Monge-Ampère Geometry and the Navier-Stokes Equations

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Joint with Bertrand Banos and Volodya Roubtsov (*J Phys A* 2016), and more recent work with Martin Wolf and Jock McOrist (Surrey)

New Trends in Applied Geometric Mechanics, "DarrylFest",

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Outline

- Monge-Ampère equations and the 2d incompressible Navier-Stokes equations
- Monge-Ampère geometry
- Burgers' vortices and symmetry reduction
- Complex structures and the 3d Navier-Stokes equations

Nonlinear stability analysis of inviscid flows in three dimensions: Incompressible fluids and barotropic fluids

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Lyapunov stability of ideal stratified fluid equilibria in hydrostatic balance

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On the Hamiltonian formulation of the quasi-hydrostatic equations

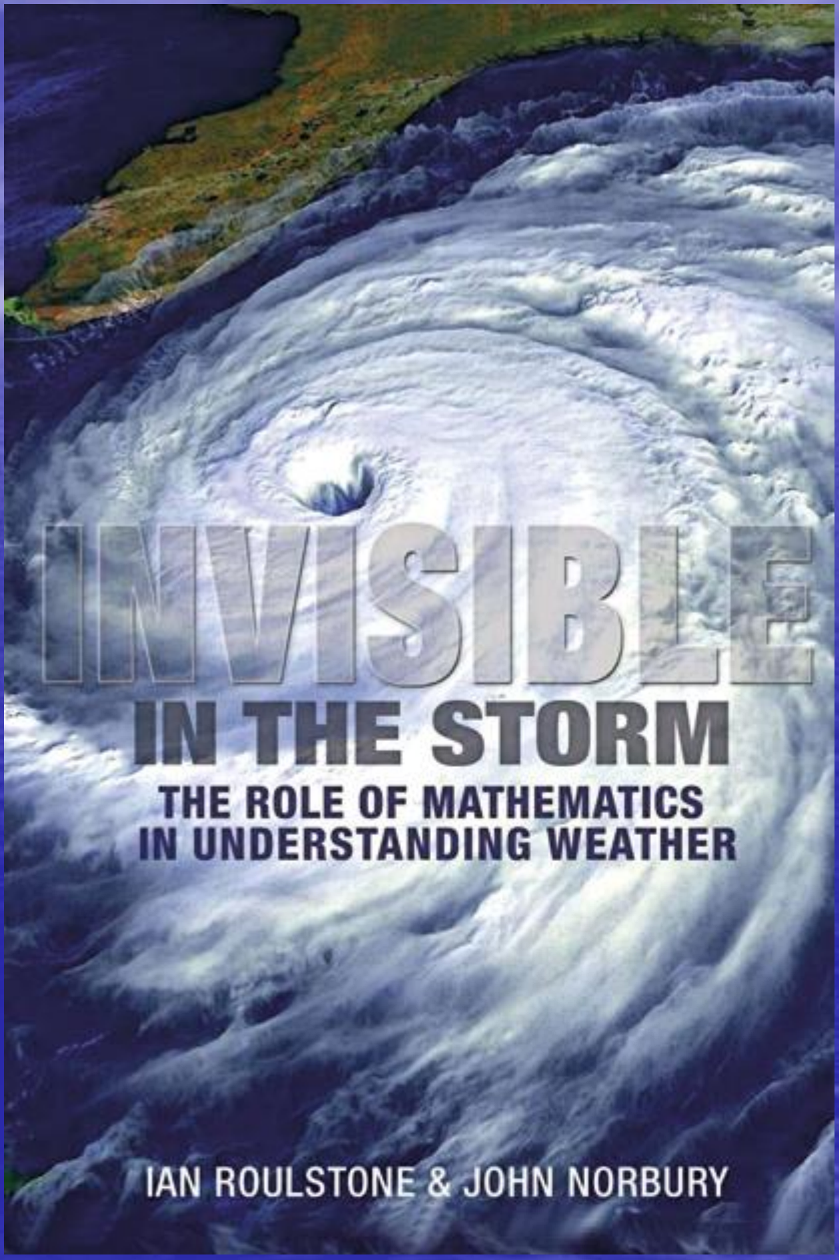
I. Roulstone, S. J. Brice
First published: QJRMS April 1995

Quaternions and particle dynamics in the Euler fluid equations

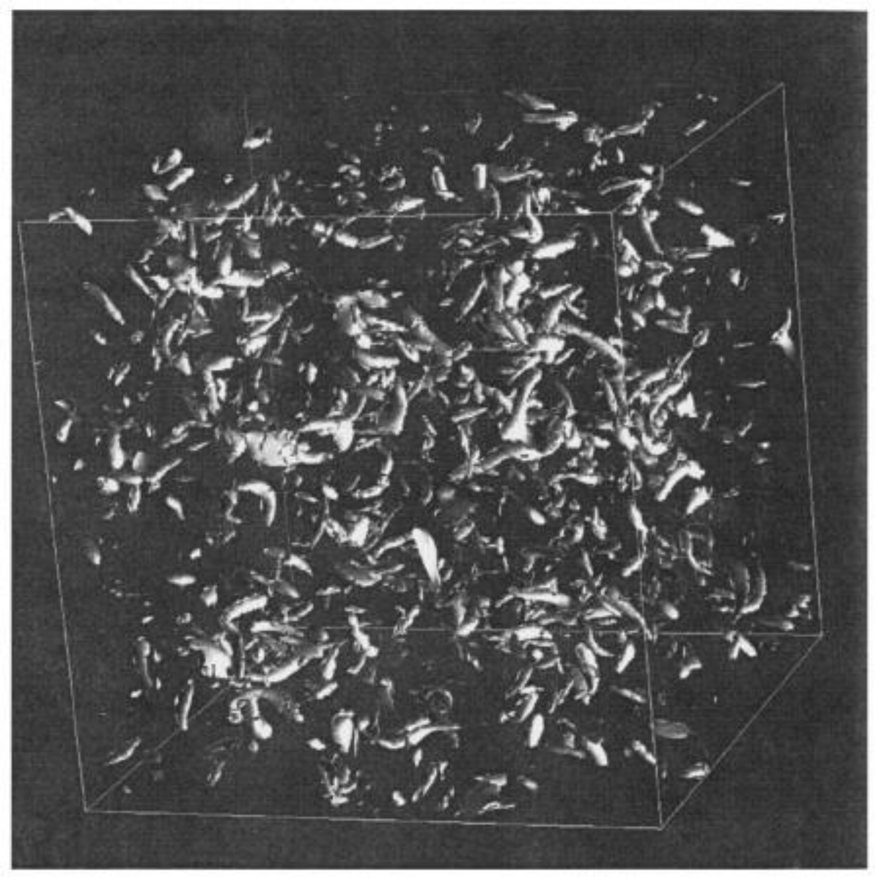
J D Gibbon[1], D D Holm[1], R M Kerr[2] and I Roulstone[3]

The pivotal role of Kelvin's Theorem!

Princeton University Press 2013



Vortex tubes – "the sinews of turbulence" (Moffatt)



Semi-Geostrophic Theory

- Potential vorticity advection and inversion – Hamiltonian system and Monge-Ampère equation
- Legendre duality, singularities – contact geometry
- Symplectic and contact geometries – Kähler geometry
- Optimal transport, minimal surfaces – calibrated geometry

Incompressible Navier-Stokes (2d/3d)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Apply div **u** = 0

$$-\nabla^2 p = u_{i,j} u_{j,i}$$

2d: Stream function – Poisson eqn for *p* or Monge-Ampère eqn for ψ

$$\boldsymbol{u} = \boldsymbol{k} \times \nabla \psi$$

$$\nabla^2 p = -2(\psi_{xy}^2 - \psi_{xx}\psi_{yy})$$

Vorticity and Rate of Strain (Okubo-Weiss Criterion)

$$\mathcal{Q} = \frac{1}{2}(W_{ij}W_{ij} - S_{ij}S_{ij}) = \frac{1}{4}(\zeta^2 - 2S_{ij}S_{ij})$$

$$S_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$W_{ij} = \tfrac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

$$\zeta = \nabla \times \boldsymbol{u}$$

$$\mathcal{Q} = -\tfrac{1}{2}u_{i,j}u_{j,i}$$

$$-\nabla^2 p = u_{i,j}u_{j,i}$$

$Q > 0$ => vorticity dominates over rate of strain, Monge-Ampère equation is elliptic

$$\Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\zeta^2 - \mathrm{Tr}\ S^2$$

- J.D. Gibbon (Physica D 2008 – *Euler, 250 years on*): "The elliptic equation for the pressure is by no means fully understood and *locally* holds the key to the formation of vortical structures through the sign of the Laplacian of pressure. In this relation, which is often thought of as a constraint, may lie a deeper knowledge of the geometry of both the Euler and Navier-Stokes equations…The fact that vortex structures are dynamically favoured may be explained by inherent geometrical properties of the Euler equations but little is known about these features."

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Monge-Ampère Geometry

Introduce a symplectic structure

$$(x, y, \hat{p}, \hat{q}), T^*\mathbb{R}^2, \Omega$$

$$\Omega = \mathrm{d}x \wedge \mathrm{d}\hat{p} + \mathrm{d}y \wedge \mathrm{d}\hat{q}$$

and a two-form

$$\omega = A\mathrm{d}\hat{p} \wedge \mathrm{d}y + B(\mathrm{d}x \wedge \mathrm{d}\hat{p} - \mathrm{d}y \wedge \mathrm{d}\hat{q}) + C\mathrm{d}x \wedge \mathrm{d}\hat{q} + D\mathrm{d}\hat{p} \wedge \mathrm{d}\hat{q} + E\mathrm{d}x \wedge \mathrm{d}y$$

On the graph of a function φ(x,y)

$$\hat{p} = \phi_x, \ \hat{q} = \phi_y$$

$$\omega|_\phi = 0$$

Monge-Ampère eqn

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + E = 0$$

Define the Pfaffian

$$\omega \wedge \omega = \mathrm{pf}(\omega)\Omega \wedge \Omega$$

$$\mathrm{pf}(\omega) = AC - B^2 - DE$$

$\mathrm{pf}(\omega) > 0$ then ω (M-A eqn) is *elliptic*, and

$$I_{\mu\nu} = \frac{1}{\sqrt{\mathrm{pf}(\omega)}}\Omega_{\mu\sigma}\omega_{\sigma\nu}$$

$$I_\omega = \frac{1}{\sqrt{\mathrm{pf}(\omega)}}\begin{pmatrix} B & -A & 0 & -D \\ C & -B & D & 0 \\ 0 & E & B & C \\ -E & 0 & -A & -B \end{pmatrix}$$

is an *almost-complex* structure $I_\omega^2 = -Id$

$T^*\mathbb{R}^2, \omega, I_\omega$ is an almost-Kähler manifold

a) $\Delta_\omega =$ is elliptic $\Leftrightarrow \mathrm{pf}(\omega) > 0 \Leftrightarrow I_\omega^2 = -Id$

b) $\Delta_\omega =$ is hyperbolic $\Leftrightarrow \mathrm{pf}(\omega) < 0 \Leftrightarrow I_\omega^2 = Id$

PROPOSITION 1. *(Lychagin-Roubtsov theorem) The three following assertions are equivalent:*

1. *$\Delta_\omega = 0$ is locally equivalent to one of the two equations*

$$\begin{cases} \Delta\phi = 0 \\ \Box\phi = 0 \end{cases}$$

2. *the almost complex (or product) structure I_ω is integrable*

3. *the form $\frac{\omega}{\sqrt{|\mathrm{pf}(\omega)|}}$ is closed.*

Complex structure: 2d Euler

$$\omega \equiv \nabla^2 p \, \mathrm{d}x \wedge \mathrm{d}y - 2\mathrm{d}u \wedge \mathrm{d}v$$

Poisson eqn

$$u = -\hat{q}, \quad v = \hat{p}$$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \nabla^2 p & 0 & 0 \\ -\nabla^2 p & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Complex structure

$$I_{\mu\nu} = \frac{1}{\sqrt{\mathrm{pf}(\omega)}} \Omega_{\mu\sigma} \omega_{\sigma\nu}$$

$$I_\omega \equiv I_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

$$\nabla^2 p = 2\alpha^2$$

$$I_\omega^2 = -I \text{ if } \nabla^2 p > 0$$

Fixing a volume form in terms of the symplectic structure, we define a metric

$$g_\omega(X,Y) = \frac{\iota_X\Omega \wedge \iota_Y\omega + \iota_Y\Omega \wedge \iota_X\omega}{\Omega \wedge \Omega} \wedge \pi^*(vol), \;\; X,Y \in T\mathbb{R}^n,$$

We may construct a metrically-dual two-form for the *hyperbolic* MA equation.

In general, we have an almost hyper-symplectic structure.

Generalized Solutions

A generalized solution of (4) is a 2d-submanifold $L^2 \subset M^4$ which is bilagrangian with respect to ω and Ω. Since $\omega = \Omega(I, \cdot, \cdot)$, it is equivalent to saying that L is closed under I: for any non vanishing vector field X on L, $\{X, IX\}$ is a local frame of L.

PROPOSITION 2. *Let L be a generalized solution and h_ω the restriction of g_ω on L.*

1. *if $\Delta p > 0$, the metric h_ω has signature $(2,0)$ or $(0,2)$*
2. *if $\Delta p < 0$, the metric h_ω has signature $(1,1)$.*

Induced metric

REMARK 1. *If $L = L_\psi$ is a regular solution, then the induced metric h_ω is affine. Indeed its tangent space is generated by $X_1 = \partial_{x_1} - \psi_{x_1 x_2}\partial_{u_1} + \psi_{x_1 x_1}\partial_{u_2}$ and $X_2 = \partial_{x_2} - \psi_{x_2 x_2}\partial_{u_1} + \psi_{x_1 x_2}\partial_{u_2}$. The induced metric on L_ψ is therefore*

$$h_\omega = 2\begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix} \tag{11}$$

and consequently the invariants of this tensor are

$$\det(h_\omega) = 2\Delta p, \quad \mathrm{tr}(h_\omega) = 2\Delta\psi.$$

Evolution of the Pfaffian

(cf. R., Clough & White *QJRMS* 2014)

- Cantwell et al 1988….Invariants of the velocity gradient tensor – analysis of critical points
- Laplacian of pressure is proportional to the second invariant of the VGT, *Q*(u,v), which in 2d is the Jacobian determinant:

$$\mathrm{Pf} = 2\triangle p = 4Q$$

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = p_{xy}\Box\psi - \psi_{xy}\Box p$$

A Geometric Flow

Let ψ_{ij} and p_{ij} denote the hessian matrices of the stream function and the pressure, then

$$\frac{\mathrm{D}\psi_{ij}}{\mathrm{D}t} = \nu\Delta\psi_{ij} + \underbrace{\epsilon_{ij}|\psi_{mn}| - \epsilon_{ik}p_{jk}}_{\int K_{ij}(x-y)|\psi_{mn}(y)|\mathrm{d}y}$$

2.8. Singular integral operators

Using (2.29), we may define a linear operator

$$T_{ij} : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

that gives the second derivatives of a function in terms of its Laplacian,

$$\partial_{ij}u = T_{ij}\Delta u.$$

Explicitly,

$$T_{ij}f(x) = \int_{B_R(x)} K_{ij}(x-y)\left[f(y) - f(x)\right] dy + \frac{1}{n}f(x)\delta_{ij} \qquad (2.35)$$

where $B_R(x) \supset \operatorname{supp} f$ and $K_{ij} = -\partial_{ij}\Gamma$ is given by

$$K_{ij}(x) = \frac{1}{\alpha_n|x|^n}\left(\frac{1}{n}\delta_{ij} - \frac{x_i x_j}{|x|^2}\right). \qquad (2.36)$$

Finite Deformation of Complex Structure

Let $U \subseteq \mathbb{R}^2$ and consider T^*U with coordinates (x, y, p, q). Furthermore, let us introduce complex coordinates

$$z^1 \; := \; \mathrm{i}\sqrt{\alpha}x - \sqrt{2}q \quad \text{and} \quad z^2 \; := \; \sqrt{\alpha}y - \mathrm{i}\sqrt{2}p \tag{4.21}$$

for constant $\alpha \in \mathbb{R}^+$ so that

$$\partial_1 \;=\; -\tfrac{1}{2}\left(\tfrac{\mathrm{i}}{\sqrt{\alpha}}\partial_x + \tfrac{1}{\sqrt{2}}\partial_q\right) \quad \text{and} \quad \partial_2 \;=\; \tfrac{1}{2}\left(\tfrac{1}{\sqrt{\alpha}}\partial_y + \tfrac{\mathrm{i}}{\sqrt{2}}\partial_p\right) . \tag{4.22}$$

Hence,

$$J_0 \;=\; \begin{pmatrix} 0 & 0 & 0 & -\sqrt{\frac{2}{\alpha}} \\ 0 & 0 & \sqrt{\frac{2}{\alpha}} & 0 \\ 0 & -\sqrt{\frac{\alpha}{2}} & 0 & 0 \\ \sqrt{\frac{\alpha}{2}} & 0 & 0 & 0 \end{pmatrix} \tag{4.23}$$

with $J_0^2 = -\mathbb{1}_4$ and $J_0\partial_a = \mathrm{i}\partial_a$. In addition, we define a holomorphic symplectic structure by

$$\omega_0 \; := \; \tfrac{1}{2}\varepsilon_{ab}\mathrm{d}z^a \wedge \mathrm{d}z^b \;=\; \mathrm{d}z^1 \wedge \mathrm{d}z^2 \quad \Rightarrow \quad \mathfrak{Im}\,\omega_0 \;=\; \alpha\mathrm{d}x \wedge \mathrm{d}y - 2\mathrm{d}p \wedge \mathrm{d}q \;, \tag{4.24}$$

$$Dz^a \ := \ \mathrm{d}z^a + (-1)^m \omega^{aba_1\cdots a_{m-2}} \partial_b A_{a_1\cdots a_{m-2}} \ .$$

$$\begin{aligned}\omega \ &= \ \frac{1}{m!}\omega_{a_1\cdots a_m} Dz^{a_1} \wedge \cdots \wedge Dz^{a_m} \\ &= \ \omega_0 - \partial_0 A + \sum_{i=2}^{m} \frac{1}{i!}\mu_i(A, \ldots, A) \ ,\end{aligned}$$

As an example, consider the special choice

$$A \ = \ A_{\bar{a}} \mathrm{d}\bar{z}^{\bar{a}} \quad \text{with} \quad A_{\bar{1}} \ := \ a(x,y) + \mathrm{i}\beta x \quad \text{and} \quad A_{\bar{2}} \ := \ \mathrm{i}b(x,y) + \beta y$$

$$\mathfrak{Im}\,\omega \ = \ [\alpha + \sqrt{\alpha}(a_y + b_x)]\mathrm{d}x \wedge \mathrm{d}y - 2\mathrm{d}p \wedge \mathrm{d}q - \sqrt{2}\beta(\mathrm{d}x \wedge \mathrm{d}q + \mathrm{d}p \wedge \mathrm{d}y) \ .$$

$$2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) + \sqrt{2}\beta(\psi_{xx} + \psi_{yy}) \ = \ \alpha + \sqrt{\alpha}(a_y + b_x)$$

Burgers' vortices I

In this section we are concerned with solutions of the incompressible Euler equations of Burgers'-type. That is, we consider flows of the form

$$u_i = (-\gamma(t)x_1/2 - \psi_{x_2}, -\gamma(t)x_2/2 + \psi_{x_1}, \gamma(t)x_3)^T, \tag{12}$$

Consider a stream function

$$\Psi = \psi(x_1, x_2, t) - \frac{3}{8}\gamma(t)^2 x_3^2. \tag{13}$$

Using (12) and (13), (3) becomes

$$\Psi_{x_1x_1}\Psi_{x_2x_2} - \Psi_{x_1x_2}^2 + \Psi_{x_3x_3} = \frac{\Delta p}{2}$$

Geometry of 3-forms I

Lychagin *et al.* (1993) associated with a Monge-Ampère structure (Ω, ω) an invariant symmetric form

$$g_\omega(X,Y) = \frac{\iota_X\omega \wedge \iota_Y\omega \wedge \Omega}{\mathrm{vol}}$$

whose signature distinguishes the different orbits of the symplectic group action.

In a seminal paper on the geometry of 3-forms, Hitchin (2001) defined the notion of nondegenerate 3-forms on a 6-dimensional space and constructed a scalar invariant, which we call the *Hitchin pfaffian*, which is non zero for such nondegenerate 3-forms. Hitchin also defined an invariant tensor A_ω on the phase space satisfying

$$A_\omega^2 = \lambda(\omega)\, Id.$$

Note that in the nondegenerate case, $K_\omega = \dfrac{A_\omega}{\sqrt{|\lambda(\omega)|}}$ is a product structure if $\lambda(\omega) > 0$ and a complex structure if $\lambda(\omega) < 0$.

Burgers' vortices II

Denote by (Ω, ϖ) the corresponding Monge-Ampère structure on $\mathbb{T}^*\mathbb{R}^3$, with Ω the canonical symplectic form

$$\Omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 + dx_3 \wedge d\xi_3,$$

where $\xi_i = \nabla_i \Psi$ and, following Roulstone *et al.* (2009), ϖ is the effective 3-form

$$\varpi = d\xi_1 \wedge d\xi_2 \wedge dx_3 - a dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge d\xi_3, \tag{15}$$

with $a(x_1, x_2) = \Delta p/2$.

The corresponding "real Calabi-Yau" structure on $T^*\mathbb{R}^6$ is

$$(g_\varpi, K_\varpi, \Omega, \varpi + \hat{\varpi}, \varpi - \hat{\varpi})$$

i) $g_{\varpi} = 2a dx_3 \otimes dx_3 + dx_1 \otimes d\xi_1 + dx_2 \otimes d\xi_2 - dx_3 \otimes d\xi_3 \qquad (\varepsilon(g_{\varpi}) = (3,3))$

ii) $$K_{\varpi} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2a & 0 & 0 & -1 \end{pmatrix} \quad \text{and thus} \quad K_{\varpi}^2 = 1$$

iii) $\varpi + \hat{\varpi} = 2 d\xi_1 \wedge d\xi_2 \wedge dx_3, \quad \varpi - \hat{\varpi} = 2 dx_1 \wedge dx_2 \wedge (d\xi_3 - a dx_3),$

and the Hitchin pfaffian is $\lambda(\varpi) = 1$.

Symplectic reduction I

Let $G = \mathbb{R}$ acting on $T^*\mathbb{R}^3$ by translation on the third coordinate:

$$\lambda_\tau(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3 + \tau, \xi_1, \xi_2, \xi_3)$$

This action is trivially hamiltonian with moment map $\mu(x, \xi) = -\xi_3$ and infinitesimal generator $X = \lambda_* \left(\frac{d}{d\tau}\right) = \frac{\partial}{\partial x_3}$.

$$\begin{cases} \Omega = \Omega_c - \dfrac{1}{2a} \iota_X \Omega \wedge \iota_Y \Omega \\ \\ \varpi = \varpi_1 \wedge \iota_X \Omega + \varpi_2 \wedge \iota_Y \Omega. \end{cases}$$

$$\begin{cases} \Omega_c = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 \\ \\ \varpi_1 = -\dfrac{1}{2a}(d\xi_1 \wedge d\xi_2 - a dx_1 \wedge dx_2), \\ \\ \varpi_2 = \dfrac{1}{2a}(d\xi_1 \wedge d\xi_2 + a dx_1 \wedge dx_2). \end{cases}$$

3d Incompressible Flows

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A geometric interpretation of coherent structures in Navier–Stokes flows

By I. Roulstone[1,*], B. Banos[2], J. D. Gibbon[3] and V. N. Roubtsov[4,5]

Geometry of 3-forms II

$$\Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\zeta^2 - \mathrm{Tr}\ S^2$$

$$\varpi = \Delta p \,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - 2(\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}x_3 + \mathrm{d}u_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}u_3 + \mathrm{d}x_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3)$$

Lychagin-Roubtsov (LR) metric

$$q_{\varpi}(w, w) \equiv -\frac{1}{4} \perp^2 (\iota(w)\varpi \wedge \iota(w)\varpi)$$

$$\perp \varpi = \iota\left(\frac{\partial}{\partial \boldsymbol{x}} \wedge \frac{\partial}{\partial \boldsymbol{u}}\right)\varpi$$

Metric and Pfaffian

$$q_\varpi = -2\begin{pmatrix} \frac{1}{2}\Delta p I & 0 \\ 0 & I \end{pmatrix}$$

Construct a linear operator, K_ω, using LR metric and symplectic structure

$$K_\varpi = -2\begin{pmatrix} 0 & I \\ -\frac{1}{2}\Delta p I & 0 \end{pmatrix}$$

The "pfaffian"

$$\lambda(\varpi) = \frac{1}{6}\mathrm{Tr}\, K_\varpi^2 = -2\Delta p$$

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Complex structure

In particular, when $\lambda(\varpi)<0$, the tensor

$$J_\varpi = \frac{1}{\sqrt{-\lambda(\varpi)}} K_\varpi$$

is an almost-complex structure and the real three-form ϖ is the real part of the complex form

$$\varpi^{\mathrm{c}} = \varpi + \mathrm{i}\hat{\varpi},$$

$$\varpi = (\mu_1 + \mathrm{i}\nu_1) \wedge (\mu_2 + \mathrm{i}\nu_2) \wedge (\mu_3 + \mathrm{i}\nu_3) + (\mu_1 - \mathrm{i}\nu_1) \wedge (\mu_2 - \mathrm{i}\nu_2) \wedge (\mu_3 - \mathrm{i}\nu_3)$$

$$\equiv \alpha + \bar{\alpha}$$

where $\mu_i = (\Delta p/2)^{1/3}\, \mathrm{d}x_i$ and $\nu_i = (\Delta p/2)^{-(1/6)}\, \mathrm{d}u_i$, when $\Delta p > 0$

Symplectic reduction II

Let $\mathbb{R}$ acting on $T^*\mathbb{R}^3$ by translation on (x_3, u_3):

$$\lambda_a(x_1, x_2, x_3, u_1, u_2, u_3) = (x_1, x_2, x_3 + \tau, u_1, u_2, u_3 + \gamma\tau) \,, \quad \gamma \in \mathbb{R}$$

The infinitesimal generator is

$$X = \lambda_* \left(\frac{d}{d\tau} \right) = \frac{\partial}{\partial x_3} + \gamma \frac{\partial}{\partial u_3}$$

and the moment map is

$$\mu(x, u) = \gamma \, x_3 - u_3.$$

We observe that $\mu =$ constant yields the linearity of u_3 in x_3, as defined in (12).

For $c \in \mathbb{R}$, the reduced space M_c is trivially $T^*\mathbb{R}^2$:

$$M_c = \{(x_1, x_2, x_3, u_1, u_2, \gamma x_3 - c\} / x_3 \sim x_3 + t$$

$$= \{(x_1, x_2, 0, u_1, u_2, -c)\}$$

and the pair $(\omega_c, \theta_c) = (\iota_X \omega, \iota_X \theta)$ is

$$\begin{cases} \omega_c = a\, dx_1 \wedge dx_2 - du_1 \wedge du_2 - \gamma du_1 \wedge dx_2 - \gamma dx_1 \wedge du_2 \\ \theta_c = du_1 \wedge dx_2 + dx_1 \wedge du_2 + \gamma dx_1 \wedge dx_2 \end{cases}$$

Considering the following change of variables

$$X_1 = x_1 \ , \qquad U_1 = \frac{\gamma}{2}x_2 + u_2,$$
$$X_2 = x_2 \ , \qquad U_2 = -\frac{\gamma}{2}x_1 - u_1,$$

we obtain

$$\theta_c = dX_1 \wedge dU_1 + dX_2 \wedge dU_2 \tag{23}$$

and

$$\omega_c = \omega_0 - \frac{\gamma}{2}\theta_c \tag{24}$$

with

$$\omega_0 = \left(a + \frac{3}{4}\gamma^2\right) dX_1 \wedge dX_2 - dU_1 \wedge dU_2. \tag{25}$$

Burgers' vortices via reduction

PROPOSITION 5. *If $\psi(x_1, x_2, t)$ is solution of the Monge-Ampère equation in two dimensions*

$$\psi_{x_1x_1}\psi_{x_2x_2} - \psi^2_{x_1x_2} = \frac{\Delta p}{2} + \frac{3}{4}\gamma^2(t). \qquad (26)$$

then the velocity $u(x_1, x_2, x_3, t)$ defined by

$$u_1 = -\frac{\gamma(t)}{2}x_1 - \psi_{x_2},$$
$$u_2 = -\frac{\gamma(t)}{2}x_2 + \psi_{x_1},$$
$$u_3 = \gamma(t)x_3 - c(t),$$

is solution of $-\Delta p = u_{ij}u_{ji}$ and $u_{ii} = 0$ in three dimensions.

Summary and Outlook

- Vorticity-dominated incompressible Euler flows in 2D are associated with almost-Kähler structure – a geometric version of the "Weiss criterion", much studied in turbulence
- Using the geometry of 3-forms in six dimensions, we are able to generalize this criterion to 3D incompressible flows

- These ideas originate in models are large-scale atmospheric flows, in which rotation dominates and an elliptic pde relates the flow velocity to the pressure field
- McIntrye and R (1996), Roubtsov and R (1997, 2001), Delahaies and R (2009) showed how hyper-Kähler structures provide a geometric foundation for understanding Legendre duality (singularity theory), Hamiltonian structure and Monge-Ampère equations, in semi-geostrophic theory and related models

- In semi-geostrophic theory, physical assumptions dictate that the Monge-Ampère equation should remain elliptic: in Euler/Navier-Stokes no such conditions exist – 2/3d E/N-S may be describable in terms of Hitchin’s <u>generalized geometry</u> (R., Wolf & McOrist)
- Further, the geometry of N-S is parameterized by time: a <u>geometric flow</u> (of advection-diffusion type) emerges in a very natural way