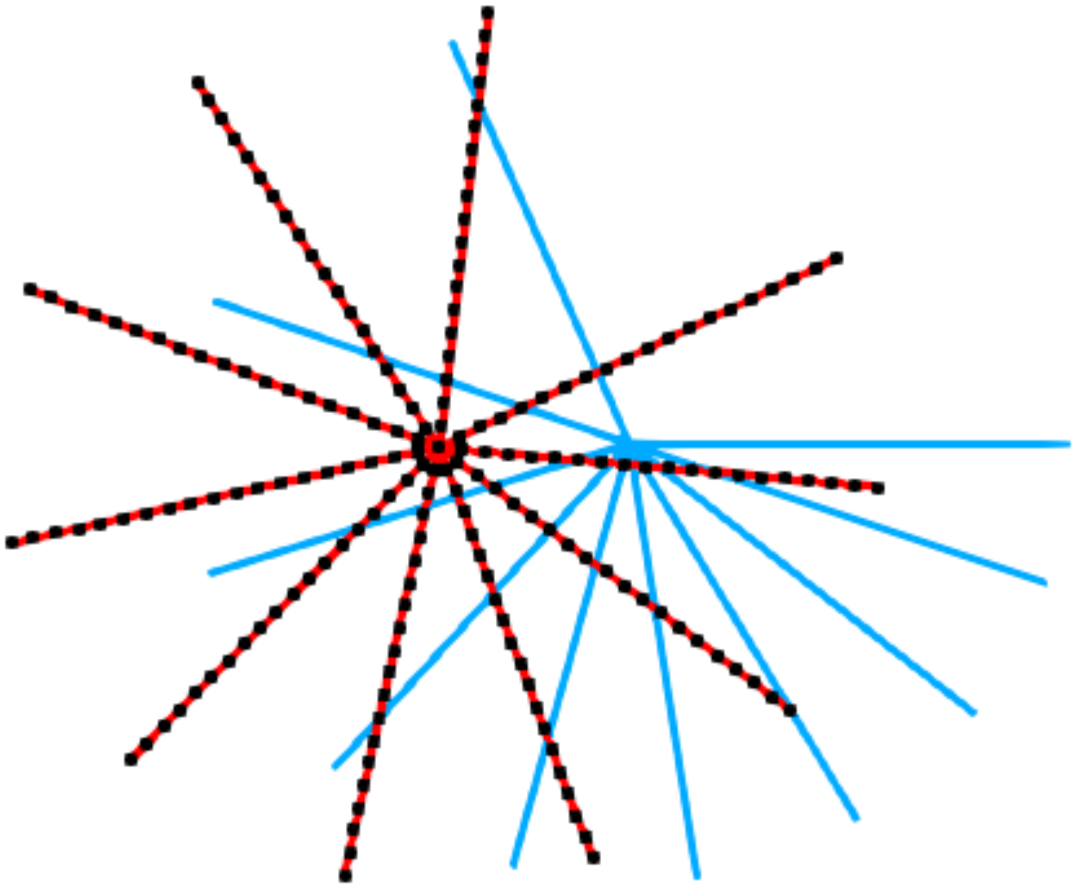


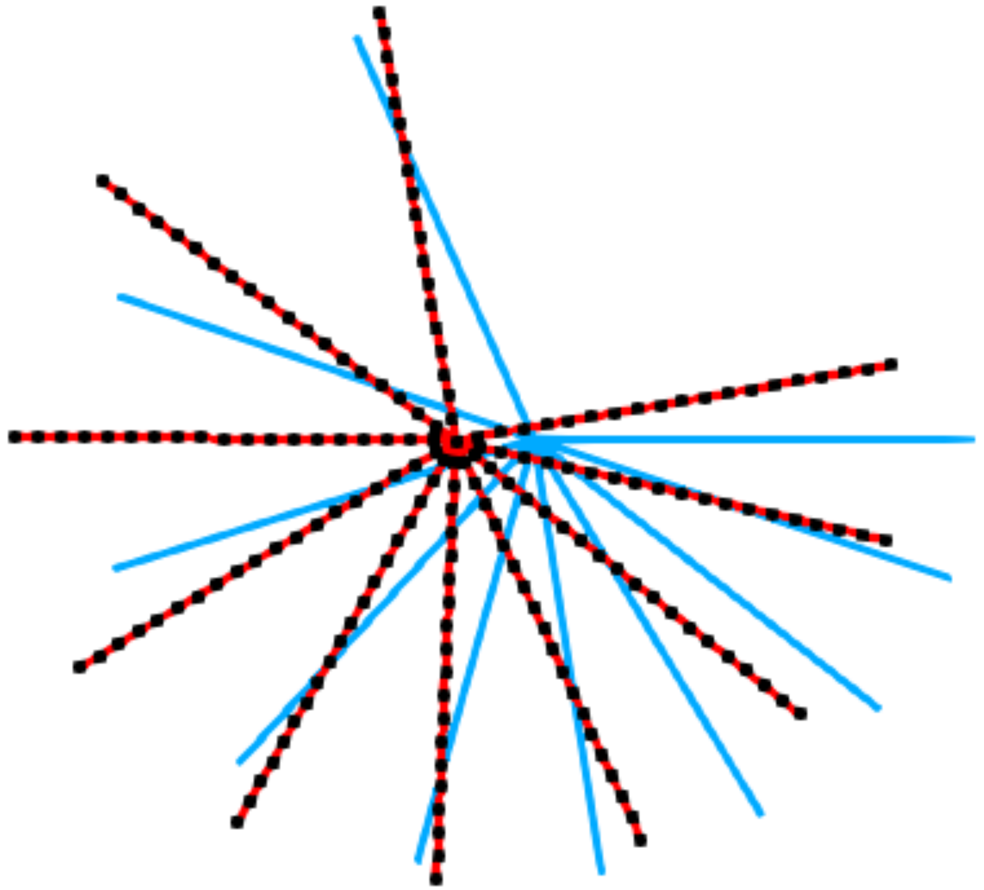


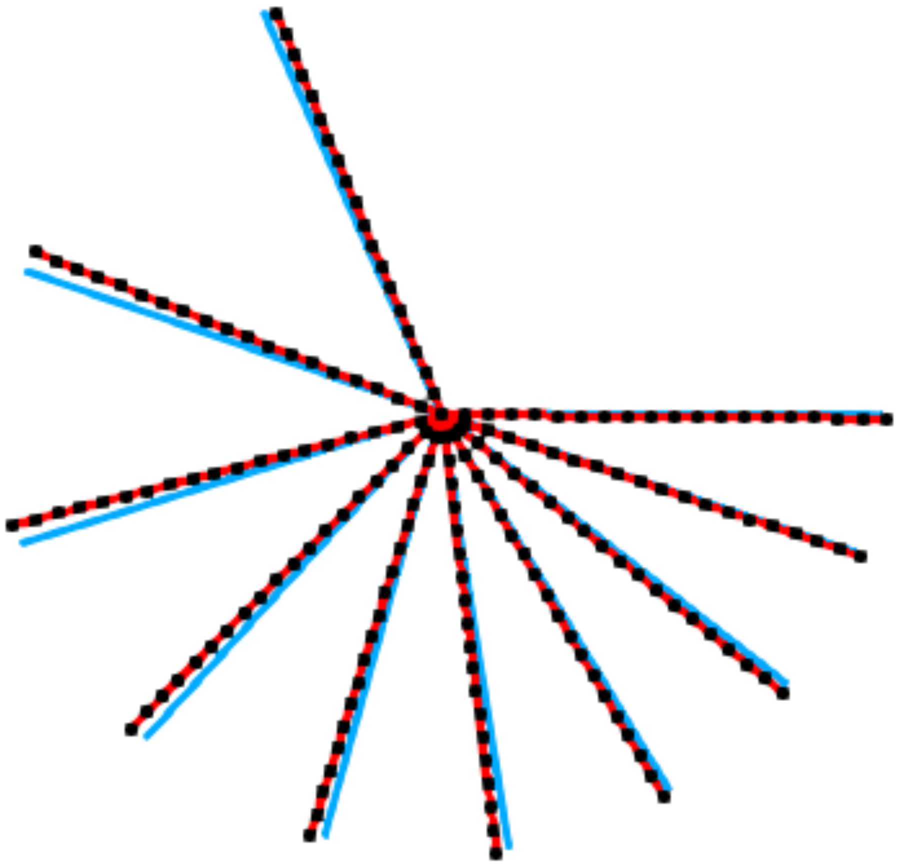
H_LDDMM

H'_invariant



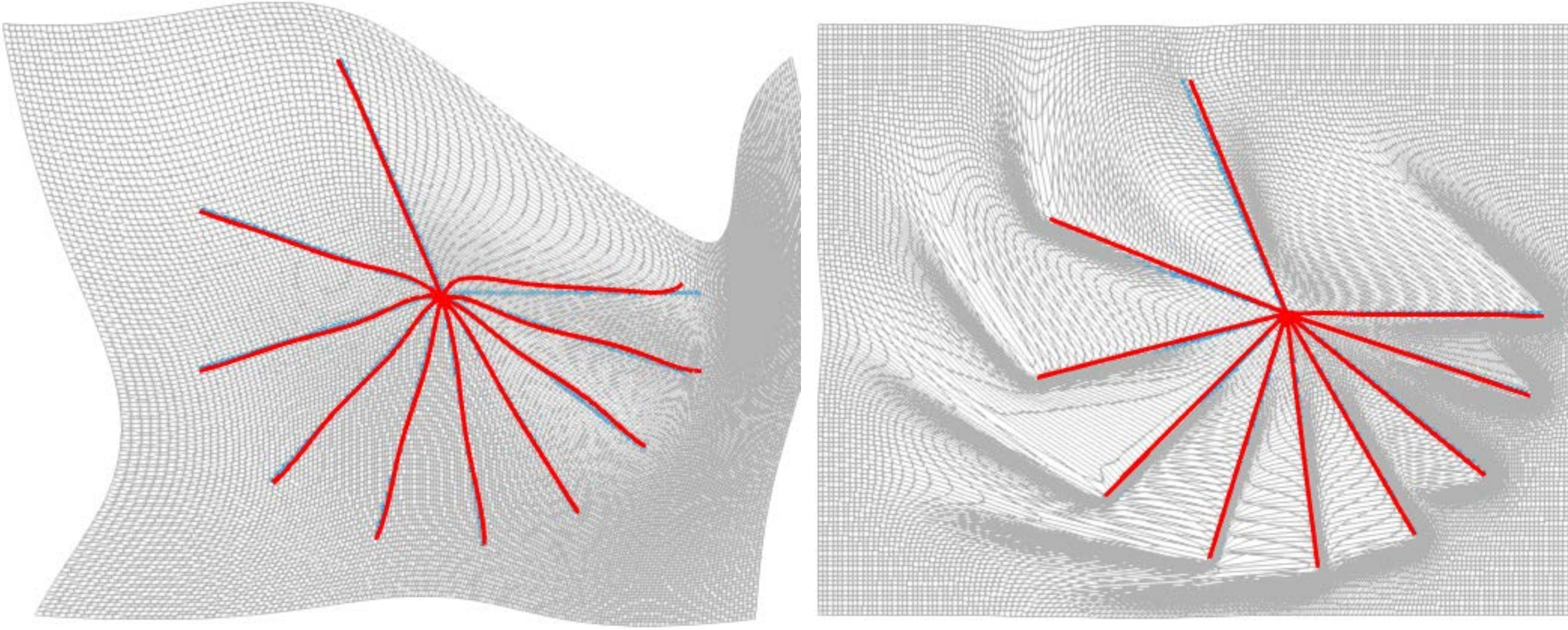




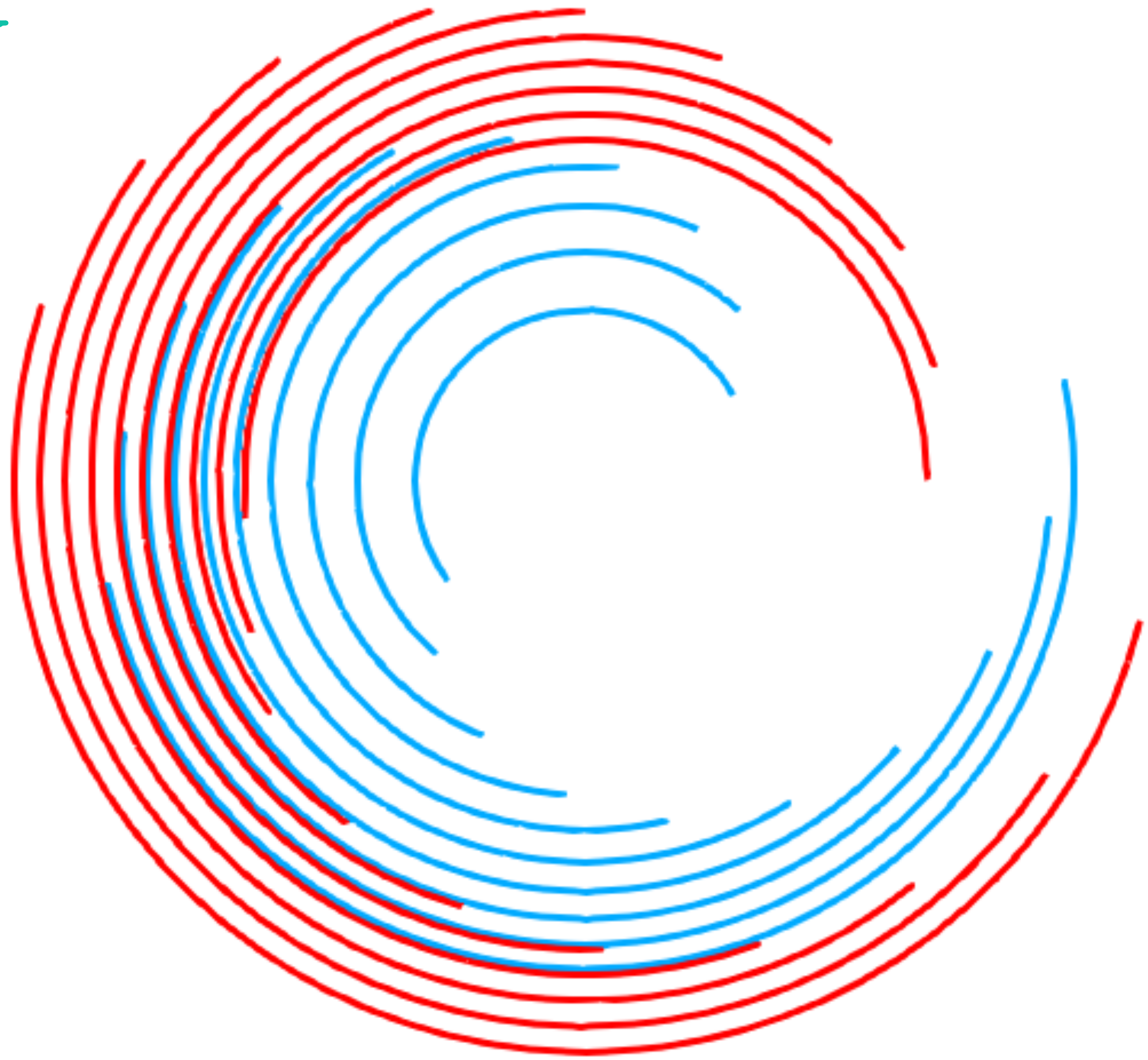


LDDMM

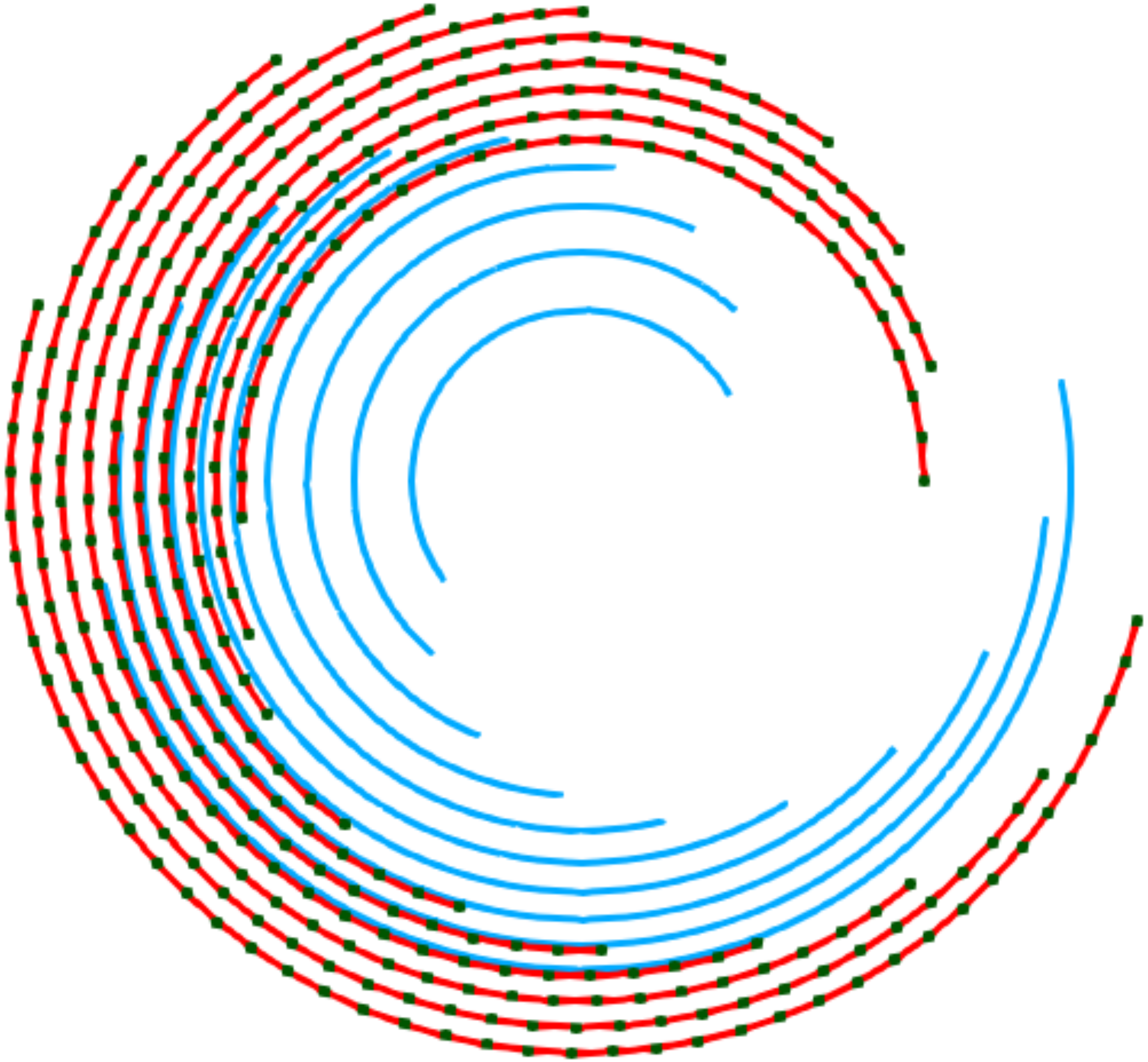
H_LDDMM

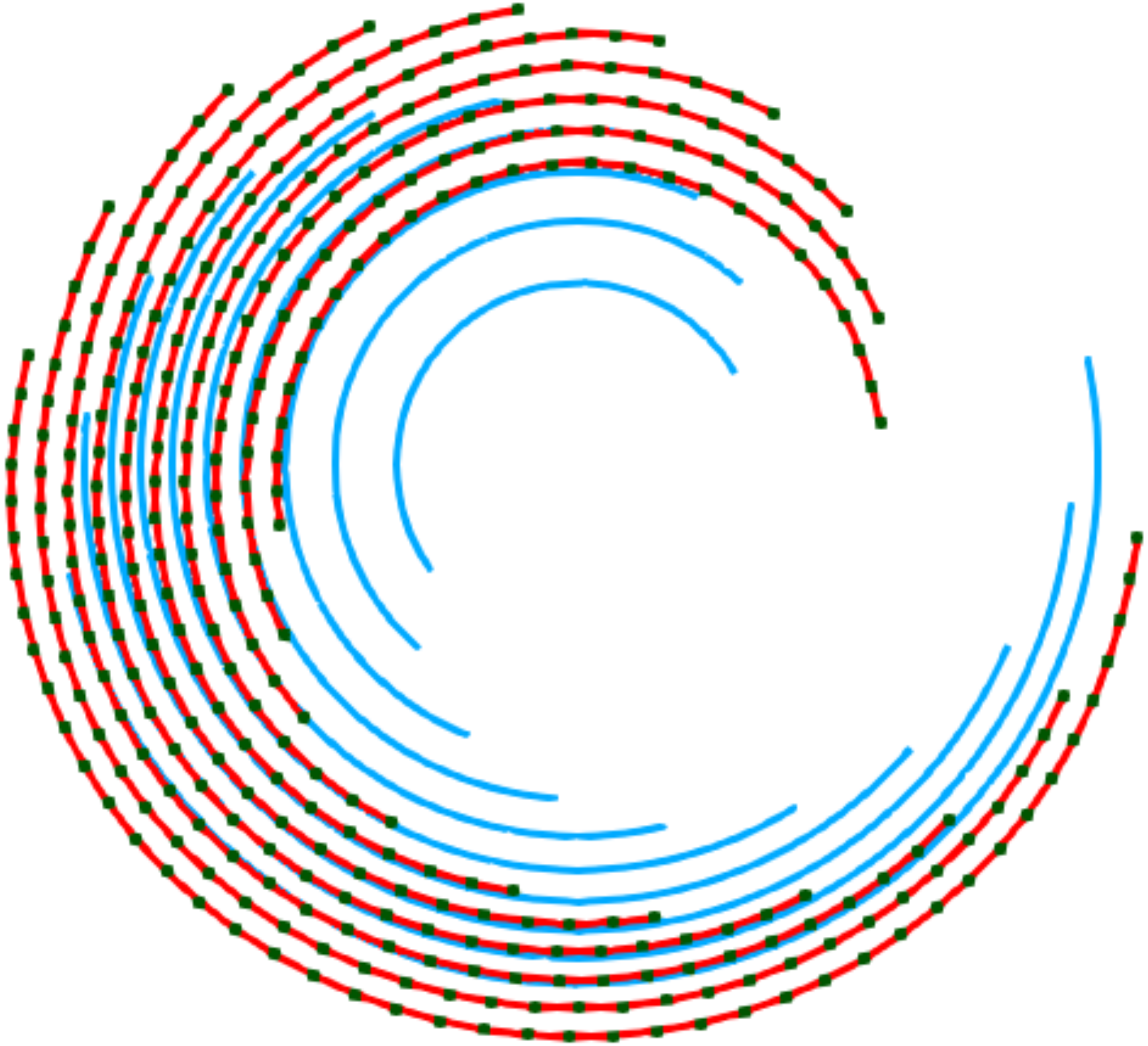


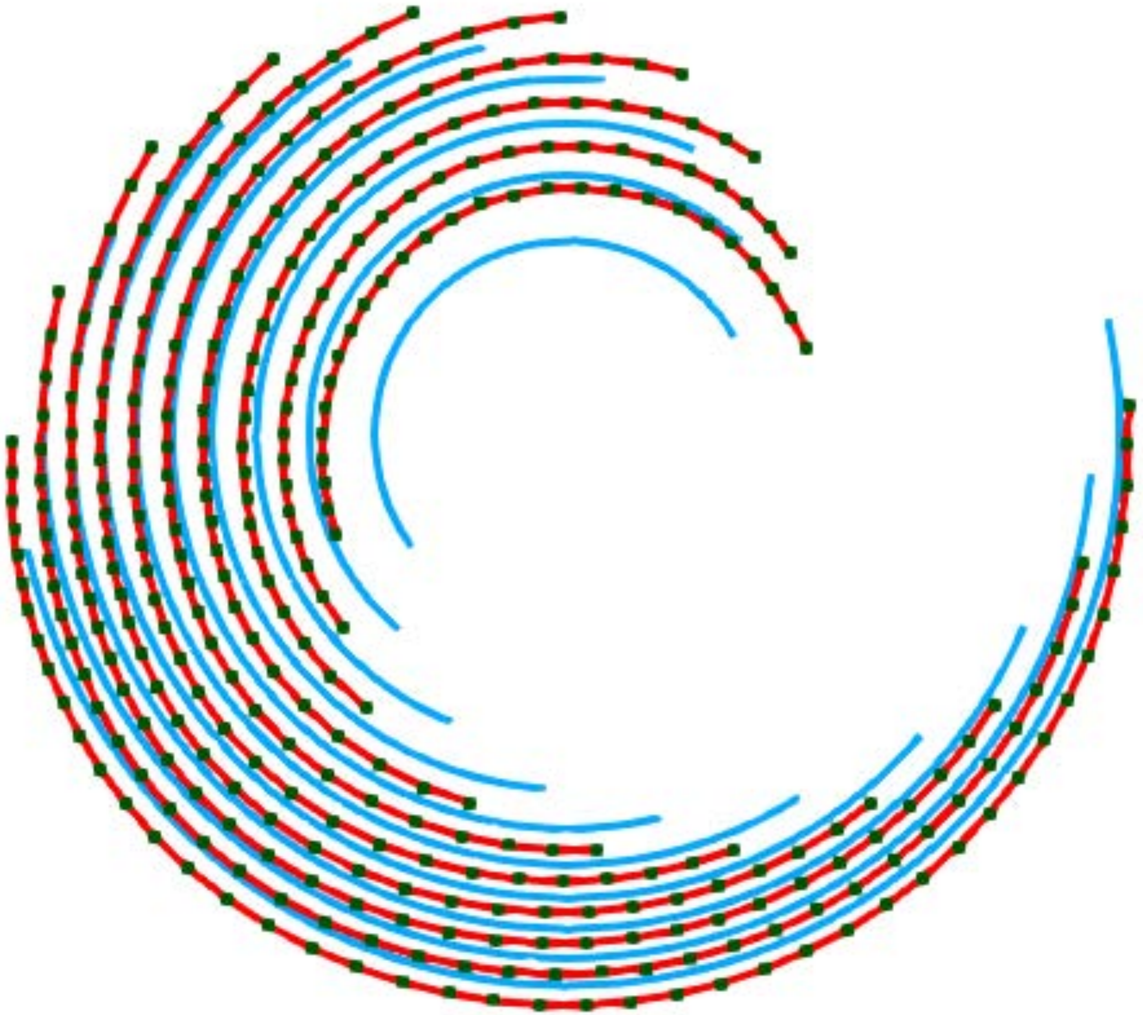
Arcs of circle

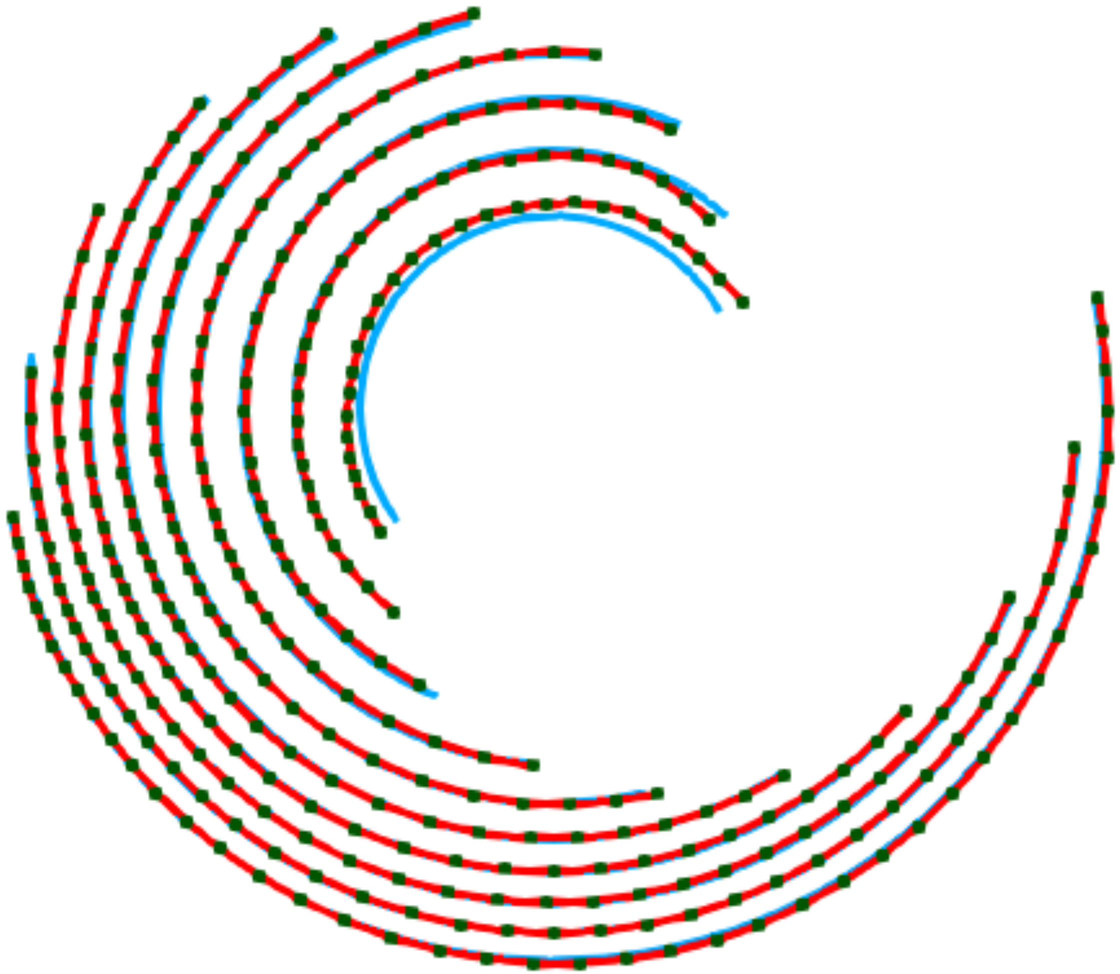


LDDMM

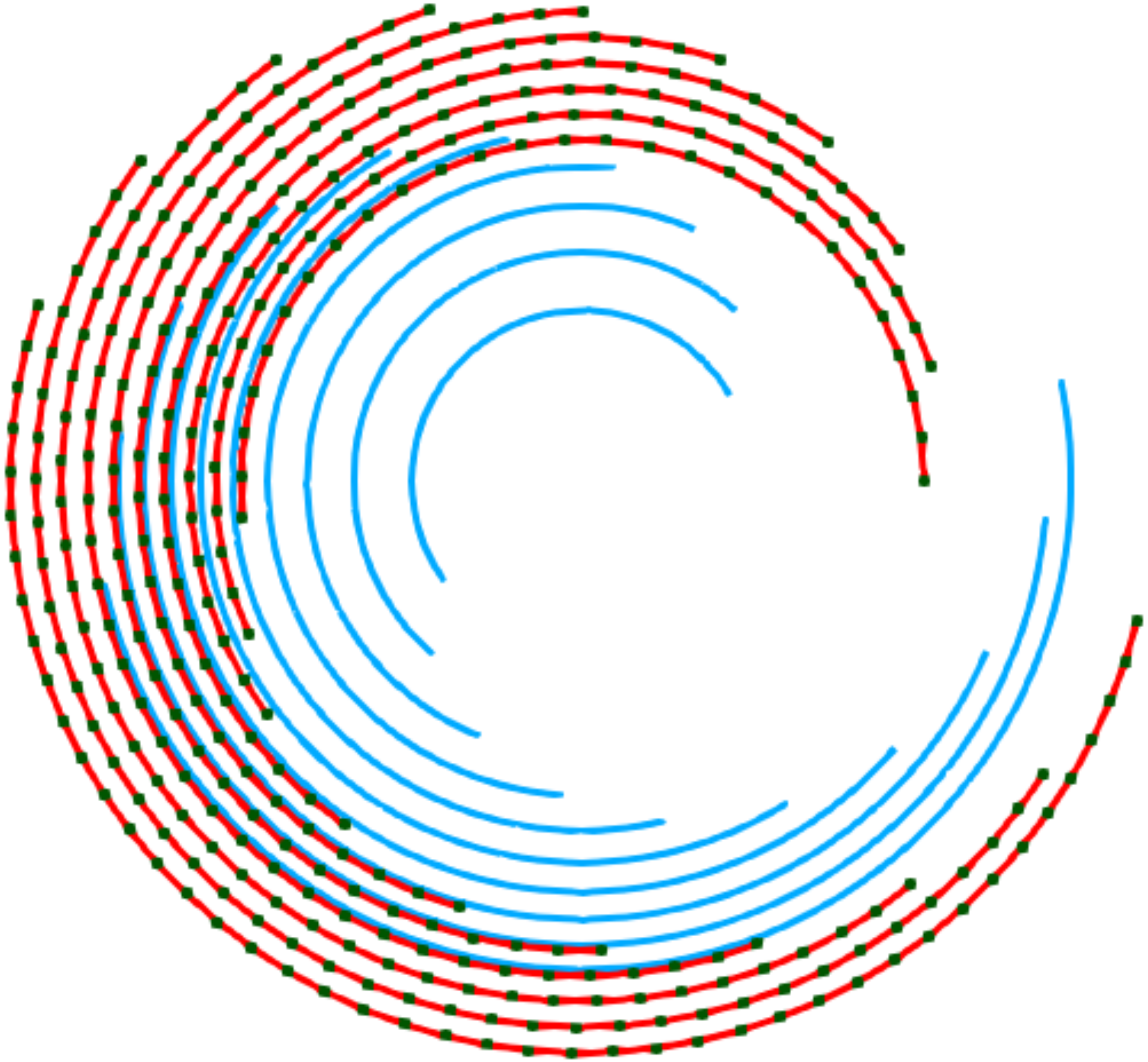


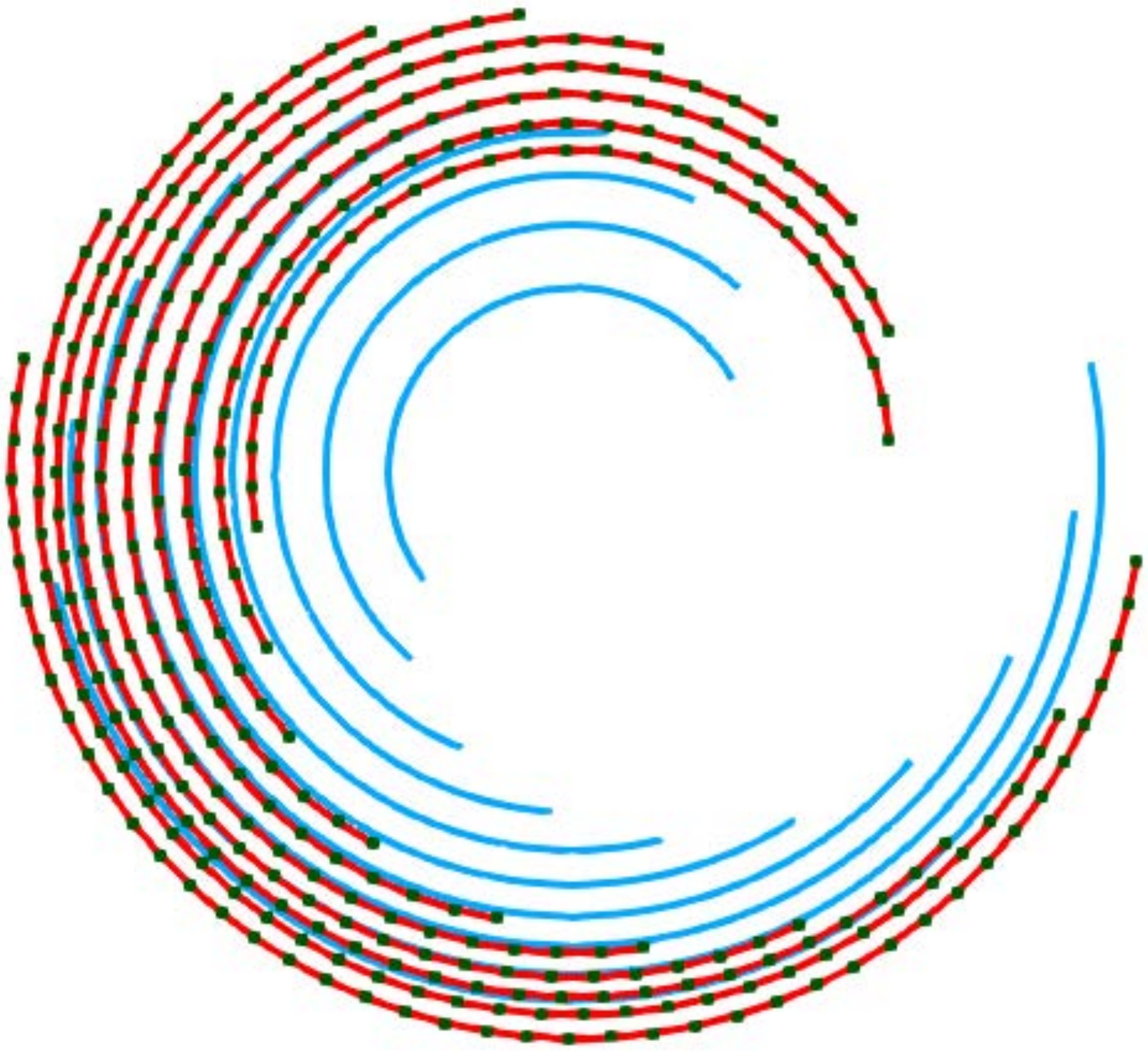


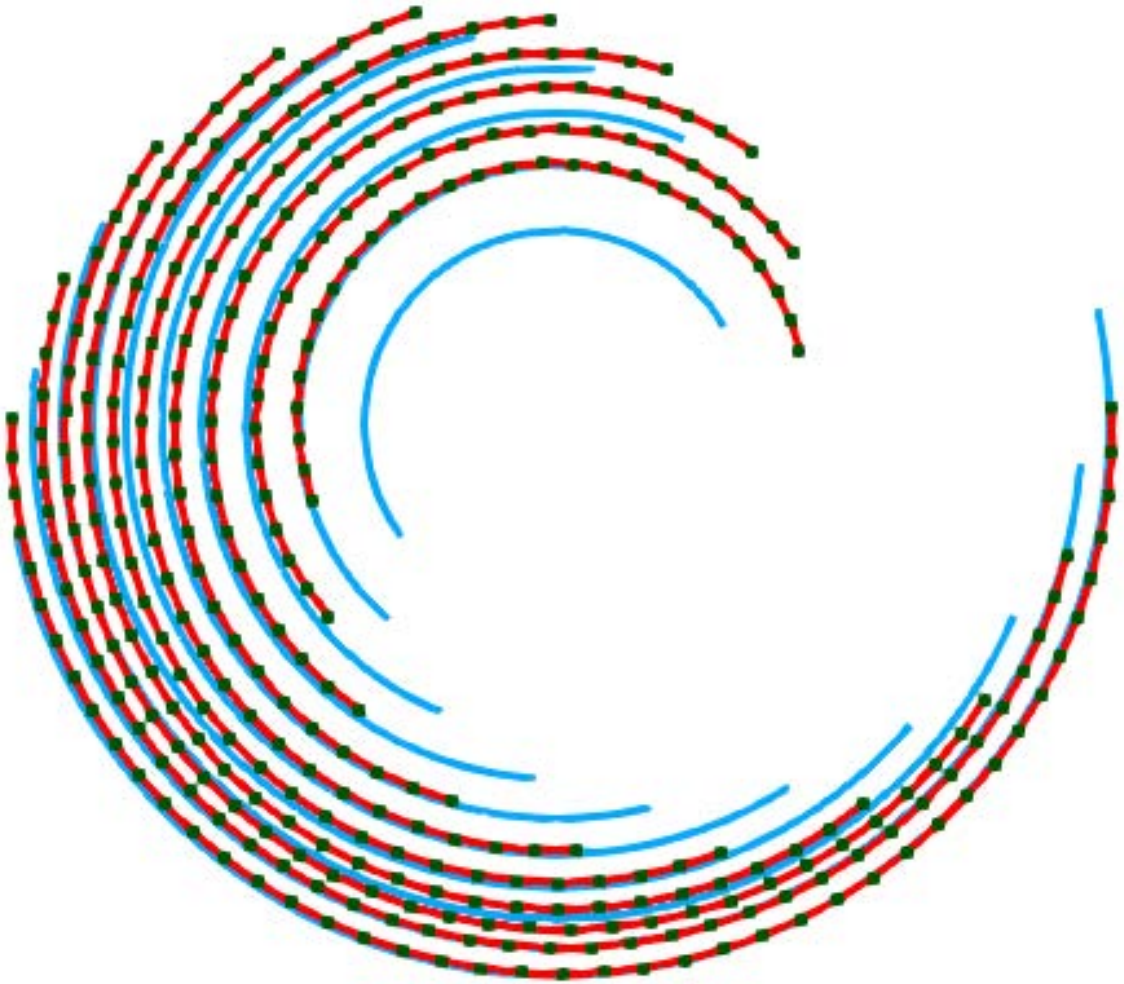


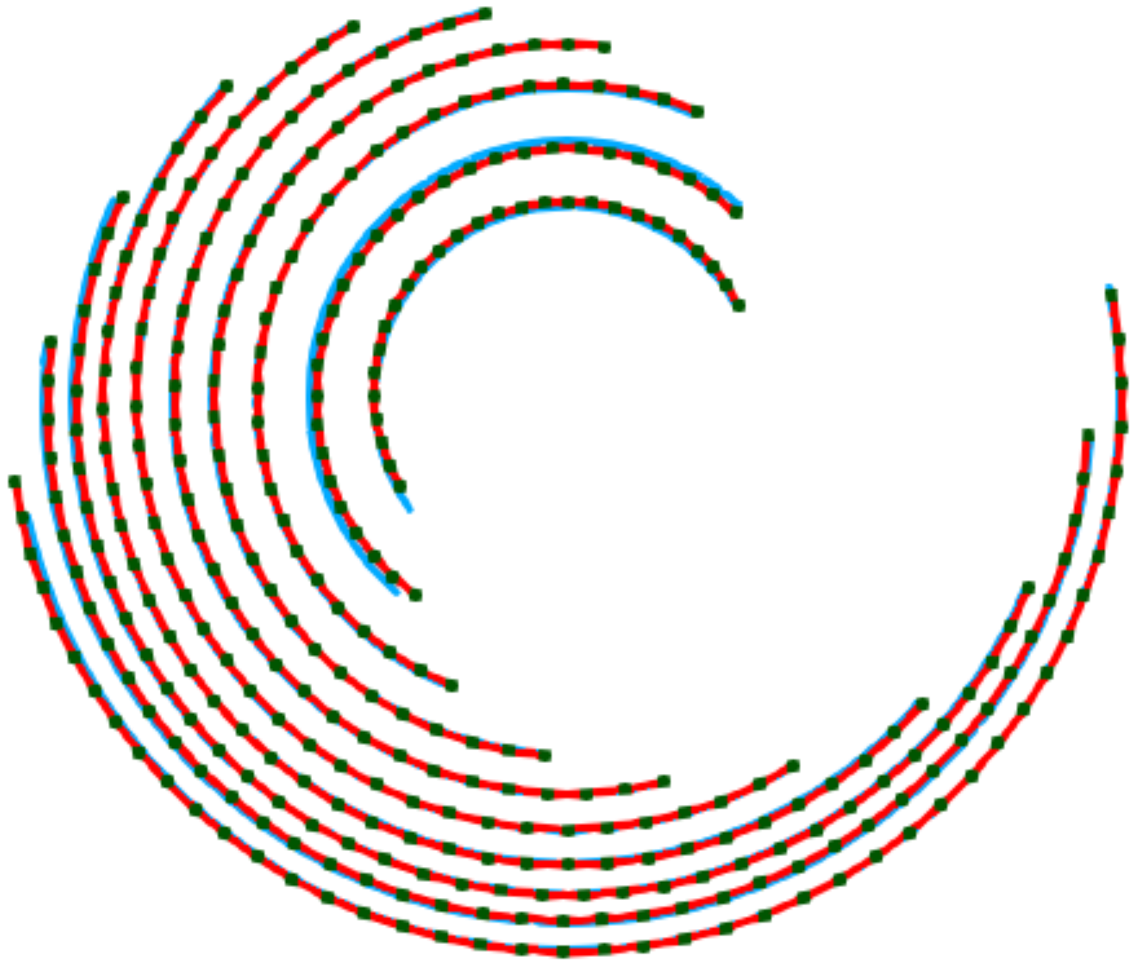


H_LDDПM

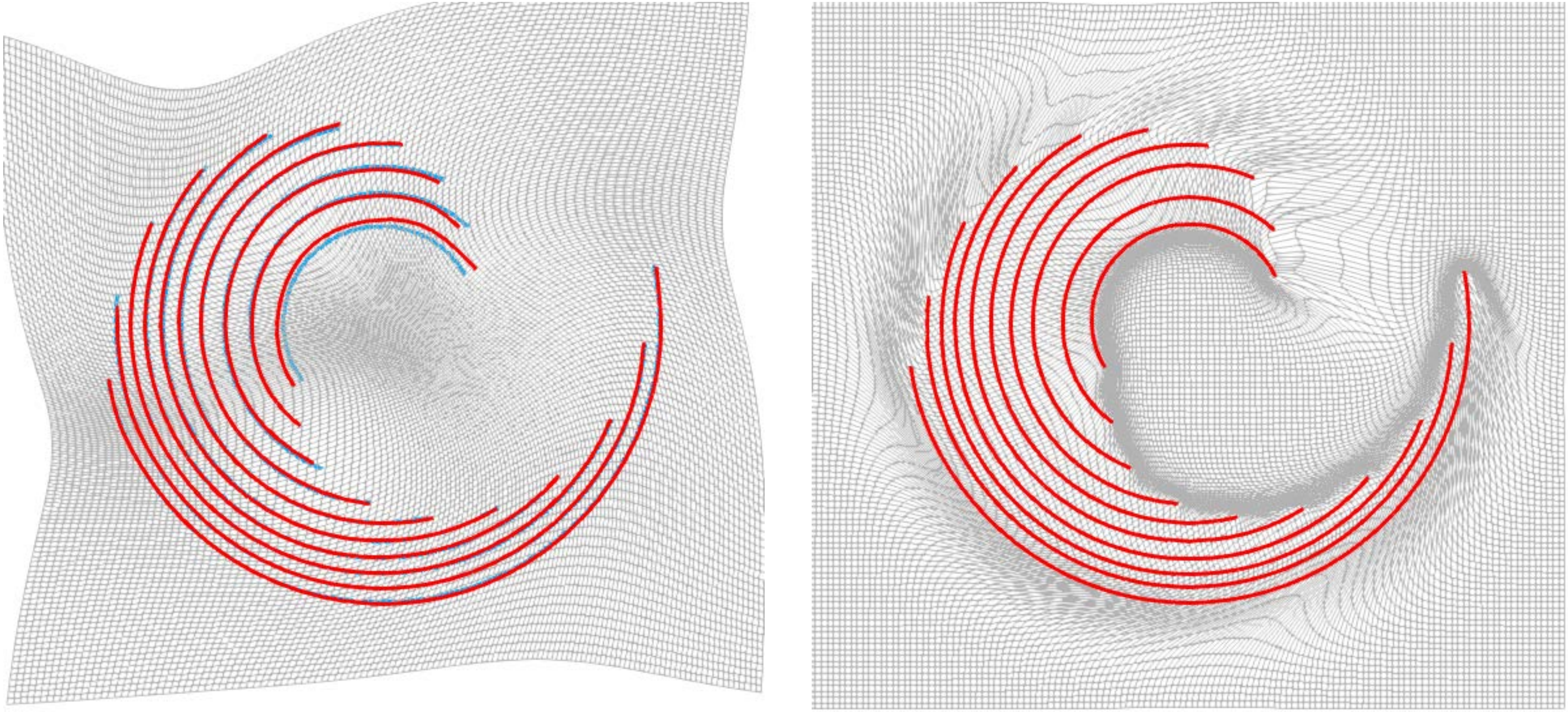








LDDMM H_LDDMM

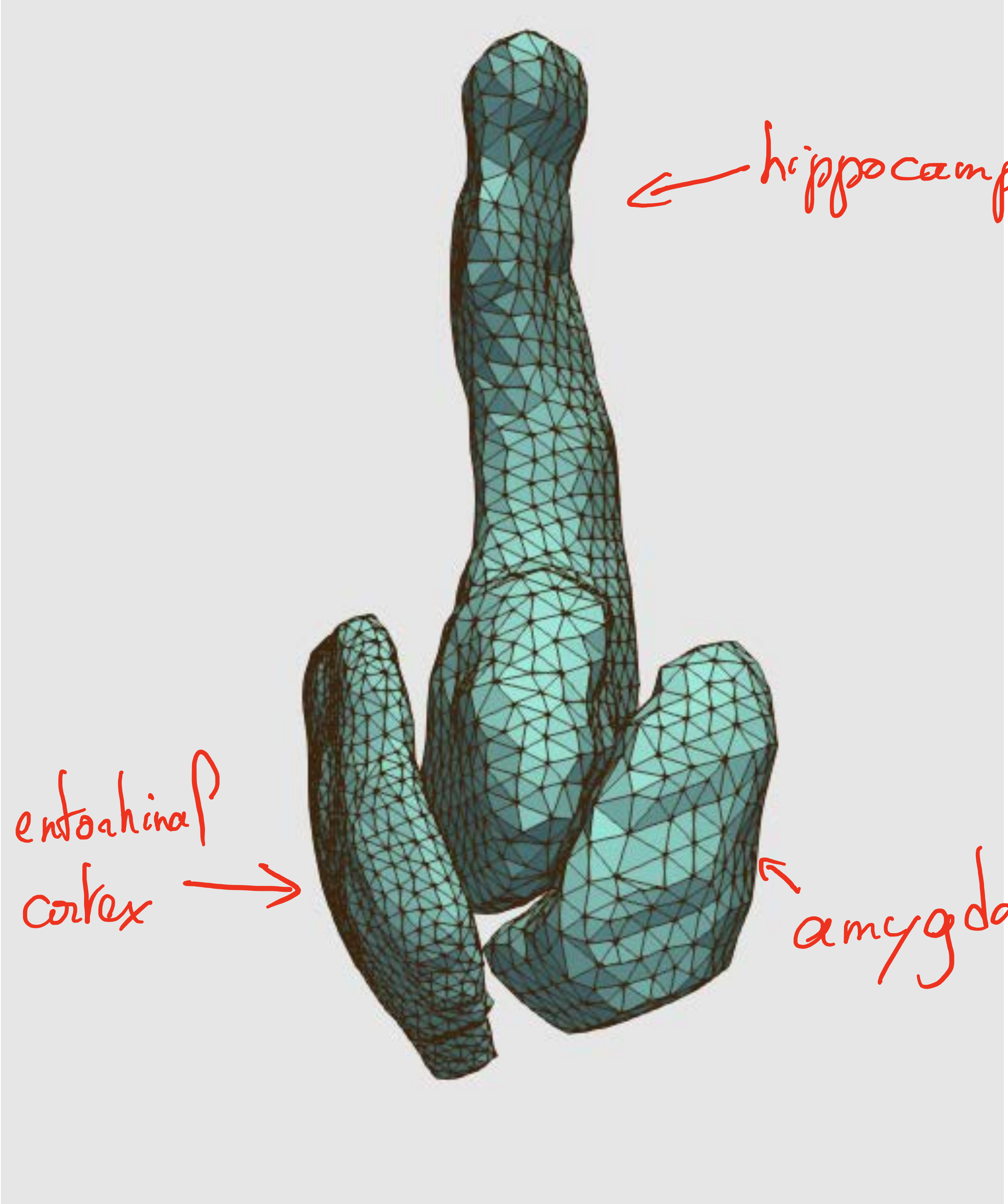


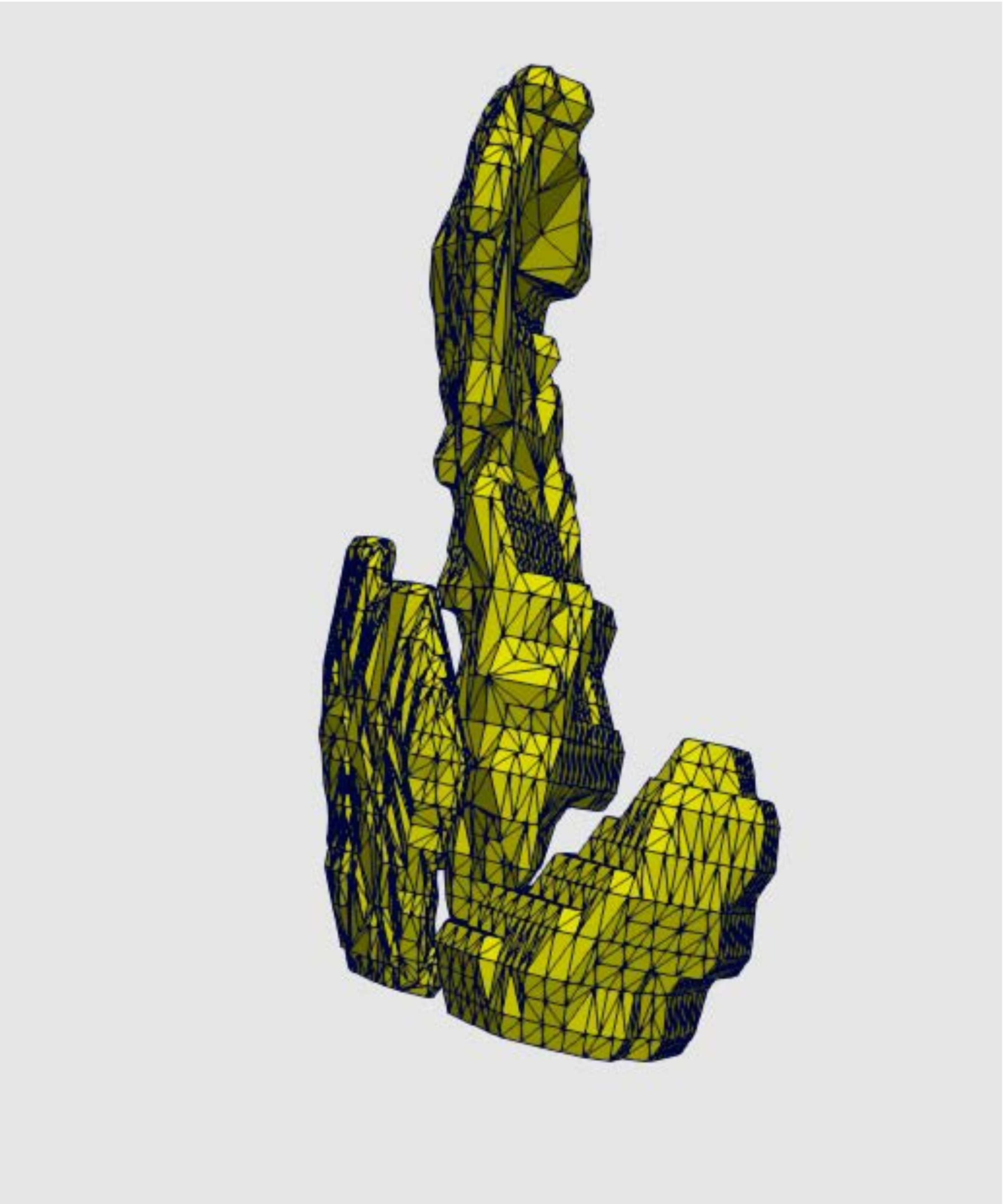
Multi_surfaces

Hippocampus

Amygdala

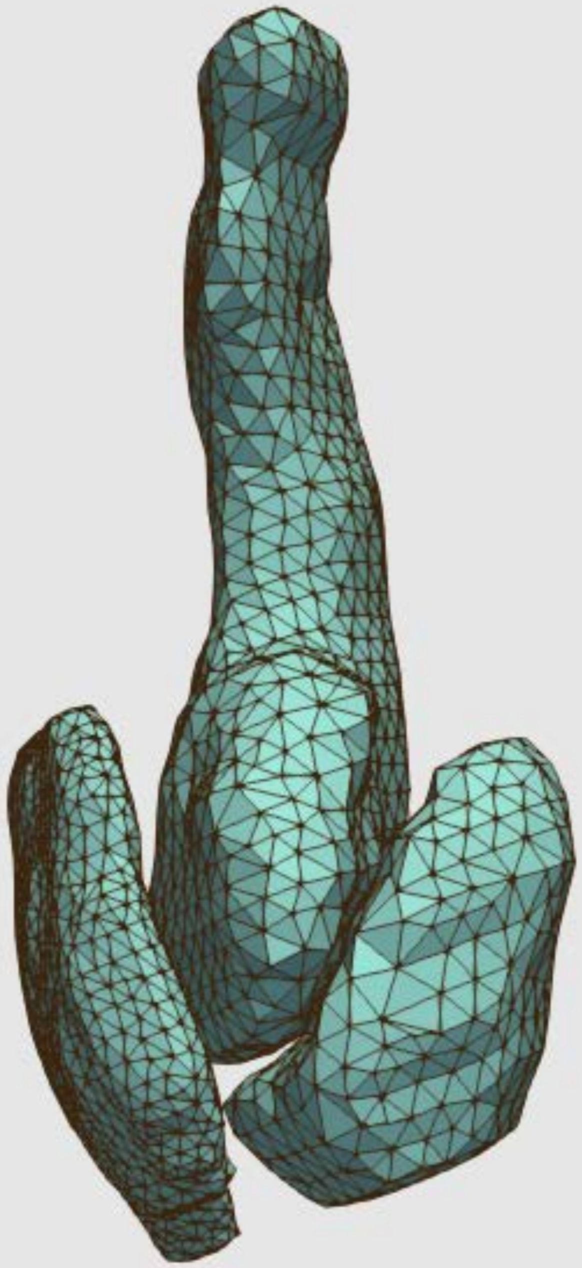
Entorhinal cortex

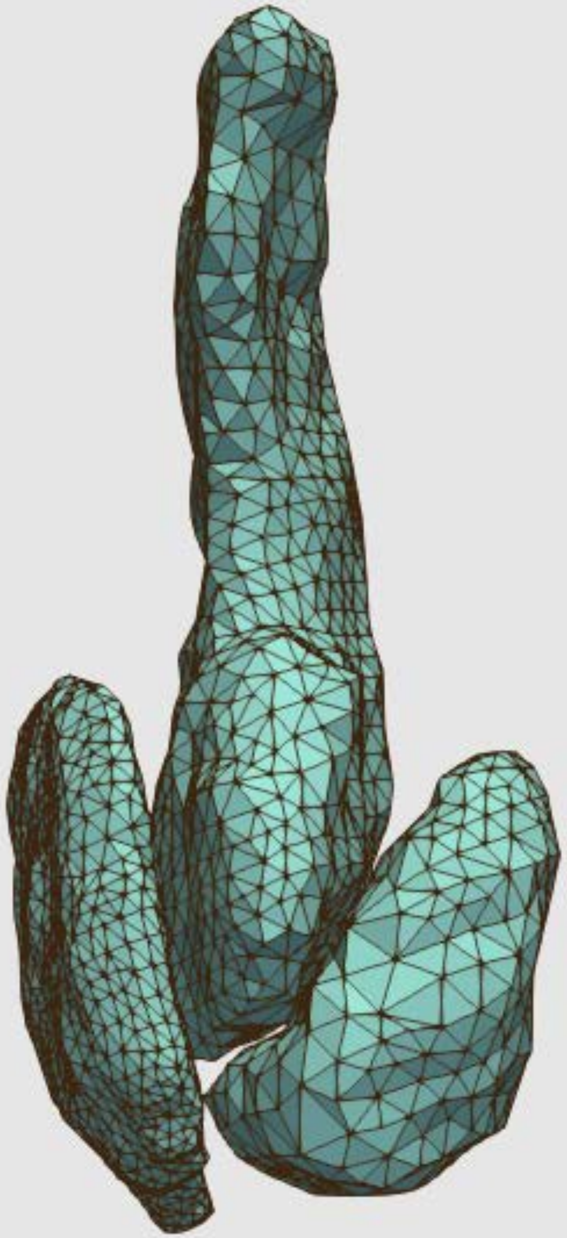

hippocampus
entorhinal cortex
amygdala

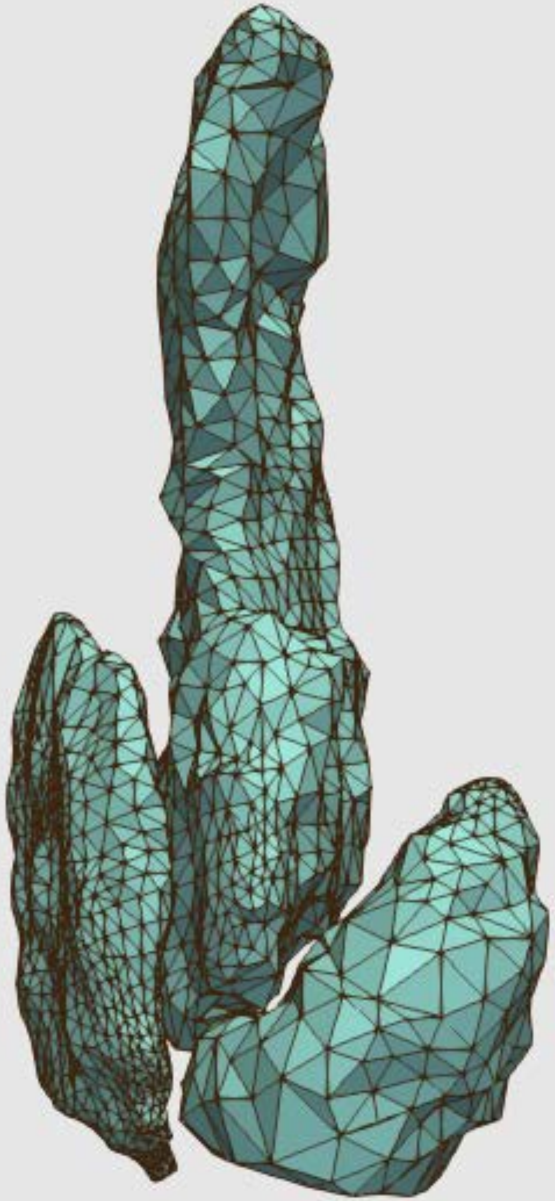


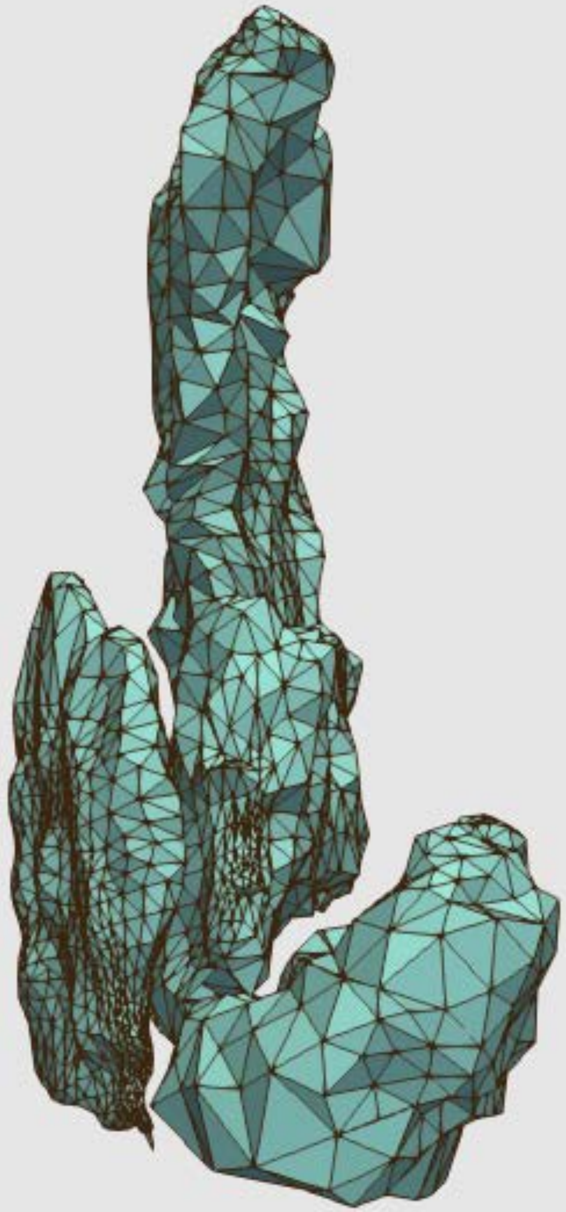
LDDMM

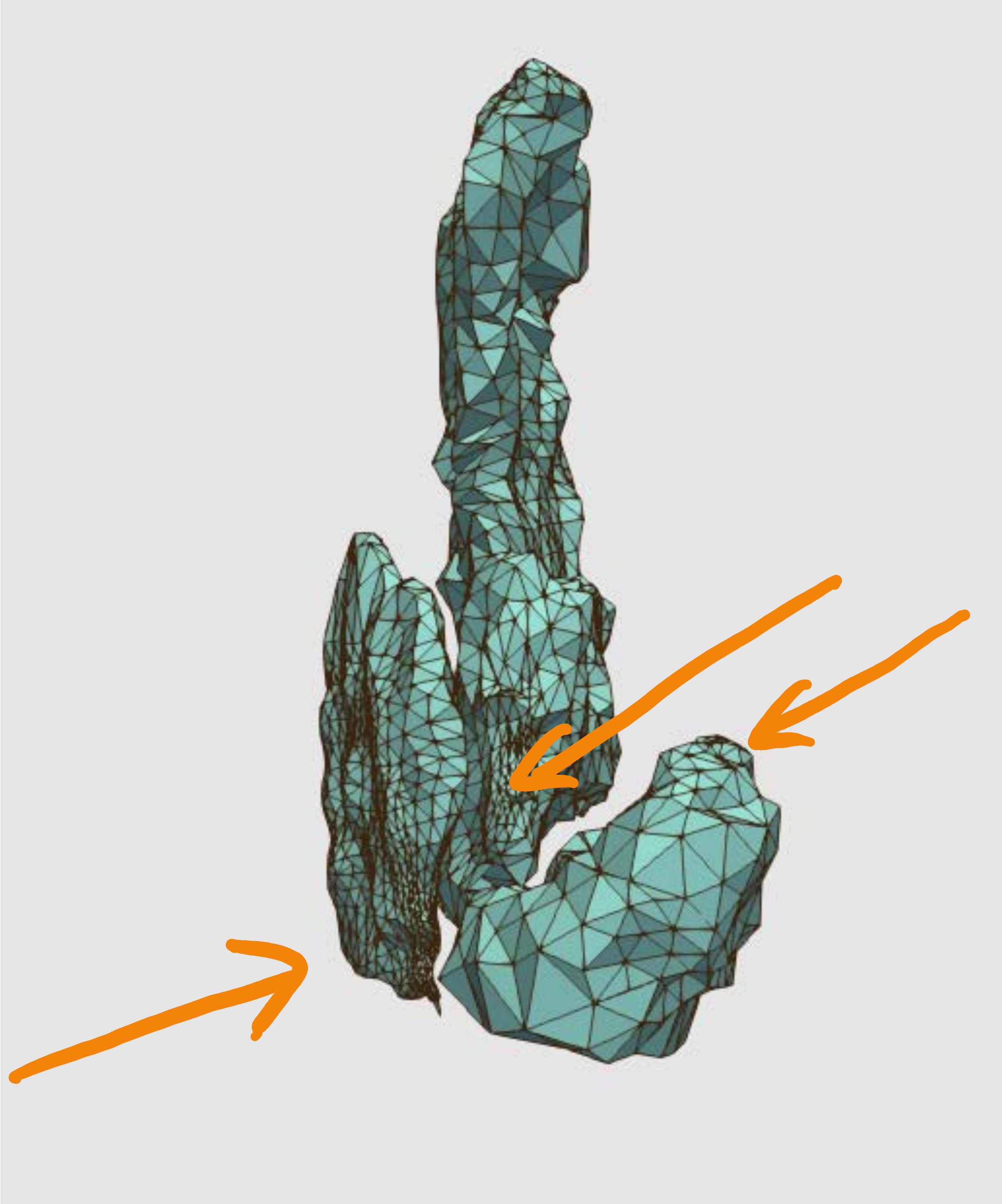
(small kernel width)



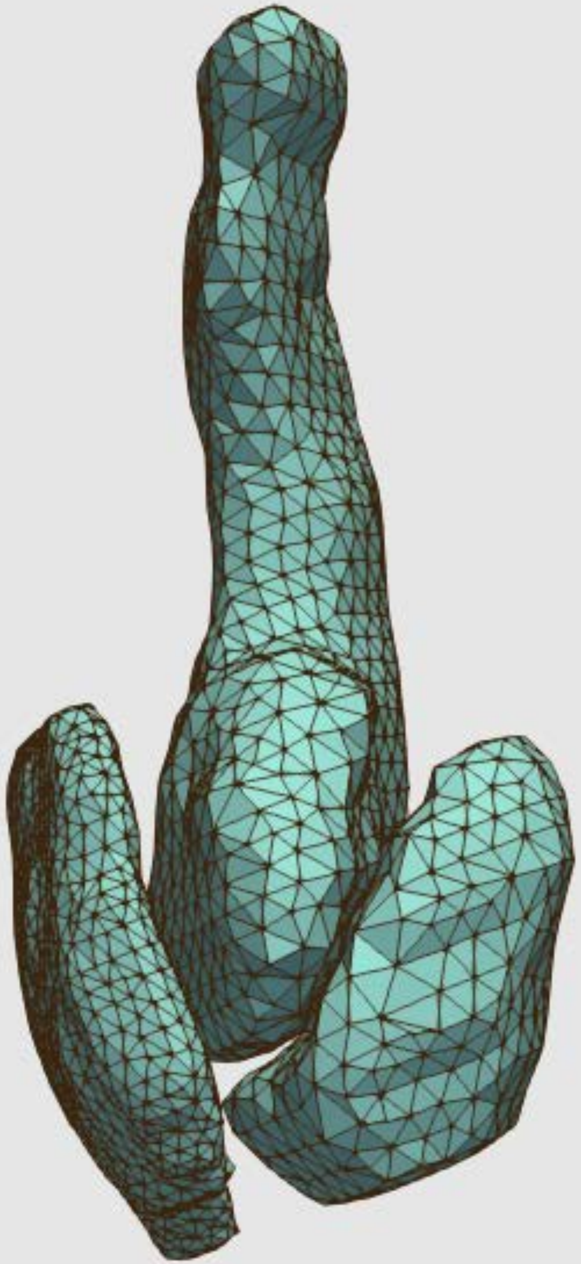


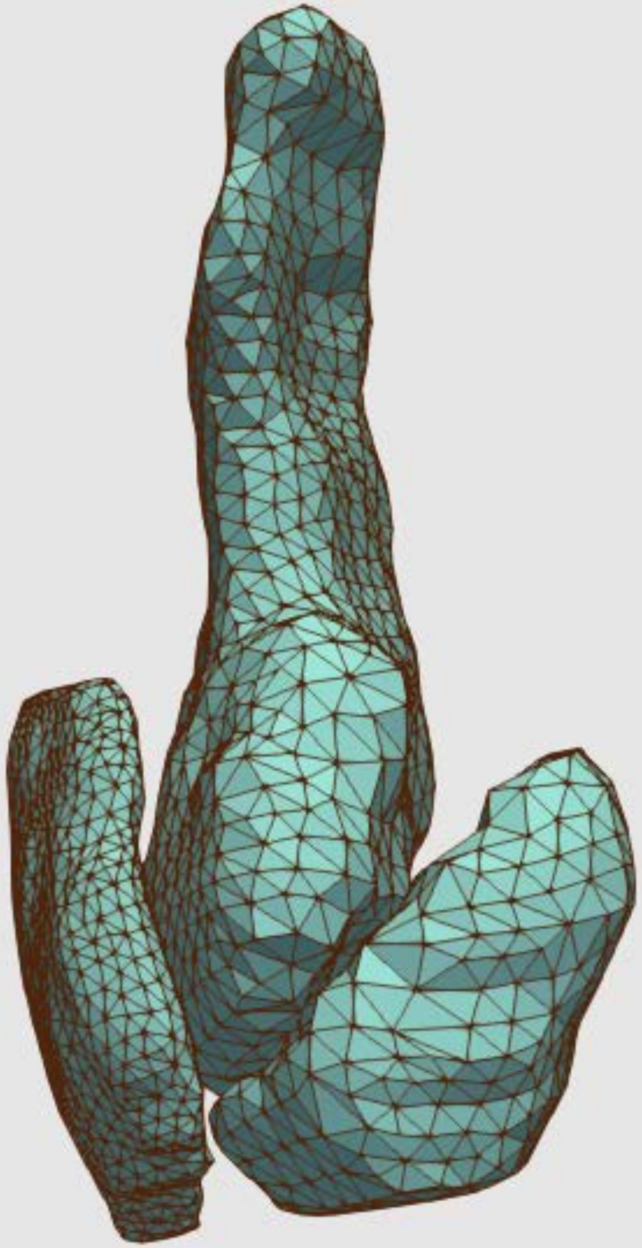


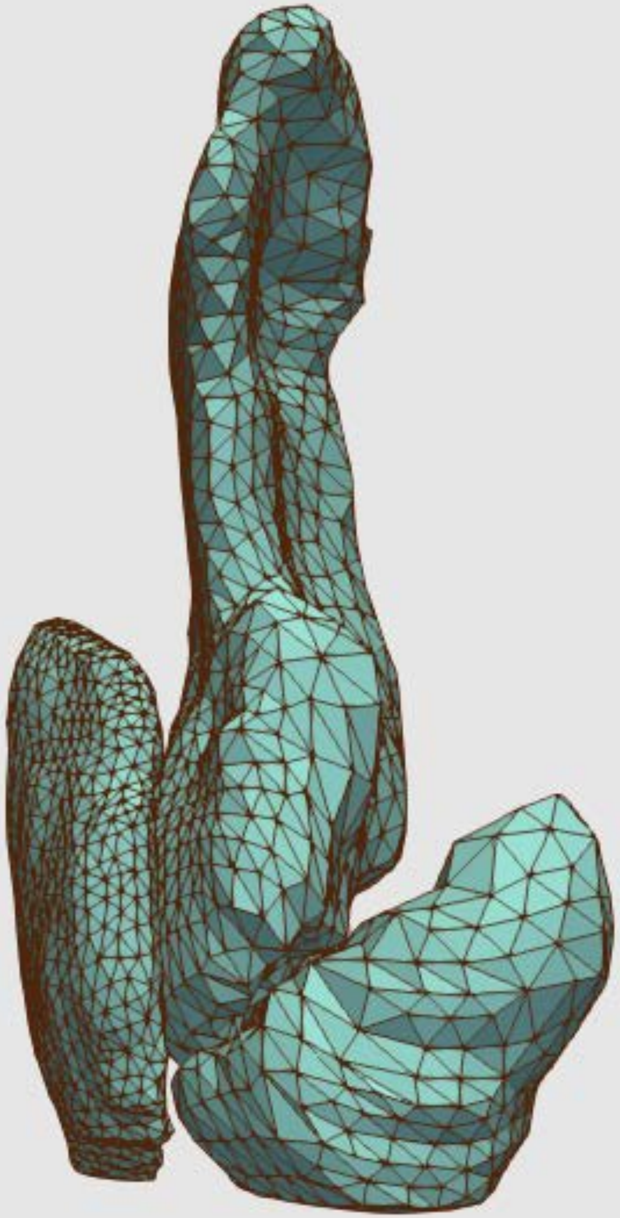


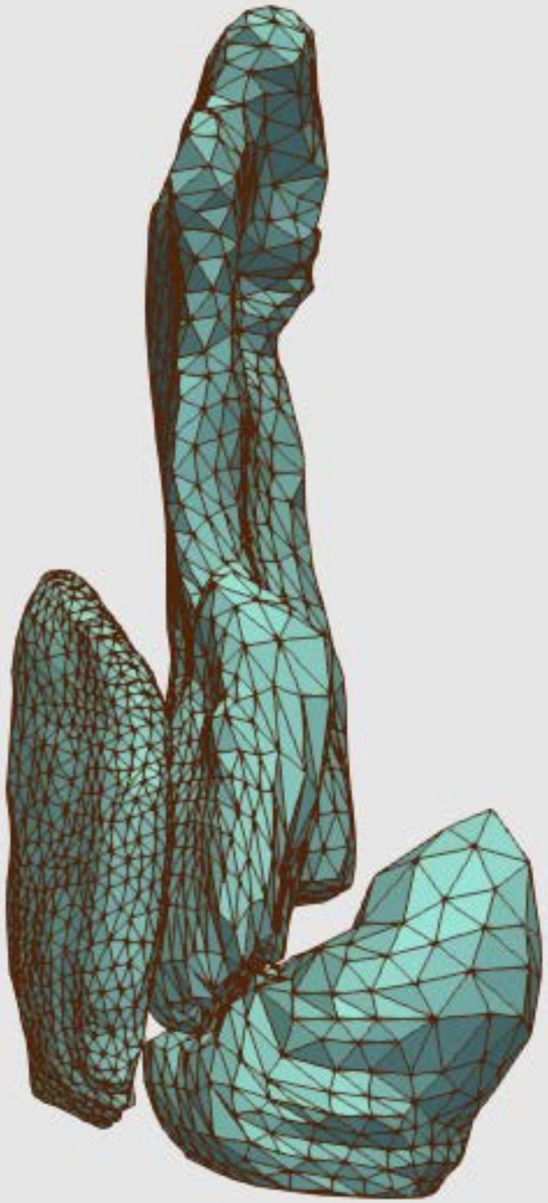


LDDMM
(larger kernel size)









Remark: When using LDDMM in computational anatomy, each structure is studied separately.

Deformation artifacts are avoided but relative structure positions are ignored.

(May be okay in that case.)

HLDDΠM

H'_{norm}