From unbalanced optimal transport to the Camassa-Holm equation

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Darryl's 70^{th} birthday, ICMAT Madrid

Darryl's 70^{th} birthday

Talk based on:

P1 *Unbalanced Optimal Transport: Geometry and Kantorovich formulation*, with L. Chizat, B. Schmitzer, G. Peyré. (2015)

P2 *From unbalanced optimal transport to the Camassa-Holm equation*, with T. Gallouet. (2016)

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A somewhat surprising result

Theorem

Solutions $u(t) \in C^\infty(S_1, \mathbb{R})$ to the Camassa-Holm equation

$$\partial_t u - \frac{1}{4}\partial_{txx} u + 3\partial_x u\, u - \frac{1}{2}\partial_{xx} u\, \partial_x u - \frac{1}{4}\partial_{xxx} u\, u = 0 \tag{1}$$

are particular solutions of an incompressible Euler equation on $R^2 \setminus \{0\}$ for a density $\rho(r,\theta) = \frac{1}{r^3}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{r^4}\,\mathrm{Leb}$

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p\,, \\ \nabla \cdot (\rho v) = 0\,. \end{cases} \tag{2}$$

The correspondence is given, on the Lagrangian flow, by

$$\mathcal{M}(\varphi) = \sqrt{\partial_x \varphi}\, e^{i\varphi}\,. \tag{3}$$

Arnold's remark on incompressible Euler

Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier, 1966.

Theorem

The incompressible Euler equation is the geodesic flow of the (right-invariant) L^2 Riemannian metric on $\mathrm{SDiff}(M)$ (volume preserving diffeomorphisms).

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Theorem

The incompressible Euler equation is the geodesic flow of the (right-invariant) L^2 *Riemannian metric on* SDiff(M) *(volume preserving diffeomorphisms).*

- An intrinsic point of view by Ebin and Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math., 1970. Short time existence results for smooth initial conditions.
- An extrinsic point of view by Brenier, relaxation of the variational problem, optimal transport, polar factorization.

Arnold's remark continued

The incompressible Euler equation on M (Eulerian form),

$$\begin{cases} \partial_t v(t,x) + v(t,x) \cdot \nabla v(t,x) = -\nabla p(t,x), \quad t > 0,\ x \in M\,, \\ \mathrm{div}(v) = 0\,, \\ v(0,x) = v_0(x)\,, \end{cases} \tag{4}$$

is the Euler-Lagrange equation for the action

$$\int_0^1 \int_M |v(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t\,, \tag{5}$$

under the flow constraint

$$\begin{aligned} &\partial_t \varphi(t,x) = v(t,\varphi(t,x))\,, \\ &\mathrm{div}(v) = 0\,. \end{aligned}$$

and time boundary value constraints:

$$\varphi(0,\cdot) = \varphi_0 \in \mathrm{SDiff}(M) \text{ and } \varphi(1,\cdot) = \varphi_1 \in \mathrm{SDiff}(M)\,. \tag{6}$$

Arnold's remark continued

Rewritten in terms of the flow φ, the action reads

$$\int_0^1 \int_M |\partial_t \varphi(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \,, \tag{7}$$

under the constraint

$$\varphi(t) \in \mathrm{SDiff}(M) \text{ for all } t \in [0,1] \,. \tag{8}$$

Arnold's remark continued

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Riemannian submanifold point of view:

Let $M \hookrightarrow \mathbb{R}^d$ be isometrically embedded: A smooth curve $c(t) \in M$ is a geodesic if and only if $\ddot{c} \perp T_c M$.

Incompressible Euler in Lagrangian form:

$$\begin{cases} \ddot{\varphi} = -\nabla p \circ \varphi \\ \varphi(t) \in \mathrm{SDiff}(M)\,. \end{cases} \tag{9}$$

About Brenier's approach to incompressible Euler

Variational approach to minimizing geodesics on $\mathrm{SDiff}(M)$ isometrically embedded in a Hilbert space.

- Projection onto $\mathrm{SDiff}(\mathbb{R}^d)$ leads to his polar factorization theorem:

Polar factorization, Y. Brenier 1991

Let $\psi \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ s.t. $\psi_*(\mathrm{Leb}) \ll \mathrm{Leb}$, then there exists a unique couple (p, φ) (up to cste on p) s.t.

$$\psi = \nabla p \circ \varphi \,, \tag{10}$$

and $\varphi_*(\mathrm{Leb}) = \mathrm{Leb}$ and p is a convex function. Moreover,

$$\|\psi - \varphi\|_{L^2} = \inf_f \{\|\psi - f\|_{L^2} \,:\, f_*(\mathrm{Leb}) = \mathrm{Leb}\} \tag{11}$$

- Smooth solutions of Euler are minimizing (for $t \in [0,1]$) if $\nabla^2 p$ is bounded in L^∞ (by π).
- In general, relaxation of the boundary value problem as (infinite) multimarginal optimal transport.

A geometric picture: Otto's Riemannian submersion

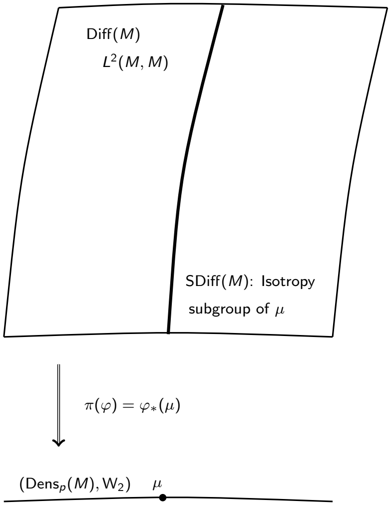

Figure – A Riemannian submersion: $\mathrm{SDiff}(M)$ as a Riemannian submanifold of $L^2(M, M)$: Incompressible Euler equation on $\mathrm{SDiff}(M)$

Reminders: Riemannian submersion

Let (M, g_M) and (N, g_N) be two Riemannian manifolds and $f : M \mapsto N$ a differentiable mapping.

Definition

The map f is a Riemannian submersion if f is a submersion and for any $x \in M$, the map $df_x : \mathrm{Ker}(df_x)^{\perp} \mapsto T_{f(x)}N$ is an isometry.

- $\mathrm{Vert}_x := \mathrm{Ker}(df(x))$ is the vertical space.
- $\mathrm{Hor}_x \overset{\text{def.}}{=} \mathrm{Ker}(df(x))^{\perp}$ is the horizontal space.
- Geodesics on N can be lifted "horizontally" to geodesics on M.

Theorem (O'Neill's formula)

Let f be a Riemannian submersion and X, Y be two orthonormal vector fields on M with horizontal lifts $\tilde{X}$ and $\tilde{Y}$, then

$$K_N(X, Y) = K_M(\tilde{X}, \tilde{Y}) + \frac{3}{4} \| \mathrm{vert}([\tilde{X}, \tilde{Y}]) \|_M^2 \,, \tag{12}$$

where K denotes the sectional curvature and vert *the orthogonal projection on the vertical space.*

A pre-formulation of the polar factorization

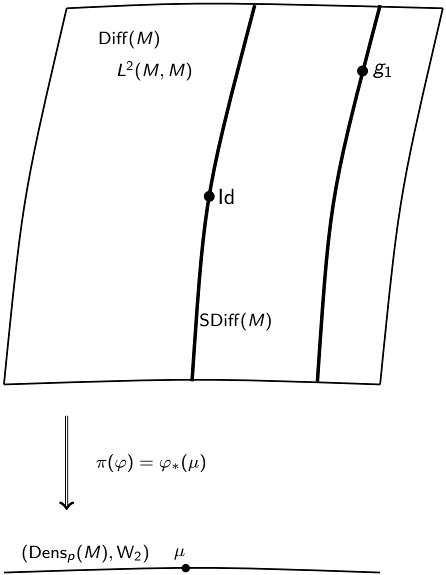

Figure – A Riemannian submersion: SDiff(M) as a Riemannian submanifold of $L^2(M, M)$: Incompressible Euler equation on SDiff(M)

A pre-formulation of the polar factorization

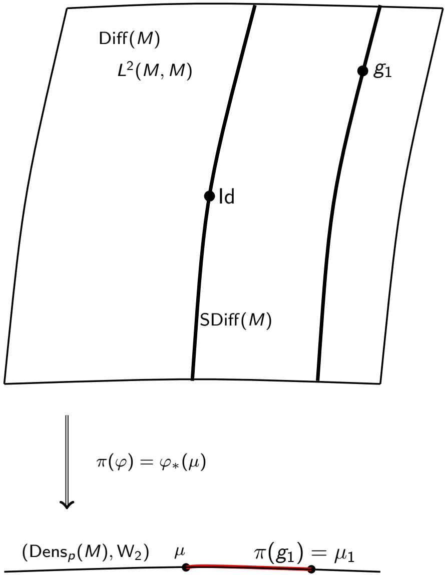

Figure – A pre polar factorization

A pre-formulation of the polar factorization

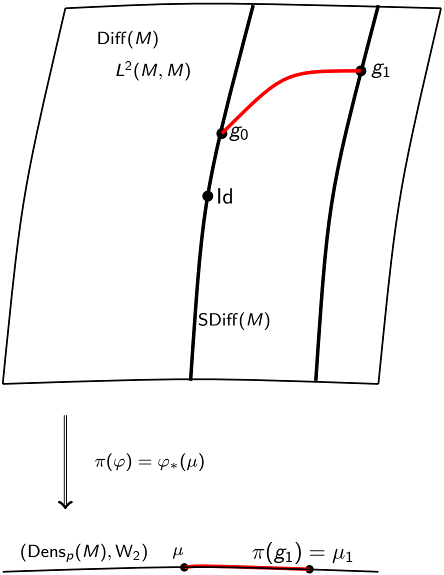

Figure – Polar factorization: $g_0 = \arg\min_{g \in \mathrm{SDiff}} \|g_1 - g\|_{L^2}$

Outline

Contents

Reminders: Static Formulation

Monge formulation (1781)

Let $\mu, \nu \in \mathcal{P}_+(M)$,

$$\textbf{Minimize} \int_M c(x, \varphi(x)) d\mu \tag{13}$$

among the map s.t. $\varphi_*(\mu) = \nu$.

Reminders: Static Formulation

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among the map s.t. $\varphi_*(\mu) = \nu$.

1. ill posed problem, the constraint may not be satisfied.
2. the constraint can hardly be made weakly closed.

$\rightarrow$ Relaxation of the Monge problem.

Reminders: Static Formulation

Kantorovich formulation (1942)

Let $\mu, \nu \in \mathcal{P}_+(M)$, define D by

$$D(\mu,\nu) = \inf_{\gamma \in \mathcal{P}(M^2)} \left\{ \int_{M^2} c(x,y)\,\mathrm{d}\gamma(x,y) : \pi^1_* \gamma = \mu \text{ and } \pi^2_* \gamma = \nu \right\}$$

1. Existence result: c lower semi-continuous and bounded from below.
2. Also valid in Polish spaces.
3. If $c(x,y) = \frac{1}{p}|x-y|^p$, $D^{1/p}$ is the Wasserstein distance denoted by W_p.

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Linear optimization problem and associated numerical methods. Recently introduced, entropic regularization. (C. Léonard, M. Cuturi, J.C. Zambrini <- Schrödinger)

Reminders: Dynamic formulation (Benamou-Brenier)

For geodesic costs, for instance $c(x,y) = \frac{1}{2}|x-y|^2$

$$\inf \mathcal{E}(v) = \frac{1}{2} \int_0^1 \int_M |v(x)|^2 \rho(x) \, \mathrm{d}x \, \mathrm{d}t \ , \tag{14}$$

s.t.

$$\begin{cases} \dot{\rho} + \nabla \cdot (v\rho) = 0 \\ \rho(0) = \mu_0 \text{ and } \rho(1) = \mu_1 \, . \end{cases} \tag{15}$$

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Convex reformulation: Change of variable: momentum $m = \rho v$,

$$\inf \mathcal{E}(m) = \frac{1}{2}\int_0^1 \int_M \frac{|m(x)|^2}{\rho(x)}\, \mathrm{d}x\, \mathrm{d}t \,, \tag{16}$$

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where $(\rho, m) \in \mathcal{M}([0,1] \times M, \mathbb{R} \times \mathbb{R}^d)$.

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where $(\rho, m) \in \mathcal{M}([0,1]\times M, \mathbb{R}\times\mathbb{R}^d)$.

Existence of minimizers: Fenchel-Rockafellar.
Numerics: First-order splitting algorithm: Douglas-Rachford.

Starting point and initial motivation

- Extend the Wasserstein L^2 distance to positive Radon measures.
- Develop associated numerical algorithms.

Possible applications: Imaging, machine learning, gradient flows, ...

Unbalanced optimal transport

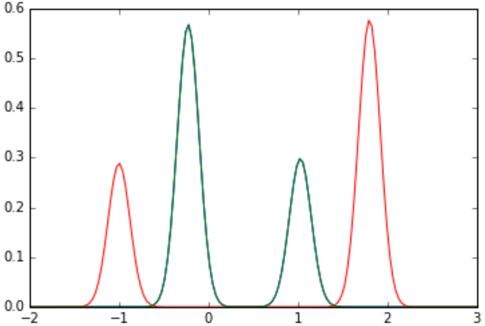

Figure – Optimal transport between bimodal densities

Unbalanced optimal transport

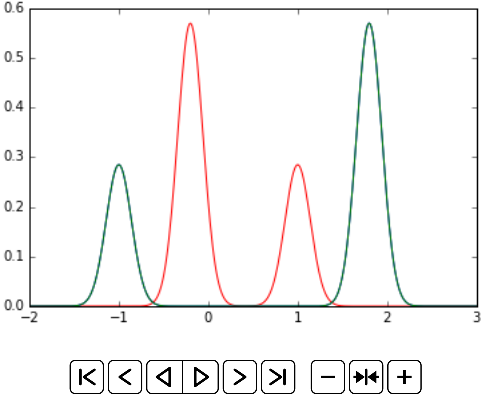

Figure – Another transformation

Bibliography before (june) 2015

Taking into account locally the change of mass:

Two directions: Static and dynamic.

- Static, Partial Optimal Transport [Figalli & Gigli, 2010]
- Static, Hanin 1992, Benamou and Brenier 2001.
- Dynamic, Numerics, Metamorphoses [Maas *et al.* , 2015]
- Dynamic, Numerics, Growth model [Lombardi & Maitre, 2013]
- Dynamic and static, [Piccoli & Rossi, 2013, Piccoli & Rossi, 2014]
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No equivalent of L^2 Wasserstein distance on positive Radon measures.

Bibliography after june 2015

More than 300 pages on the same model!

Starting point: Dynamic formulation

- Dynamic, Numerics, Imaging [Chizat *et al.* , 2015]
- Dynamic, Geometry and Static [Chizat *et al.* , 2015]
- Dynamic, Gradient flow [Kondratyev *et al.* , 2015]
- Dynamic, Gradient flow [Liero *et al.* , 2015b]
- Static and more [Liero *et al.* , 2015a]
- Optimal transport for contact forms [Rezakhanlou, 2015]
- Static relaxation of OT, machine learning [Frogner *et al.* , 2015]

Two possible directions

Pros and cons:

- Extend static formulation: Frogner et al.

$$\min \lambda KL(\mathrm{Proj}^1_* \gamma, \rho_1) + \lambda KL(\mathrm{Proj}^2_* \gamma, \rho_2) + \int_{M^2} \gamma(x,y) d(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \quad (18)$$

  Good for numerics, but is it a distance ?

- Extend dynamic formulation: on the tangent space of a density, choose a metric on the transverse direction. Built-in metric property but does there exist a static formulation ?

An extension of Benamou-Brenier formulation

Add a source term in the constraint: (weak sense)

$$\dot{\rho} = -\nabla \cdot (\rho v) + \alpha\rho\,,$$

where α can be understood as the growth rate.

$$\mathrm{WF}(m,\alpha)^2 = \frac{1}{2}\int_0^1 \int_M |v(x,t)|^2 \rho(x,t)\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta^2}{2}\int_0^1 \int_M \alpha(x,t)^2 \rho(x,t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

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where δ is a length parameter.
Remark: very natural and not studied before.

Convex reformulation

Add a source term in the constraint: (weak sense)

$$\dot{\rho} = -\nabla \cdot m + \mu \,.$$

The Wasserstein-Fisher-Rao metric:

$$\mathrm{WF}(m,\mu)^2 = \frac{1}{2}\int_0^1 \int_M \frac{|m(x,t)|^2}{\rho(x,t)} \,\mathrm{d}x\,\mathrm{d}t + \frac{\delta^2}{2}\int_0^1 \int_M \frac{\mu(x,t)^2}{\rho(x,t)} \,\mathrm{d}x\,\mathrm{d}t \,.$$

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- Fisher-Rao metric: Hessian of the Boltzmann entropy/ Kullback-Leibler divergence and reparametrization invariant. Wasserstein metric on the space of variances in 1D.
- Convex and 1-homogeneous: convex analysis (existence and more)
- Numerics: First-order splitting algorithm: Douglas-Rachford.
- Code available at `https://github.com/lchizat/optimal-transport/`

A general framework

Definition (Infinitesimal cost)

An infinitesimal cost is $f : M \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ such that for all $x \in M$, $f(x,\cdot,\cdot,\cdot)$ is convex, positively 1-homogeneous, lower semicontinuous and satisfies

$$f(x,\rho,m,\mu) \begin{cases} = 0 & \text{if } (m,\mu) = (0,0) \text{ and } \rho \geq 0 \\ > 0 & \text{if } |m| \text{ or } |\mu| > 0 \\ = +\infty & \text{if } \rho < 0\,. \end{cases}$$

Definition (Dynamic problem)

For $(\rho, m, \mu) \in \mathcal{M}([0,1] \times M)^{1+d+1}$, let

$$J(\rho,m,\mu) \stackrel{\text{def.}}{=} \int_0^1 \int_M f(x, \tfrac{\mathrm{d}\rho}{\mathrm{d}\lambda}, \tfrac{\mathrm{d}m}{\mathrm{d}\lambda}, \tfrac{\mathrm{d}\mu}{\mathrm{d}\lambda})\, \mathrm{d}\lambda(t,x) \tag{19}$$

The dynamic problem is, for $\rho_0, \rho_1 \in \mathcal{M}_+(M)$,

$$C(\rho_0,\rho_1) \stackrel{\text{def.}}{=} \inf_{(\rho,\omega,\zeta) \in \mathcal{CE}_0^1(\rho_0,\rho_1)} J(\rho,\omega,\zeta)\,. \tag{20}$$

Existence of minimizers

Proposition (Fenchel-Rockafellar)

Let $B(x)$ be the polar set of $f(x,\cdot,\cdot,\cdot)$ for all $x \in M$ and assume it is a lower semicontinuous set-valued function. Then the minimum of (20) is attained and it holds

$$C_D(\rho_0,\rho_1) = \sup_{\varphi \in K} \int_M \varphi(1,\cdot)\,\mathrm{d}\rho_1 - \int_M \varphi(0,\cdot)\,\mathrm{d}\rho_0 \qquad (21)$$

with $K \stackrel{\text{def.}}{=} \left\{\varphi \in C^1([0,1]\times M) : (\partial_t \varphi, \nabla\varphi, \varphi) \in B(x),\ \forall (t,x) \in [0,1]\times M\right\}$.

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$$\mathrm{WF}(x,y,z) = \begin{cases} \frac{|y|^2+\delta^2 z^2}{2x} & \text{if } x>0,\\ 0 & \text{if } (x,|y|,z)=(0,0,0)\\ +\infty & \text{otherwise}\end{cases}$$

and the corresponding Hamilton-Jacobi equation is

$$\partial_t\varphi + \frac{1}{2}\left(|\nabla\varphi|^2 + \frac{\varphi^2}{\delta^2}\right) \leq 0\,.$$

Numerical simulations

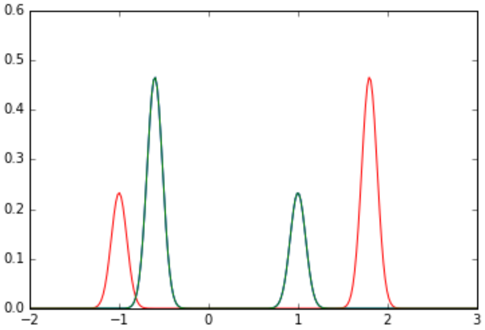

Figure – WFR geodesic between bimodal densities

Numerical simulations

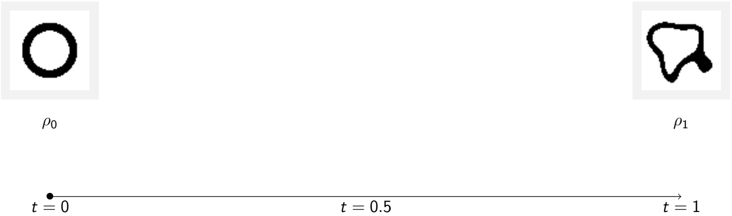

Figure – Geodesics between ρ_0 and ρ_1 for (1st row) Hellinger, (2nd row) W_2, (3rd row) partial OT, (4th row) WF.

An Interpolating Distance between Optimal Transport and Fisher-Rao, L. Chizat, B. Schmitzer, G. Peyré, and F.-X. Vialard, FoCM, 2016.

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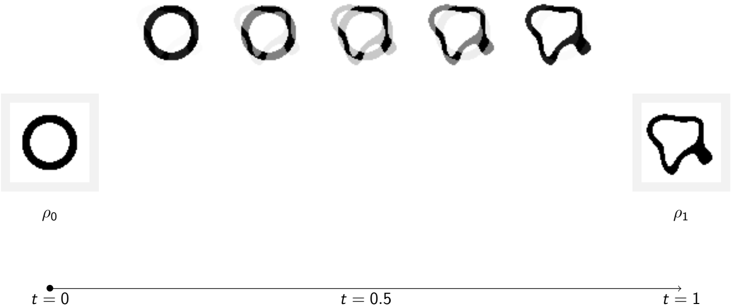

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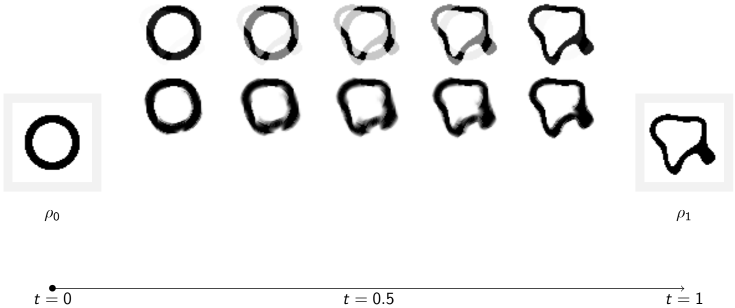

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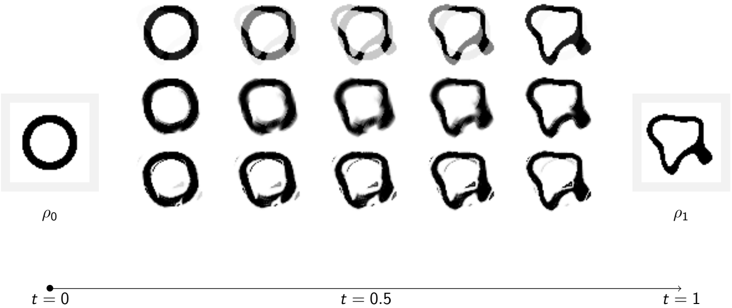

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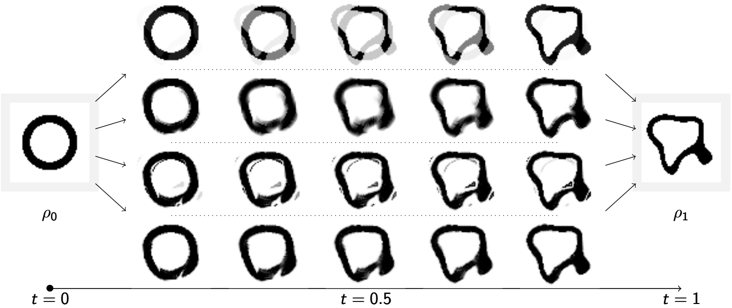

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From dynamic to static

Group action

Mass can be moved and changed: consider $m(t)\delta_{x(t)}$.

From dynamic to static

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Infinitesimal action

$$\dot{\rho} = -\nabla \cdot (v\rho) + \mu \Leftrightarrow \begin{cases} \dot{x}(t) = v(t, x(t)) \\ \dot{m}(t) = \mu(t, x(t)) \end{cases}$$

From dynamic to static

Group action

Mass can be moved and changed: consider $m(t)\delta_{x(t)}$.

Infinitesimal action

$$\dot{\rho} = -\nabla \cdot (v\rho) + \mu \Leftrightarrow \begin{cases} \dot{x}(t) = v(t, x(t)) \\ \dot{m}(t) = \mu(t, x(t)) \end{cases}$$

A cone metric

$$\mathrm{WF}^2(x, m)\left((\dot{x}, \dot{m}), (\dot{x}, \dot{m})\right) = \frac{1}{2}(m\dot{x}^2 + \frac{\dot{m}^2}{m}),$$

Change of variable: $r^2 = m$...

Riemannian cone

Definition

Let (M, g) be a Riemannian manifold. The cone over (M, g) is the Riemannian manifold $(M \times \mathbb{R}_+^*, r^2 g + \mathrm{d}r^2)$.

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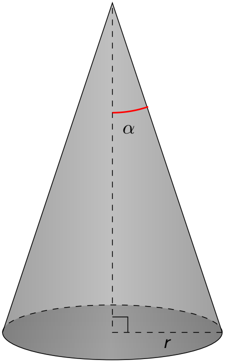


For $M = S_1(r)$, radius $r \leq 1$. One has $\sin(\alpha) = r$.

Geometry of a cone

- Change of variable: $\mathrm{WF}^2 = \frac{1}{2} r^2 g + 2\,\mathrm{d}r^2$.
- Non complete metric space: add the vertex $M \times \{0\}$.
- The distance:

$$d((x_1, m_1), (x_2, m_2))^2 = \\ m_2 + m_1 - 2\sqrt{m_1 m_2} \cos\left(\frac{1}{2} d_M(x_1, x_2) \wedge \pi\right) . \quad (22)$$

- Curvature tensor: $R(\tilde{X}, e) = 0$ and $R(\tilde{X}, \tilde{Y})\tilde{Z} = (R_g(X, Y)Z - g(Y, Z)X + g(X, Z)Y, 0)$.
- $M = \mathbb{R}$ then $(x, m) \mapsto \sqrt{m} e^{ix/2} \in \mathbb{C}$ local isometry.

Corollary

If (M, g) has sectional curvature greater than 1, then $(M \times \mathbb{R}_+^*, m\, g + \frac{1}{4m}\,\mathrm{d}m^2)$ has non-negative sectional curvature. For X, Y two orthornormal vector fields on M,

$$K(\tilde{X}, \tilde{Y}) = (K_g(X, Y) - 1) \quad (23)$$

where K and K_g denote respectively the sectional curvatures of $M \times \mathbb{R}_+^*$ and M.

Visualize geodesics for $r^2 g + \mathrm{d}r^2$

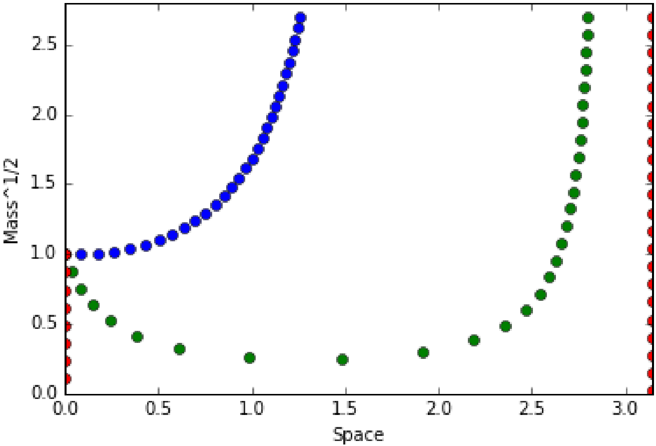
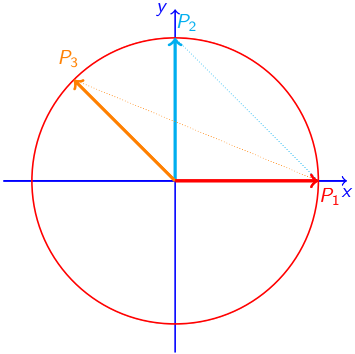


Figure – Geodesics on the cone

Distance between Diracs



$$\frac{1}{4} WF(m_1\delta_{x_1}, m_2\delta_{x_2})^2 = m_2 + m_1 - 2\sqrt{m_1 m_2} \cos\left(\frac{1}{2} d_M(x_1, x_2) \wedge \pi/2\right) .$$

Proof: prove that an explicit geodesic is a critical point of the convex functional.

Properties: positively 1-homogeneous and convex in (m_1, m_2).

Generalization of Otto's Riemannian submersion

Idea of a left group action:

$$\pi : \left(\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)\right) \times \mathrm{Dens}(M) \mapsto \mathrm{Dens}(M)$$
$$\pi\left((\varphi, \lambda), \rho\right) := \varphi_*(\lambda^2 \rho)$$

Group law:

$$(\varphi_1, \lambda_1) \cdot (\varphi_2, \lambda_2) = (\varphi_1 \circ \varphi_2, (\lambda_1 \circ \varphi_2)\lambda_2) \tag{24}$$

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Theorem (P1)

Let $\rho_0 \in \mathrm{Dens}(M)$ *and* $\pi_0 : \mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*) \mapsto \mathrm{Dens}(M)$ *defined by* $\pi_0(\varphi, \lambda) := \varphi_*(\lambda^2 \rho_0)$. *It is a Riemannian submersion*

$$(\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*), L^2(M, M \times \mathbb{R}_+^*)) \xrightarrow{\pi_0} (\mathrm{Dens}(M), \mathrm{WF})$$

(where $M \times \mathbb{R}_+^*$ *is endowed with the cone metric).*

O'Neill's formula: sectional curvature of $(\mathrm{Dens}(M), \mathrm{WF})$.

Horizontal lift

Proposition (Horizontal lift)

Let $\rho \in \mathrm{Dens}^s(M)$ *be a smooth density and* $X_\rho \in H^s(M, \mathbb{R})$ *be a tangent vector at the density* ρ*. The horizontal lift at* $(Id, 1)$ *of* X_ρ *is given by* $(\frac{1}{2}\nabla Z, Z)$ *where* Z *is the solution to the elliptic partial differential equation:*

$$-\operatorname{div}(\rho \nabla Z) + 2Z\rho = X_\rho \,. \tag{25}$$

By elliptic regularity, the unique solution Z *belongs to* $H^{s+2}(M)$*.*

Geometric consequence

The sectional curvature of Dens(M) at point ρ is: (Z being the horizontal lift)

$$K(\rho)(X_1, X_2) = \int_M k(x,1)(Z_1(x), Z_2(x)) w(Z_1(x), Z_2(x)) \rho(x) \, \mathrm{d}\nu(x) + \frac{3}{4} \left\| [Z_1, Z_2]^V \right\|^2 \quad (26)$$

where

$$w(Z_1(x), Z_2(x)) = g(x)(Z_1(x), Z_1(x)) g(x)(Z_2(x), Z_2(x)) - g(x)(Z_1(x), Z_2(x))^2$$

and $[Z_1, Z_2]^V$ denotes the vertical projection of $[Z_1, Z_2]$ at identity and $\| \cdot \|$ denotes the norm at identity.

Geometric consequence

The sectional curvature of $\mathrm{Dens}(M)$ at point ρ is: (Z being the horizontal lift)

$$K(\rho)(X_1, X_2) = \int_M k(x,1)(Z_1(x), Z_2(x))w(Z_1(x), Z_2(x))\rho(x)\,\mathrm{d}\nu(x) + \frac{3}{4}\left\|[Z_1, Z_2]^V\right\|^2 \quad (26)$$

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and $[Z_1, Z_2]^V$ denotes the vertical projection of $[Z_1, Z_2]$ at identity and $\|\cdot\|$ denotes the norm at identity.

Corollary

Let (M, g) be a compact Riemannian manifold of sectional curvature bounded below by 1*, then the sectional curvature of* $(\mathrm{Dens}(M), \mathrm{WF})$ *is non-negative.*

Consequences

Monge formulation

$$WF(\rho_0,\rho_1) = \inf_{(\varphi,\lambda)} \left\{ \|(\varphi,\lambda) - (Id,1)\|_{L^2(\rho_0)} \,:\, \varphi_*(\lambda^2 \rho_0) = \rho_1 \right\} \quad (27)$$

Under existence and smoothness of the minimizer, there exists a function $p \in C^\infty(M,\mathbb{R})$ such that

$$(\varphi(x),\lambda(x)) = \exp_x^{\mathcal{C}(M)}\left(\frac{1}{2}\nabla p(x), p(x)\right)\,, \quad (28)$$

Equivalent to Monge-Ampère equation

With $z \overset{\text{def.}}{=} \log(1+p)$ one has

$$(1+|\nabla z|^2)e^{2z}\rho_0 = \det(D\varphi)\rho_1 \circ \varphi \quad (29)$$

and

$$\varphi(x) = \exp^M_{(x,1)}\left(\arctan\left(\frac{1}{2}|\nabla z|\right)\frac{\nabla z(x)}{|\nabla z(x)|}\right)\,.$$

Equivalence static/dynamic

Definition

The path-based cost c_s is

$$c_s(x_0, m_0, x_1, m_1) \stackrel{\text{def.}}{=} \inf_{(x(t),m(t))} \int_0^1 f(x(t), m(t), m(t)\, x'(t), m'(t)) \, \mathrm{d}t \tag{30}$$

for $(x(t), m(t)) \in C^1([0,1], \Omega \times [0, +\infty[)$ such that $(x(i), m(i)) = (x_i, m_i)$ for $i \in \{0, 1\}$.

Consequence: $c_d \leq c_s$.

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Consequence: $c_d \leq c_s$.

Theorem

If C_K weak continuous and c_d l.s.c. then $c_d = c_s^{**}$ and $C_K = C_D$.*

Kantorovich formulation

Recall

$$\frac{1}{4}c_d^2(x_1, m_1, x_2, m_2) = m_2 + m_1 - 2\sqrt{m_1 m_2}\cos\left(\frac{1}{2}d_M(x_1, x_2) \wedge \pi/2\right).$$

then

$$WF(\rho_1, \rho_2)^2 = \inf_{(\gamma_1,\gamma_2)\in\Gamma(\rho_1,\rho_2)} \int_{M^2} c_d^2\left((x, \frac{\mathrm{d}\gamma_1}{\mathrm{d}\gamma}), (y, \frac{\mathrm{d}\gamma_2}{\mathrm{d}\gamma})\right) \mathrm{d}\gamma(x, y),$$

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Theorem (Dual formulation)

$$WF^2(\rho_0, \rho_1) = \sup_{(\phi,\psi)\in C(M)^2} \int_M \phi(x)\,\mathrm{d}\rho_0 + \int_M \psi(y)\,\mathrm{d}\rho_1$$

subject to $\forall (x, y) \in M^2$,

$$\begin{cases} \phi(x) \leq 1\,, \quad \psi(y) \leq 1\,, \\ (1 - \phi(x))(1 - \psi(y)) \geq \cos^2(|x - y|/2 \wedge \pi/2) \end{cases}$$

A relaxed static OT formulation

Define

$$KL(\gamma,\nu) = \int \frac{\mathrm{d}\gamma}{\mathrm{d}\nu} \log\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\nu}\right) \mathrm{d}\nu + |\nu| - |\gamma|$$

Theorem (Dual formulation, P1)

$$WF^2(\rho_0,\rho_1) = \sup_{(\phi,\psi)\in C(M)^2} \int_M \phi(x)\,\mathrm{d}\rho_0 + \int_M \psi(y)\,\mathrm{d}\rho_1$$

subject to $\forall (x,y) \in M^2$, $\phi(x) \leq 1\,, \quad \psi(y) \leq 1$ *and*

$$(1-\phi(x))(1-\psi(y)) \geq \cos^2\left(|x-y|/2 \wedge \pi/2\right)$$

The corresponding primal formulation

$$WF^2(\rho_1,\rho_2) = \inf_{\gamma} KL(\mathrm{Proj}^1_*\,\gamma,\rho_1) + KL(\mathrm{Proj}^2_*\,\gamma,\rho_2)$$
$$- \int_{M^2} \gamma(x,y)\log(\cos^2(d(x,y)/2 \wedge \pi/2))\,\mathrm{d}x\,\mathrm{d}y$$

Theorem (P2)

On a Riemannian manifold (compact without boundary), the static and dynamic formulations are equal.

New algorithm

Scaling Algorithms for Unbalanced Transport Problems, L. Chizat, G. Peyré, B. Schmitzer, F.-X. Vialard.

- Use of entropic regularization.

$$WF^2(\rho_1, \rho_2) = \inf_{\gamma} KL(\mathrm{Proj}^1_* \gamma, \rho_1) + KL(\mathrm{Proj}^2_* \gamma, \rho_2)$$
$$- \int_{M^2} \gamma(x,y) \log(\cos^2(d(x,y)/2 \wedge \pi/2)) \,\mathrm{d}x\,\mathrm{d}y + \varepsilon KL(\gamma, \mu_0)\,.$$

- Alternate projection algorithm (contraction for a Hilbert type metric).
- Applications to color transfer, Fréchet-Karcher mean (barycenters).
- Similarity measure in inverse problems. (Optimal transport for diffeomorphic registration, MICCAI 2017).
- Simulations for gradient flows.

Application to color transfer

Figure – Transporting the color histograms: initial and final image

Optimal transport

Range constraint

Kullback-Leibler

Total variation

Contents

The Riemannian submersion for WFR

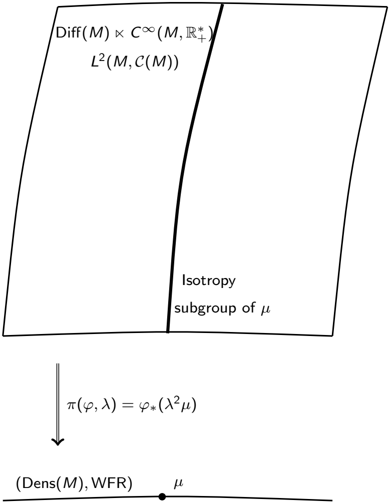

Figure – The same picture in our case: what is the corresponding equation to Euler?

The isotropy subgroup for unbalanced optimal transport

Recall that

$$\pi_0^{-1}(\{\rho_0\}) = \{(\varphi, \lambda) \in \mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*) \; : \; \varphi_*(\lambda^2 \rho_0) = \rho_0\}$$

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$$\pi_0^{-1}(\{\rho_0\}) = \{(\varphi, \sqrt{\mathrm{Jac}(\varphi)}) \in \mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*) \, : \, \varphi \in \mathrm{Diff}(M)\}.$$

The vertical space is

$$\mathrm{Vert}_{(\varphi,\lambda)} = \{(v, \alpha) \circ (\varphi, \lambda) \, ; \, \mathrm{div}(\rho v) = 2\alpha\rho\} \, , \qquad (31)$$

where $(v, \alpha) \in \mathrm{Vect}(M) \times C^\infty(M, \mathbb{R})$. The horizontal space is

$$\mathrm{Hor}_{(\varphi,\lambda)} = \left\{ \left(\frac{1}{2} \nabla p, p \right) \circ (\varphi, \lambda) \, ; \, p \in C^\infty(M, \mathbb{R}) \right\} . \qquad (32)$$

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The induced metric is

$$G(v, \mathrm{div}\, v) = \int_M |v|^2 \,\mathrm{d}\mu + \frac{1}{4} \int_M |\,\mathrm{div}\, v|^2 \,\mathrm{d}\mu\,. \tag{33}$$

The H^{div} right-invariant metric on the group of diffeomorphisms.

Contents

Right-invariant metric on a Lie group

Definition (Right-invariant metric)

Let $g_1, g_2 \in G$ be two group elements, the distance between g_1 and g_2 can be defined by:

$$d^2(g_1, g_2) = \inf_{g(t)} \left\{ \int_0^1 \|v(t)\|_{\mathfrak{g}}^2 \, dt \, | g(0) = g_0 \text{ and } g(1) = g_1 \right\}$$

where $\partial_t g(t) g(t)^{-1} = v(t) \in \mathfrak{g}$, with $\mathfrak{g}$ the Lie algebra.

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where $\partial_t g(t) g(t)^{-1} = v(t) \in \mathfrak{g}$, with $\mathfrak{g}$ the Lie algebra.

Right-invariance means:

$$d^2(g_1 g, g_2 g) = d(g_1, g_2) \, .$$

It comes from:

$$\partial_t (g(t) g_0)(g(t) g_0)^{-1} = \partial_t g(t) g_0 g_0^{-1} g(t)^{-1} = \partial_t g(t) g(t)^{-1} \, .$$

Euler-Arnold-Poincaré equation

Compute the Euler-Lagrange equation of the distance functional:

$$\frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}} = 0$$

Euler-Arnold-Poincaré equation

Compute the Euler-Lagrange equation of the distance functional:

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In the case of $\int_0^1 L(g, \dot{g})dt = \int_0^1 \|u\|^2 dt$,
Euler-Poincaré-Arnold equation

$$\begin{cases} \dot{g} = u \circ g \\ \dot{u} + \mathrm{ad}_u^* u = 0 \end{cases} \tag{34}$$

where ad_u^* is the (metric) adjoint of $\mathrm{ad}_u v = [v, u]$.

Proof.

Compute variations of $v(t)$ in terms of $u(t) = \delta g(t) g(t)^{-1}$. Find that admissible variations on $\mathfrak{g}$ can be written as:
$\delta v(t) = \dot{u} - \mathrm{ad}_v u$ for any u vanishing at 0 and 1. □

Fluid dynamics examples of Euler-Arnold equations

- Incompressible Euler equation.
- Korteweg-de-Vries equation.
- Camassa-Holm equation 1981/1993. *An integrable shallow water equation with peaked solitons*

Consider $\mathrm{Diff}(S_1)$ endowed with the H^1 right-invariant metric $\|v\|_{L^2}^2 + \frac{1}{4}\|\partial_x v\|_{L^2}^2$. One has

$$\begin{cases} \partial_t u - \frac{1}{4}\partial_{txx} u\, u + 3\partial_x u\, u - \frac{1}{2}\partial_{xx} u\, \partial_x u - \frac{1}{4}\partial_{xxx} u\, u = 0 \\ \partial_t \varphi(t,x) = u(t, \varphi(t,x))\,. \end{cases} \quad (35)$$

- Model for waves in shallow water.
- Completely integrable system (bi-Hamiltonian).
- Exhibits particular solutions named as peakons. (geodesics as collective Hamiltonian).
- Blow-up of solutions which gives a model for wave breaking.

Ebin-Marsden and Michor-Mumford

Rewrite the metric in Lagrangian coordinates φ and a tangent vector X_φ and realize that it is smooth...

- The right-invariant H^{div} metric:

$$G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi \circ \varphi^{-1}|^2 + b^2 \operatorname{div}(X_\varphi \circ \varphi^{-1})^2 \,\mathrm{d}\mu\,. \quad (36)$$

Smooth weak metric on an infinite dimensional Riemannian manifold when $M = S_1$.

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Smooth weak metric on an infinite dimensional Riemannian manifold when $M = S_1$. Consequences:

- Geodesic equations is a simple ODE (No need for a Riemannian connection)
- Gauss lemma on H^s for $s > d/2 + 2$
- Geodesics are minimizing within H^s topology.

Theorem (Consequence of Ebin and Marsden)

Local well-posedness of the geodesics for the $H^1(S_1)$ right-invariant metric on $\mathrm{Diff}^s(S_1)$ for $s > 1/2 + 2$.

Theorem (Michor-Mumford)

Local well-posedness of the geodesics for the H^{div} right-invariant metric on $\mathrm{Diff}^s(\mathbb{R}^d)$ for s high enough.

Metric properties

Theorem (Michor and Mumford, 2005)

The distance on $\mathrm{Diff}(M)$ *endowed with the right-invariant metric* L^2 *is degenerate; i.e.* $d(\varphi_0, \varphi_1) = 0$ *for every* $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$.

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Theorem (Michor and Mumford, 2005)

The distance on $\mathrm{Diff}(M)$ *endowed with the right-invariant metric* H^{Div} *is non degenerate.*

Proof.

Direct using the isometric injection. □

An isometric embedding

We have

$$\begin{aligned} \mathrm{inj} : (\mathrm{Diff}(M), H^{\mathrm{div}}) &\hookrightarrow L^2(M, \mathcal{C}(M)) \\ \varphi &\mapsto (\varphi, \sqrt{\mathrm{Jac}(\varphi)})\,. \end{aligned}$$

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The geodesic equations can be written in Lagrangian coordinates

$$\begin{cases} \frac{D}{Dt}\dot{\varphi} + 2\frac{\dot{\lambda}}{\lambda}\dot{\varphi} = -\nabla^g P \circ \varphi \\ \ddot{\lambda} r - \lambda r g(\dot{\varphi}, \dot{\varphi}) = -2\lambda r P \circ \varphi\,. \end{cases} \tag{37}$$

In Eulerian coordinates,

$$\begin{cases} \dot{v} + \nabla^g_v v + 2v\alpha = -\nabla^g P \\ \dot{\alpha} + \langle \nabla \alpha, v \rangle + \alpha^2 - g(v, v) = -2P\,, \end{cases} \tag{38}$$

where $\alpha = \frac{\dot{\lambda}}{\lambda} \circ \varphi^{-1}$ and $v = \partial_t \varphi \circ \varphi^{-1}$.

Consequences of the isometric embedding

$$(\mathrm{Diff}(M), H^{\mathrm{div}}) \hookrightarrow L^2(M, \mathcal{C}(M)) \qquad (39)$$

Consequences of the isometric embedding

$$(\mathrm{Diff}(M), H^{\mathrm{div}}) \hookrightarrow L^2(M, \mathcal{C}(M)) \tag{39}$$

1. Using Gauss-Codazzi formula, it generalizes a curvature formula by Khesin et al. obtained on $\mathrm{Diff}(S_1)$.
2. Smooth geodesics are length minimizing for a short enough time under mild conditions (generalization of Brenier's proof).
3. The Camassa-Holm equation as incompressible Euler.
4. A new polar factorization theorem.

In short:

Gain w.r.t. Ebin and Marsden

- Ebin and Marsden proved that: *Smooth solutions are minimizing in a* $H^{d/2+2+\varepsilon}$ *neighborhood.*
- We have: *Smooth solutions are minimizing in a* $W^{1,\infty}$ *neighborhood.*

In short:

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- Ebin and Marsden proved that: *Smooth solutions are minimizing in a $H^{d/2+2+\varepsilon}$ neighborhood.*
- We have: *Smooth solutions are minimizing in a $W^{1,\infty}$ neighborhood.*

Theorem (P2)

When $M = S_1$, smooth solutions to the Camassa-Holm equation

$$\begin{cases} \partial_t u - \frac{1}{4}\partial_{txx} u + 3\partial_x u\, u - \frac{1}{2}\partial_{xx} u\, \partial_x u - \frac{1}{4}\partial_{xxx} u\, u = 0 \\ \partial_t \varphi(t,x) = u(t,\varphi(t,x))\,. \end{cases} \tag{40}$$

are length minimizing for short times.

Generalisation of Brenier's proof

Theorem (P2)

Let $(\varphi(t), r(t))$ be a smooth solution to the geodesic equations on the time interval $[t_0, t_1]$. If $(t_1 - t_0)^2 \langle w, \nabla^2 \Psi_{P(t)}(x, r) w \rangle < \pi^2 \|w\|^2$ holds for all $t \in [t_0, t_1]$ and $(x, r) \in \mathcal{C}(M)$ and $w \in T_{(x,r)}\mathcal{C}(M)$, then for every smooth curve $(\varphi_0(t), r_0(t)) \in \mathrm{Aut}_{\mathrm{vol}}(\mathcal{C}(M))$ satisfying $(\varphi_0(t_i), r_0(t_i)) = (\varphi(t_i), r(t_i))$ for $i = 0, 1$ and the condition $(*)$, one has

$$\int_{t_0}^{t_1} \|(\dot{\varphi}, \dot{r})\|^2 \, \mathrm{d}t \leq \int_{t_0}^{t_1} \|(\dot{\varphi}_0, \dot{r}_0)\|^2 \, \mathrm{d}t \,, \tag{41}$$

with equality if and only if the two paths coincide on $[t_0, t_1]$.

Define $\delta_0 \stackrel{\text{def.}}{=} \min\{r(x, t) \, : \,$ injectivity radius at $(\varphi(t, x), r(t, x))\}$, then the condition $(*)$ is:

1. *If the sectional curvature of $\mathcal{C}(M)$ can assume both signs or if $\mathrm{diam}(M) \geq \pi$, there exists δ satisfying $0 < \delta < \delta_0$ such that the curve $(\varphi_0(t), r_0(t))$ has to belong to a δ-neighborhood of $(\varphi(t), r(t))$, namely*

   $$d_{\mathcal{C}(M)}\left((\varphi_0(t, x), r_0(t, x)), (\varphi(t, x), r(t, x))\right) \leq \delta$$

   for all $(x, t) \in M \times [t_0, t_1]$ where $d_{\mathcal{C}(M)}$ is the distance on the cone.
2. *If $\mathcal{C}(M)$ has non positive sectional curvature, then, for every δ as above, there exists a short enough time interval on which the geodesic will be length minimizing.*
3. *If $M = S_d(1)$, the result is valid for every path $(\dot{\varphi}_0, \dot{r}_0)$.*

Contents

Toward the incompressible Euler equation

Why? Unbalanced OT is linked to standard OT on the cone.

Toward the incompressible Euler equation

Why? Unbalanced OT is linked to standard OT on the cone.

Question

Understand $\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$ *as a subgroup of* $\mathrm{Diff}(\mathcal{C}(M))$*?*

Answer

The cone $\mathcal{C}(M)$ is a trivial principal fibre bundle over M.
The automorphism group $\mathrm{Aut}(\mathcal{C}(M)) \subset \mathrm{Diff}(\mathcal{C}(M))$ can be identified with $\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$. One has $(\varphi, \lambda) : (x, r) \mapsto (\varphi(x), \lambda(x) r)$.

Recall that $\psi \in \mathrm{Aut}(\mathcal{C}(M))$ if $\psi \in \mathrm{Diff}(\mathcal{C}(M))$ and $\forall \lambda \in \mathbb{R}_+^*$ one has $\psi(\lambda \cdot (x, r)) = \lambda \cdot \psi(x, r)$ where $\lambda \cdot (x, r) \stackrel{\text{def.}}{=} (x, \lambda r)$.

CH as an incompressible Euler equation

The geodesic equation on $\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$

$$\begin{cases} \frac{D}{Dt}\dot{\varphi} + 2\frac{\dot{\lambda}}{\lambda}\dot{\varphi} = -\nabla^g P \circ \varphi \\ \ddot{\lambda} r - \lambda r g(\dot{\varphi}, \dot{\varphi}) = -2\lambda r P \circ \varphi \,. \end{cases} \tag{42}$$

can be extended to $\mathrm{Aut}(\mathcal{C}(M))$ as

$$\frac{D}{Dt}(\dot{\varphi}, \dot{\lambda r}) = -\nabla \Psi_P \circ (\varphi, \lambda r)\,, \tag{43}$$

where $\Psi_P(x, r) \stackrel{\text{def.}}{=} r^2 P(x)$.

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Question

Does there exist a density $\tilde{\mu}$ on the cone such that $\mathrm{inj}(\mathrm{Diff}(M)) \subset \mathrm{SDiff}_{\tilde{\mu}}(\mathcal{C}(M))$*? (answer: yes)*

CH as an incompressible Euler equation

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Proof.

The measure $\tilde{\mu} \stackrel{\text{def.}}{=} r^{-3}\,\mathrm{d}r\,\mathrm{d}\mu$ where μ denotes the volume form on M. □

A new geometric picture

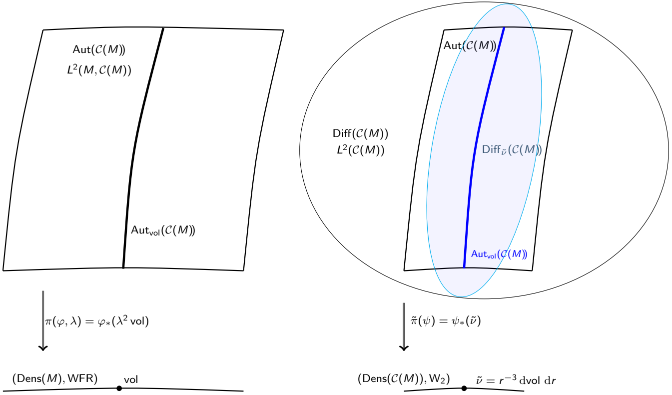

Figure – On the left, the picture represents the Riemannian submersion between Aut($\mathcal{C}(M)$) and the space of positive densities on M and the fiber above the volume form is Aut$_{\text{vol}}$($\mathcal{C}(M)$). On the right, the picture represents the automorphism group Aut($\mathcal{C}(M)$) isometrically embedded in Diff($\mathcal{C}(M)$) and the intersection of Diff$_{\tilde\nu}$($\mathcal{C}(M)$) and Aut($\mathcal{C}(M)$) is equal to Aut$_{\text{vol}}$($\mathcal{C}(M)$).

Results

Theorem (P2)

Let φ be the flow of a smooth solution to the Camassa-Holm equation then $\Psi(\theta, r) \stackrel{\text{def.}}{=} (\varphi(\theta), \sqrt{\mathrm{Jac}(\varphi(\theta))}r)$ is the flow of a solution to the incompressible Euler equation for the density $\frac{1}{r^4} r \,\mathrm{d}r \,\mathrm{d}\theta$.

Results

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Case where $M = S_1$, $\mathcal{M}(\varphi) = [(\theta, r) \mapsto r\sqrt{\partial_x \varphi(\theta)} e^{i\varphi(\theta)}]$ then the CH equation is

$$\begin{cases} \partial_t u - \frac{1}{4}\partial_{txx} u \, u + 3\partial_x u \, u - \frac{1}{2}\partial_{xx} u \, \partial_x u - \frac{1}{4}\partial_{xxx} u \, u = 0 \\ \partial_t \varphi(t,x) = u(t, \varphi(t,x)) \,. \end{cases} \tag{44}$$

The Euler equation on the cone, $\mathcal{C}(M) = \mathbb{R}^2 \setminus \{0\}$ for the density $\rho = \frac{1}{r^4}$ Leb is

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p \,, \\ \nabla \cdot (\rho v) = 0 \,. \end{cases} \tag{45}$$

where $v(\theta, r) \stackrel{\text{def.}}{=} \left(u(\theta), \frac{r}{2}\partial_x u(\theta)\right)$.

Conclusion on this link with CH:

Reformulation of CH

CH is a geodesic equation for an L^2 metric on the subgroup $\mathrm{Aut}_{\mathrm{vol}}(\mathcal{C}(M))$: automorphisms of $\mathcal{C}(M)$ which preserve $\frac{1}{r^3}\,\mathrm{d}r\,\mathrm{d}\,\mathrm{vol}_M$.

Contents

Toward polar factorization

Definition

The generalized automorphism semigroup of $C(M)$ is the set of mesurable maps (φ, λ) from M to $C(M)$

$$\overline{\mathrm{Aut}}(\mathcal{C}(M)) = \left\{(\varphi, \lambda) \in \mathcal{M}es(M, M) \ltimes \mathcal{M}es(M, \mathbb{R}_+^*)\right\} , \tag{46}$$

endowed with the semigroup law

$$(\varphi_1, \lambda_1) \cdot (\varphi_2, \lambda_2) = (\varphi_1 \circ \varphi_2, (\lambda_1 \circ \varphi_2)\lambda_2) .$$

The stabilizer of the volume measure in the automorphisms of $C(M)$ is

$$\overline{\mathrm{Aut}}_{\mathrm{vol}}(\mathcal{C}(M)) = \left\{(s, \lambda) \in \overline{\mathrm{Aut}}(\mathcal{C}(M)) \, : \, \pi\left((s, \lambda), \mathrm{vol}\right) = \mathrm{vol}\right\} . \tag{47}$$

By abuse of notation, any $(s, \lambda) \in \overline{\mathrm{Aut}}_{\mathrm{vol}}(\mathcal{C}(M))$ will be denoted $\left(s, \sqrt{\mathrm{Jac}(s)}\right)$ i.e. $f \in C(M, \mathbb{R})$

$$\int_M f(s(x)) \sqrt{\mathrm{Jac}(s)}^2 \, \mathrm{d}\,\mathrm{vol}(x) = \int_M f(x) \, \mathrm{d}\,\mathrm{vol}(x) . \tag{48}$$

Toward polar factorization

Definition (Admissible measures)

We say that a positive Radon measure ρ on M is admissible (with respect to vol) if for any $x \in M$, there exists $y \in \mathrm{Supp}(\rho)$ such that $d(x, y) < \pi/2$.

Toward polar factorization

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Consequence (Liero, Mielke, Savaré): Existence of a unique optimal potential which takes finite values a.e. between vol and ρ admissible. Recall that $c(x,y) = -\log(\cos^2(d(x,y) \wedge \pi/2))$.

$$\mathrm{WF}^2(\rho_0,\rho_1) = \sup_{(z_0,z_1)\in C(M)^2} \int_M 1 - e^{-z_0(x)}\,\mathrm{d}\rho_0(x) + \int_M 1 - e^{-z_1(y)}\,\mathrm{d}\rho_1(y) \tag{49}$$

subject to $\forall (x,y) \in M^2$,

$$z_0(x) + z_1(y) \leq -\log\left(\cos^2\left(d(x,y) \wedge (\pi/2)\right)\right). \tag{50}$$

Polar factorization

Theorem (Polar factorization, P2)

Let $(\phi, \lambda) \in \overline{\mathrm{Aut}}(\mathcal{C}(M))$ *s.t.* $\rho_1 = \pi_0 \left[(\phi, \lambda), \mathrm{vol}\right]$ *is an absolute continuous admissible measure. Then, there exist a unique minimizer, characterized by a c-concave function* z_0*, between* vol *and* ρ_1 *and a unique measure preserving generalized automorphism* $(s, \sqrt{\mathrm{Jac}(s)}) \in \overline{\mathrm{Aut}}_{\mathrm{vol}}(\mathcal{C}(M))$ *such that* vol *a.e.*

$$(\phi, \lambda) = \exp^{\mathcal{C}(M)}\left(-\frac{1}{2}\nabla p_{z_0}, -p_{z_0}\right) \circ (s, \sqrt{\mathrm{Jac}(s)}) \tag{51}$$

or equivalently

$$(\phi, \lambda) = \left(\varphi, e^{-z_0}\sqrt{1 + \|\nabla z_0\|^2}\right) \cdot (s, \sqrt{\mathrm{Jac}(s)})\,, \tag{52}$$

where $p_{z_0} = e^{z_0} - 1$ *and*

$$\varphi(x) = \exp_x^M\left(-\arctan\left(\frac{1}{2}\|\nabla z_0(x)\|\right)\frac{\nabla z_0(x)}{\|\nabla z_0(x)\|}\right). \tag{53}$$

Moreover $(s, \sqrt{\mathrm{Jac}(s)})$ *is the unique* $L^2(M, \mathcal{C}(M))$ *projection of* (ϕ, λ) *onto* $\overline{\mathrm{Aut}}_{\mathrm{vol}}(\mathcal{C}(M))$.

Polar factorization

Another formulation of the polar factorization:

Corollary (P2)

Denote by $\mathcal{M}es^1(\mathcal{C}(M))^{\mathbb{R}_+^*}$ *the space of mesurable and approximate differentiable functions* $f : \mathcal{C}(M) \mapsto \mathbb{R}$ *that satisfy* $f(x,r) = r^2 f(x,1)$ *for any* $r \in \mathbb{R}_+^*$*. Under the hypothesis of the previous theorem, there exists a unique couple* $\left((s, \sqrt{\mathrm{Jac}(s)}), \Psi_P\right) \in \overline{\mathrm{Aut}}_{\mathrm{vol}} \times \mathcal{M}es^1(\mathcal{C}(M))^{\mathbb{R}_+^*}$ *such that*

$$(\phi, \lambda) = \exp^{\mathcal{C}(M)}(-\nabla \Psi_P) \circ (s, \sqrt{\mathrm{Jac}(s)})\,, \tag{54}$$

where $\Psi(x,r) = r^2 z_0(x)$*.*

Generalized solutions to Incompressible Euler.

$$\inf_{\mu \in \mathcal{P}([0,1],M)} \langle \mu, \dot{x}^2 \rangle \text{ s.t. } [e_t]_*(\mu) = \rho_0 \text{ and } [e_{0,1}]_*(\mu) = \delta_{x,\varphi(x)} .$$

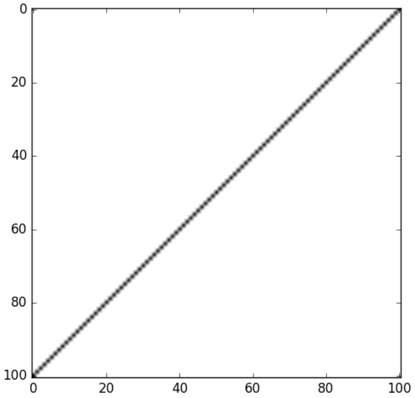

Figure – (Entropic/Schrödinger) Multimarginal Euler - Bre(ö)dinger !

Perspectives

- Study the generalized geodesics for CH (uniqueness of the pressure, how the angle of the cone affects the results...)
- Develop numerical approaches following Mérigot et al.
- Treat other fluid dynamic equations ?

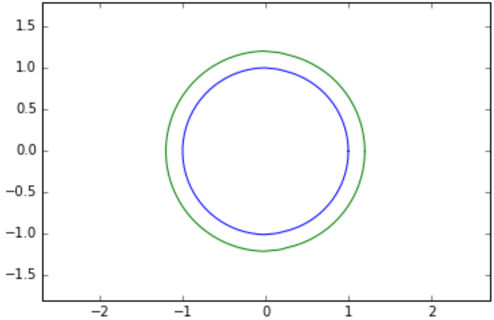

Figure – CH equation after the "Madelung transform"

Corresponding decomposition of vector fields

Polar factorization as extension of the Hodge-Helmholtz decomposition:

$$v = w + \nabla p \ \text{ where } \ \mathrm{div}(v) = 0\,. \tag{55}$$

In our case,

$$(v(\theta), r\lambda(\theta)) = \left(w(\theta), \frac{r}{2}\,\mathrm{div}(w(\theta))\right) + \left(\frac{1}{2}\,\nabla p(\theta), rp(\theta)\right)\,. \tag{56}$$

A word about smoothness: Monge-Ampère equation

The corresponding Monge-Ampère equation can be written as

$$\det\left[-\nabla^2 z(x) + (\nabla^2_{xx} c)(x,\varphi(x))\right] = \left|\det\left[(\nabla_{x,y} c)(x,\varphi(x))\right]\right| e^{-2z(x)} \left(1 + \frac{1}{4}\|\nabla z(x)\|^2\right) \frac{f(x)}{g \circ \varphi(x)}, \tag{57}$$

where φ is the $c-$exponential of $-z$:

$$\varphi(x) = \exp_x^M\left(-\arctan\left(\frac{1}{2}\|\nabla z(x)\|\right)\frac{\nabla z(x)}{\|\nabla z(x)\|}\right). \tag{58}$$

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For the cost $c(x,y) = -\log(\cos^2(d(x,y)\wedge\pi/2))$,

- On the plane, there exist $(x,y)\in M^2$ and $(v,w)\in T_xM\times T_yM$, $\mathrm{MTW}(x,y,v,w)<0$.
- On the sphere of radius $r=1$, as well.
- If r small enough, then numerically, $\mathrm{MTW}\geq 0$.

Link with the reflector problem

Consider the sphere of radius $1/2$, then $d(x,y) = \frac{1}{2}\arccos(x \cdot y)$:

$$\begin{aligned} -\log(\cos^2(d(x,y))) &= -\log(1+\cos(2d(x,y))) + \log(2) \\ &= -\log(1 + x\cdot y) + \log(2) \\ &= -2\log(|x+y|) = 2c_r(x,-y) \end{aligned}$$

The cost for the reflector antenna is $c_r(x,y) = -\log(|x-y|)$. Clearly,

$$\operatorname{sgn}(\mathrm{MTW}(c_r(\cdot,\cdot))) = \operatorname{sgn}(\mathrm{MTW}(c_r(\cdot,-\cdot)))$$

Therefore, $\mathrm{MTW}(-\log(\cos^2(d))) \geq 0$ on the sphere of radius $1/2$. (Loeper, Lee and Li).

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