Variational methods in fluid-structure interactions: Dynamics, dissipation, constraints, and Darcy's law in moving media

Vakhtang Putkaradze

Mathematical and Statistical Sciences, University of Alberta, Canada

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Joint work with Francois Gay-Balmaz, (CNRS and ENS, Paris) Akif Ibragimov (Texas Tech) and Dmitry Zenkov (NCSU)

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[2]FGB & VP, *Comptes Rendus Mécanique* **342**, 79-84 (2014)
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Outline

1. Problem formulation: tubes conveying fluid
2. Variational derivation of tube-fluid equations
3. Discretization of tube-fluid equations with examples
4. Compressible gas in stretchable tube
5. Introduction of dissipation
6. A simple problem: tube pendulum with a droplet
7. Poromechanics and Darcy's law as dynamical limit
8. Conclusions and open questions

Variational treatment of a tube conveying fluid

Figure: Image of a garden hose and its mathematical description

- No friction in the system for now, incompressible fluid, Reynolds numbers $\sim 10^4$ (much higher in some applications), general 3D motions
- Hose can stretch and bend arbitrarily (inextensible also possible)
- Cross-section of the hose changes dynamically with deformations: *collapsible tube*

Previous work

- *Constant fluid velocity in the tube*, 2D dynamics:
  English: Benjamin (1961); Gregory, Païdoussis (1966); Païdoussis (1998); Doare, De Langre (2002); Flores, Cros (2009), . . .
  Russian: Bolotin (?) (1956), Svetlitskii (monographs 1982, 1987), Danilin (2005), Zhermolenko (2008), Akulenko *et al.* (2015) . . .
  Hard to generalize to general 3D motions
  Not possible to consistently incorporate the cross-sectional dynamics
- Elastic rod with directional (tangent) momentum source at the end – the follower-force method, see Bou-Rabee, Romero, Salinger (2002), critiqued by Elishakoff (2005).
- Shell models: Paidoussis & Denise (1972), Matsuzaki & Fung (1977), Heil (1996), Heil & Pedley (1996) , . . . : Complex, computationally intensive, difficult (impossible) to perform analytic work for non-straight tubes.
- 3D dynamics from Cosserat's model (Beauregard, Goriely & Tabor 2010): Force balance, not variational, cannot accommodate dynamical change of the cross-section.
- Variational derivation: FGB & VP (2014,2015).

Variational treatment of changing cross-sections dynamics

Mathematical preliminaries:

1. Rod dynamics is described by $SE(3)$-valued functions (rotations and translations in space) $\pi(s,t) = (\Lambda, \mathbf{r})(s,t)$.
2. Fluid dynamics inside the rod is described by 1D diffeomorphisms $s = \varphi(a,t)$, where a is the Lagrangian label.
3. Conservation of 1-form volume element (fluid incompressibility) defined through a holonomic constraint:

$$Q := A \left| \frac{d\mathbf{r}}{ds} \right| = \left(Q_0 \circ \varphi^{-1}(s,t) \right) \partial_s \varphi^{-1}(s,t) \tag{1}$$

   where area A depends on the deformations of the tube.

4. Alternatively, evolution equation for Q is $\partial_t Q + \partial_s (Qu) = 0$.
5. Note that commonly used $Au =$const does not conserve volume for time-dependent flow. See *e.g.* [Kudryashov *et al*, Nonlinear dynamics (2008)] for correct derivation in 1D.

Mathematical preliminaries: Geometric rod theory for elastic rods I

- Purely elastic Lagrangian

$$\mathcal{L} = \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{r}', \Lambda, \dot{\Lambda}, \Lambda')$$

- Use $SE(3)$ symmetry reduction [Simo, Marsden, Krishnaprasad 1988] (SMK) to reduce the Lagrangian to $\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma})$ of the following coordinate-invariant variables (prime= ∂_s, dot=∂_t):

$$\boldsymbol{\Gamma} = \Lambda^{-1}\mathbf{r}' \, , \; \boldsymbol{\Omega} = \Lambda^{-1}\Lambda' \, , \tag{2}$$

$$\boldsymbol{\gamma} = \Lambda^{-1}\dot{\mathbf{r}} \, , \; \boldsymbol{\omega} = \Lambda^{-1}\dot{\Lambda} \, . \tag{3}$$

- Note that symmetry reduction for elastic rods is *left-invariant* (reduces to body variables).
- Notation: small letters (e.g. $\boldsymbol{\omega}, \boldsymbol{\gamma}$) denote time derivatives; capital letters (e.g. $\boldsymbol{\Omega}, \boldsymbol{\Gamma}$) denote the *s*-derivatives.

Mathematical preliminaries: Geometric rod theory for elastic rods II

- Euler Poincaré theory: [Holm, Marsden, Ratiu 1998].
  For elastic rods: compute variations as in [Ellis, Holm, Gay-Balmaz, VP and Ratiu, *Arch. Rat.Mech. Anal.*, (2010)]: consider $\boldsymbol{\Sigma} = \Lambda^{-1}\delta\Lambda \in \mathfrak{so}(3)$ and $\boldsymbol{\Psi} = \Lambda^{-1}\delta\mathbf{r} \in \mathbb{R}^3$, and $(\boldsymbol{\Sigma}, \boldsymbol{\Psi}) \in \mathfrak{se}(3)$.
  $$\delta\boldsymbol{\omega} = \frac{\partial\boldsymbol{\Sigma}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\gamma} = \frac{\partial\psi}{\partial t} + \boldsymbol{\gamma}\times\boldsymbol{\Sigma} + \boldsymbol{\omega}\times\psi \tag{4}$$
  $$\delta\boldsymbol{\Omega} = \frac{\partial\boldsymbol{\Sigma}}{\partial s} + \boldsymbol{\Omega}\times\boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\Gamma} = \frac{\partial\psi}{\partial s} + \boldsymbol{\Gamma}\times\boldsymbol{\Sigma} + \boldsymbol{\Omega}\times\psi, \tag{5}$$
- Compatibility conditions (cross-derivatives in s and t are equal)
  $$\boldsymbol{\Omega}_t - \boldsymbol{\omega}_s = \boldsymbol{\Omega}\times\boldsymbol{\omega}\,, \quad \boldsymbol{\Gamma}_t + \boldsymbol{\omega}\times\boldsymbol{\Gamma} = \boldsymbol{\gamma}_s + \boldsymbol{\Omega}\times\boldsymbol{\gamma}\,.$$
- Critical action principle $\delta\int \ell \,\mathrm{d}t\mathrm{d}s = 0+$ (4,5) give SMK equations.
  $$0 = \delta\int \ell\,\mathrm{d}t\mathrm{d}s = \int\left\langle \frac{\delta\ell}{\delta\boldsymbol{\omega}}\,,\,\delta\boldsymbol{\omega}\right\rangle + \int\left\langle \frac{\delta\ell}{\delta\boldsymbol{\Omega}}\,,\,\delta\boldsymbol{\Omega}\right\rangle + \ldots$$
  $$= \int \langle \text{linear momentum eq}, \boldsymbol{\Psi}\rangle + \langle \text{angular momentum eq}, \boldsymbol{\Sigma}\rangle\,\mathrm{d}t\mathrm{d}s$$

Mathematics preliminaries: incompressible fluid motion

- Following Arnold (1966), describe a 3D incompressible fluid motion by Diff_{Vol} group $\mathbf{r} = \varphi(\mathbf{a}, t)$.
- Eulerian fluid velocity is $u = \varphi_t \circ \varphi^{-1}$; symmetry-reduced Lagrangian is $\ell = 1/2 \int u^2 \mathrm{d}\mathbf{r}$.
- Variations of velocity are computed as

$$\eta = \delta\varphi \circ \varphi^{-1}(s, t) \, , \quad \delta u = \eta_t + u \nabla \eta - \eta \nabla u \, . \tag{6}$$

- Incompressibility condition

$$J = \left| \frac{\partial \mathbf{r}}{\partial \mathbf{a}} \right| = 1 \Rightarrow \text{Lagrange multiplier } p \, .$$

- Euler equations: $\delta \int \ell \, \mathrm{d}V \, \mathrm{d}t = 0$ with (6) and (**??**)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \, , \quad \mathrm{div} u = 0$$

Further considerations: α-model, Complex fluids *etc.*: D. D. Holm & many others

Garden hoses: Lagrangian and symmetry reductions

1. Symmetry group of the system (ignoring gravity for now)
$$G = SE(3) \times \mathrm{Diff}_A(\mathbb{R}) = SO(3) \circledS \mathbb{R} \times \mathrm{Diff}_A(\mathbb{R})\,. \qquad (7)$$
2. Position of elastic tube and fluid:
$$(\pi,\varphi)\cdot\Big(\big(\Lambda_0,\mathbf{r}_{t,0}\big),\mathbf{r}_f\Big) = \Big(\underbrace{\pi\cdot\big(\Lambda_0,\mathbf{r}_{t,0}\big)}_{\text{left invariant}},\underbrace{\pi\cdot\mathbf{r}_f\circ\varphi^{-1}(s,t)}_{\text{right invariant}}\Big)\,.$$
3. Velocities:
$$\begin{aligned}\big(\mathbf{v}_r\,,\,\mathbf{v}_f\big) &= \frac{d}{dt}\Big(\mathbf{r}(s,t)\,,\,\mathbf{r}\circ\varphi^{-1}(s,t)\Big)\\ &= \Big(\dot{\mathbf{r}}(s,t),\dot{\mathbf{r}}\circ\varphi^{-1}(s,t)+\mathbf{r}'(s,t)u(s,t)\Big)\,. \end{aligned} \qquad (8)$$
4. Change in cross-section $A = A(\boldsymbol{\Omega},\boldsymbol{\Gamma})$
5. Incompressibility condition $J = A(s,t)\frac{\partial a}{\partial s}|\boldsymbol{\Gamma}| = 1$ with Lagrange multiplier μ (pressure)
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s}(Qu) = 0\,, \quad \text{with} \quad Q = A|\boldsymbol{\Gamma}|\,. \qquad (9)$$

Equations of motion

$$\begin{cases} (\partial_t + \boldsymbol{\omega}\times)\dfrac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \dfrac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\delta \ell}{\delta \boldsymbol{\Omega}} - \dfrac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma} \times \left(\dfrac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = 0 \\ (\partial_t + \boldsymbol{\omega}\times)\dfrac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = 0 \\ m_t + \partial_s (mu - \mu) = 0, \quad m := \dfrac{1}{Q}\dfrac{\delta \ell}{\delta u} \\ \partial_t Q + \partial_s (Qu) = 0, \quad Q = A|\boldsymbol{\Gamma}| \end{cases}$$

Compatibility condition: $\Lambda_{st} = \Lambda_{ts}$, $\mathbf{r}_{st} = \mathbf{r}_{ts}$

$$\partial_t \boldsymbol{\Omega} = \boldsymbol{\omega} \times \boldsymbol{\Omega} + \partial_s \boldsymbol{\omega}\,, \quad \partial_t \boldsymbol{\Gamma} + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \partial_s \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \boldsymbol{\gamma}$$

Assume $A = A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$, symmetric tube with axis $\mathbf{E}_1$ for Lagrangian

$$\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) = \frac{1}{2}\int \Big(\alpha|\boldsymbol{\gamma}|^2 + \left\langle \mathbb{I}\boldsymbol{\omega}, \boldsymbol{\omega}\right\rangle + \rho A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})|\boldsymbol{\gamma} + \boldsymbol{\Gamma} u|^2 - \left\langle \mathbb{J}\boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle - \lambda|\boldsymbol{\Gamma} - \mathbf{E}_1|^2\Big)|\boldsymbol{\Gamma}|\mathrm{d}s\,.$$

See FGB & VP for linear stability analysis, nonlinear solutions *etc.*

Non-conservation of energy

Define the energy function

$$e(\boldsymbol{\omega},\gamma,\boldsymbol{\Omega},\boldsymbol{\Gamma},u) = \int_0^L \left(\frac{\delta \ell}{\delta \boldsymbol{\omega}} \cdot \boldsymbol{\omega} + \frac{\delta \ell}{\delta \boldsymbol{\gamma}} \cdot \boldsymbol{\gamma} + \frac{\delta \ell}{\delta u} u \right) \mathrm{d}s - \ell(\boldsymbol{\omega},\gamma,\boldsymbol{\Omega},\boldsymbol{\Gamma},u)$$

and boundary forces at the exit (free boundary)

$$F_u := \frac{\delta \ell}{\delta u} u - \mu Q \Big|_{s=L}, \quad \mathbf{F}_{\boldsymbol{\Gamma}} := \frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \mu \frac{\partial Q}{\partial \boldsymbol{\Gamma}} \Big|_{s=L}, \quad \mathbf{F}_{\boldsymbol{\Omega}} := \frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \mu \frac{\partial Q}{\partial \boldsymbol{\Omega}} \Big|_{s=L}.$$

Then, the energy changes according to

$$\frac{d}{dt} e(\boldsymbol{\omega},\gamma,\boldsymbol{\Omega},\boldsymbol{\Gamma},u) = \int_0^T \left(\mathbf{F}_{\boldsymbol{\Omega}} \cdot \boldsymbol{\Omega} + \mathbf{F}_{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Gamma} + F_u u \right) \Big|_{s=L}^{s=0} \mathrm{d}t.$$

The system is not closed and the energy is not conserved. Similar statement is true for variational discretization.

Variational discretization of tube conveying fluid in space: definitions

- As in Demoures *et al* (2014), discretize s as $s \to (s_0, s_1, \dots, s_N)$ and define the variables $\lambda_i := \Lambda_i^{-1}\Lambda_{i+1} \in SO(3)$ (relative orientation) and $\boldsymbol{\kappa}_i = \Lambda_i^{-1}(\mathbf{r}_{i+1} - \mathbf{r}_i) \in \mathbb{R}^3$ (relative shift).
- Define the forward Lagrangian map $s = \varphi(a, t)$ and back to labels map $a = \psi(s, t) = \varphi^{-1}(s, t)$.
- Discretize $\psi(s, t)$ as $\overline{\psi}(t) = (\psi_1(t), \psi_2(t), \dots, \psi_N(t))$ with $\psi_i(t) \simeq \psi(s_i, t)$.
- Discretize the spatial derivative as $D_i\overline{\psi}(t) := \sum_{j \in J} a_j \psi_{i+j}(t)$, where J is a discrete set around 0,
- For example, we can take $D_i\overline{\psi} = (\psi_i - \psi_{i-1})/h$ (backwards derivative), in that case

$$J = (-1, 0) \quad \text{and} \quad a_{-1} = -\frac{1}{h}\,, a_0 = \frac{1}{h}\,.$$

- For more general cases, for example, variable s-step, we take $D_i\overline{\psi}(t) := \sum_{j \in i+J} A_{ij}\psi_j(t)$.

Variational discretization of a tube conveying fluid in space: definitions

- Discretize the conservation law $(Q_0 \circ \varphi^{-1})\partial_s \varphi^{-1} = Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ as

$$Q_0 D_i \overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i \quad \Rightarrow \quad \dot{F}_i + D_i \left(\overline{uF} \right) = 0$$

- Differentiate the identity $s = \varphi(\psi(s,t), t)$ with respect to time to get $u(s,t) = (\varphi_t \circ \psi)(s,t)$ as

$$u(s,t) = (\partial_t \varphi \circ \psi)(s,t) = -\frac{\partial_t \psi(s,t)}{\partial_s \psi(s,t)} \quad \Rightarrow \quad u_i(t) = -\frac{\dot{\psi}_i}{D_i \overline{\psi}}$$

- Define the approximation for the action

$$S = \int \ell(\boldsymbol{\omega}, \gamma, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) \mathrm{d}t \mathrm{d}s \to S_d = \int \sum_i \ell_d(\boldsymbol{\omega}_i, \gamma_i, \lambda_i, \boldsymbol{\kappa}_i, u_i) \mathrm{d}t$$

Variational discretization of variables: variations

- Define the discrete action principle

$$\delta \int \sum_i \left[\ell_d(\boldsymbol{\omega}_i, \boldsymbol{\gamma}_i, \lambda_i, \boldsymbol{\kappa}_i, u_i) + \mu_i \left(Q_0 D_i \overline{\psi} - F(\lambda_i, \boldsymbol{\kappa}_i) \right) \right] \mathrm{d}t = 0$$

- Compute the variations of elastic in variables terms of free variations $\xi_i = \Lambda_i^{-1} \delta \Lambda_i \in \mathfrak{so}(3)$ and $\boldsymbol{\eta}_i = \Lambda_i^{-1} \delta \mathbf{r}_i \in \mathbb{R}^3$ as

$$\delta \lambda_i = -\xi_i \lambda_i + \lambda_i \xi_{i+1} \quad \delta \boldsymbol{\kappa}_i = -\boldsymbol{\xi}_i \times \boldsymbol{\kappa}_i + \lambda_i \boldsymbol{\eta}_{i+1} - \boldsymbol{\eta}_i \,,$$

- Compute the variations of velocity in terms of $\delta \psi_i$

$$\delta u_i = -\frac{\delta \dot{\psi}_i}{D_i \overline{\psi}} + \frac{\dot{\psi}_i}{(D_i \overline{\psi})^2} \sum_{j \in J} a_j \delta \psi_{i+j} = -\frac{Q_0}{D_i \overline{\psi}} \left(\delta \dot{\psi}_i + u_i D_i \overline{\delta \psi} \right) .$$

- Terms proportional to ξ_i give angular momentum conservation law
- Terms proportional to $\boldsymbol{\eta}_i$ give linear momentum conservation law
- Terms proportional to ψ_i give a fluid momentum, but we need to use the fluid conservation law $Q_0 D_i \overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i$ to remove all $\overline{\psi}$ from equations.

Variational integrator for spatial discretization I

- Angular momentum: terms proportional to $\boldsymbol{\xi}_i = \left(\Lambda_i^{-1}\delta\Lambda_i\right)^\vee$ [1]

$$\left(\frac{d}{dt} + \boldsymbol{\omega}_i\times\right)\frac{\partial \ell_d}{\partial \boldsymbol{\omega}_i} + \boldsymbol{\gamma}_i \times \frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i} + \left[\left(\frac{\partial \ell_d}{\partial \lambda_i} - \mu_i \frac{\partial F}{\partial \lambda_i}\right)\lambda_i^T - \lambda_{i-1}^T\left(\frac{\partial \ell_d}{\partial \lambda_{i-1}} - \mu_{i-1}\frac{\partial F}{\partial \lambda_{i-1}}\right)\right]^\vee + \boldsymbol{\kappa}_i \times \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i} - \mu_i \frac{\partial F}{\partial \boldsymbol{\kappa}_i}\right) = \mathbf{0}$$

Compare with the continuum equation:

$$(\partial_t + \boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma}\times\frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \frac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma}\times\left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = \mathbf{0}$$

[1] We denote $\widehat{\mathbf{a}} = -\epsilon_{ijk}\mathbf{a}_k$ is the hat map for $\mathbb{R}^3 \to \mathfrak{so}(3)$, and $a^\vee = \mathbf{a} \in \mathbb{R}^3$ is its inverse

Variational integrator for spatial discretization I

- Angular momentum: terms proportional to $\boldsymbol{\xi}_i = \left(\Lambda_i^{-1}\delta\Lambda_i\right)^\vee$ [1]

$$\left(\frac{d}{dt} + \boldsymbol{\omega}_i \times\right)\frac{\partial \ell_d}{\partial \boldsymbol{\omega}_i} + \boldsymbol{\gamma}_i \times \frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i} + \left[\left(\frac{\partial \ell_d}{\partial \lambda_i} - \mu_i \frac{\partial F}{\partial \lambda_i}\right)\lambda_i^T - \lambda_{i-1}^T\left(\frac{\partial \ell_d}{\partial \lambda_{i-1}} - \mu_{i-1}\frac{\partial F}{\partial \lambda_{i-1}}\right)\right]^\vee + \boldsymbol{\kappa}_i \times \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i} - \mu_i \frac{\partial F}{\partial \boldsymbol{\kappa}_i}\right) = \mathbf{0}$$

Compare with the continuum equation:

$$(\partial_t + \boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \frac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma} \times \left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = \mathbf{0}$$

- Linear momentum: terms proportional to $\boldsymbol{\eta}_i = \Lambda_i^{-1}\delta \mathbf{r}_i$

$$\left(\frac{d}{dt} + \boldsymbol{\omega}_i \times\right)\frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i} + \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i} - \mu_i \frac{\partial F}{\partial \boldsymbol{\kappa}_i}\right) - \lambda_{i-1}^T\left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_{i-1}} - \mu_{i-1}\frac{\partial F}{\partial \boldsymbol{\kappa}_{i-1}}\right) = \mathbf{0}$$

Corresponding continuum equation

$$(\partial_t + \boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = \mathbf{0}$$

[1] We denote $\widehat{\mathbf{a}} = -\epsilon_{ijk}\mathbf{a}_k$ is the hat map for $\mathbb{R}^3 \to \mathfrak{so}(3)$, and $a^\vee = \mathbf{a} \in \mathbb{R}^3$ is its inverse

Variational integrator for spatial discretization II

- Fluid momentum equation: terms proportional to $\delta\psi_i$

$$\frac{d}{dt}\left(\frac{1}{F_i}\frac{\partial \ell_d}{\partial u_i}\right) + D_i^+\left(\overline{\frac{u}{F}\frac{\partial \ell_d}{\partial u}} - \overline{\mu}\right) = 0$$

where we have defined the dual discrete derivative
$D_i^+\overline{X} := -\sum_{j\in J} a_j X_{i-j}$, and $m^\vee{}_c := -\frac{1}{2}\sum_{ab}\epsilon_{abc}m_{ab}$

Continuum equation:

$$m_t + \partial_s\left(mu - \mu\right) = 0, \quad m := \frac{1}{Q}\frac{\delta \ell}{\delta u}$$

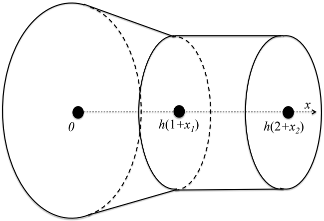
- Conservation law in the discrete form:

$$Q_0 D_i\overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i \quad \Rightarrow \quad \dot{F}_i + D_i\left(\overline{uF}\right) = 0$$

Continuum version

$$Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) := A\left|\boldsymbol{\Gamma}\right| = \left(Q_0 \circ \varphi^{-1}(s,t)\right)\, \varphi' \circ \varphi^{-1}(s,t) \Rightarrow \partial_t Q + \partial_s(Qu) = 0$$

An example: 1D stretching motion



- Assume that all motion of the tube is along the $\mathbf{E}_1$ direction, so $\mathbf{r}_k = h(k + x_k, 0, 0)^T$ and $\Lambda_i = \mathrm{Id}_{3\times 3}$, where x_k is the dimensionless deviation from equilibrium.
- Consider a simplified model with only three points, $k = 0, 1, 2$, denote $x = x_1$.
- Fixed BC on the left, $x_0 = 0$ and no deformation in the cross-section.
- Free BC on the right, $x_2 = x_1 = x$.
- Express all variables u_i, μ_i in terms of x_i and its time derivatives.
- Get a nonlinear ODE $\ddot{x} = f(x, \dot{x})$ for a single variable $x(t)$.

Numerical solutions of stretching tube equations

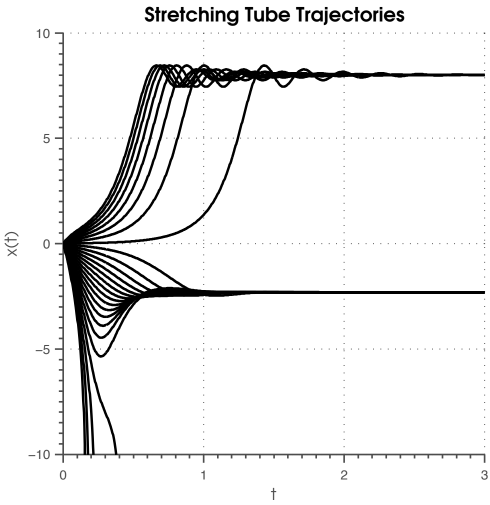


Figure: Trajectories $x(t)$ starting with $x(0) = 0$ for varying initial conditions $x'(0) = x'_0$.

Steady states and their stability as a function of u_0

Parameter values:

$$h = 0.1,\ T = 1,\ \mu_0 = 1,\ \rho = 11,\ F_1 = 2,\ \alpha = 1,\ \beta = 3,\ \xi = 1.$$

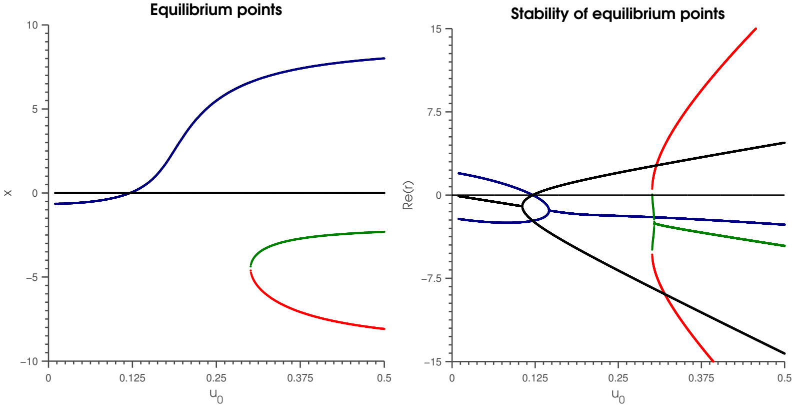


Figure: Left: Equilibrium points as a function of u_0, Right: their stability. Color labeling is the same for each equilibrium point.

Time and space discretization

- Discretize $s \to (s_0, s_1, \ldots, s_N)$ and $t \to (t_0, t_1, \ldots, t_M)$.
- Define the temporal and spatial relative orientations and shifts (first index is s, second index is t):

$$\lambda_{i,j} := \Lambda_{i,j}^{-1}\Lambda_{i+1,j}\,, \quad \boldsymbol{\kappa}_{i,j} := \Lambda_{i,j}^{-1}\left(\mathbf{r}_{i+1,j} - \mathbf{r}_{i,j}\right)$$
$$q_{i,j} := \Lambda_{i,j}^{-1}\Lambda_{i,j+1}\,, \quad \gamma_{i,j} := \Lambda_{i,j}^{-1}\left(\mathbf{r}_{i,j+1} - \mathbf{r}_{i,j}\right)\,.$$

- Define discrete spatial and temporal derivatives are
$D^s_{i,j}\overline{\psi} := \sum_{k\in K} a_j \psi_{i,j+k}\,, \quad D^t_{i,j}\overline{\psi} := \sum_{m\in M} b_m \psi_{i+m,j}$
- The velocity is given by

$$u_{i,j} = -\frac{D^t_{i,j}\overline{\psi}}{D^s_{i,j}\overline{\psi}} \qquad \left(\text{Compare with} \quad u = -\frac{\psi_t}{\psi_s}\right)$$

- Discrete conservation law is

$$Q_0 D^s_{i,j}\overline{\psi} = F_{i,j} \quad \Rightarrow \quad D^t_{i,j}\overline{F} + D^s_{i,j}(\,\overline{uF}\,) = 0\,.$$

Variational integrator in time and space

- Consider the critical discrete action principle
$$\delta \sum_{i,j} \mathcal{L}_d\left(\lambda_{i,j}, \boldsymbol{\kappa}_{i,j}, q_{i,j}, \gamma_{i,j}, u_{i,j}\right) + \mu_{i,j}\left(Q_0 D^s_{i,j}\overline{\psi} - F(\lambda_{i,j}, \boldsymbol{\kappa}_{i,j})\right) = 0$$
- Perform variations to obtain equations of motion
- Angular momentum equation: terms proportional to $\boldsymbol{\Sigma}_{i,j} = \left(\Lambda^{-1}_{i,j}\delta\Lambda_{i,j}\right)^\vee$

$$\left[\frac{\partial \mathcal{L}_d}{\partial q_{i,j}} q^T_{i,j} - q^T_{i,j-1}\frac{\partial \mathcal{L}_d}{\partial q_{i,j-1}}\right]^\vee + \left[\left(\frac{\partial \mathcal{L}_d}{\partial \lambda_{i,j}} - \mu_{i,j}\frac{\partial F}{\partial \lambda_{i,j}}\right)\lambda^T_{i,j}\right.$$
$$\left. - \lambda^T_{i-1,j}\left(\frac{\partial \mathcal{L}_d}{\partial \lambda_{i-1,j}} - \mu_{i-1,j}\frac{\partial F}{\partial \lambda_{i-1,j}}\right)\right]^\vee + \boldsymbol{\gamma}_{i,j} \times \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\gamma}_{i,j}} + \boldsymbol{\kappa}_{i,j} \times \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\kappa}_{i,j}} = \mathbf{0}$$

Continuum equation for reference

$$(\partial_t + \boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \frac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma} \times \left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = \mathbf{0}$$

Equations of motion, continued

- Linear momentum equation: terms proportional to $\boldsymbol{\Psi}_{i,j} = \Lambda_{i,j}^{-1}\delta\mathbf{r}_{i,j}$

$$\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\gamma}_{i,j}} - q_{i,j-1}^T \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\gamma}_{i,j-1}} + \left(\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\kappa}_{i,j}} - \mu_{i,j} \frac{\partial F}{\partial \boldsymbol{\kappa}_{i,j}} \right) - \lambda_{i-1,j}^T \left(\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\kappa}_{i-1,j}} - \mu_{i-1,j} \frac{\partial F}{\partial \boldsymbol{\kappa}_{i-1,j}} \right) = \mathbf{0}$$

Continuum version for reference:

$$\left(\partial_t + \boldsymbol{\omega}\times\right) \frac{\delta \ell}{\delta \boldsymbol{\gamma}} + \left(\partial_s + \boldsymbol{\Omega}\times\right) \left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}} \mu \right) = \mathbf{0}$$

- Fluid momentum equation: terms proportional to $\delta\psi_{i,j}$

$$D_{i,j}^{t,+}\overline{m} + D_{i,j}^{s,+}\left(\overline{um} - \overline{\mu}\right) = 0\,, \quad m_{i,j} := \frac{1}{F_{i,j}} \frac{\partial \mathcal{L}_d}{\partial u_{i,j}}$$

$$D_{i,j}^{s,+}\overline{X} := -\sum_{k\in K} a_k X_{i,j-k}\,, \quad D_{i,j}^{t,+}\overline{X} := -\sum_{m\in M} b_j X_{i-m,j}$$

Continuum version: $\quad m_t + \partial_s\left(mu - \mu\right) = 0, \quad m := \dfrac{1}{Q}\dfrac{\delta \ell}{\delta u}$

Tube with expandable walls filled with compressible gas

1. Add entropy S and density ρ as variables; internal energy $e(\rho, S)$

$$\mathrm{d}e = -p\,\mathrm{d}\left(\frac{1}{\rho}\right) + T\mathrm{d}S \Rightarrow p(\rho, S) = \rho^2 \frac{\partial e}{\partial \rho}(\rho, S)\,,\; T(\rho, S) = \frac{\partial e}{\partial S}(\rho, S)\,,$$

2. Changes in radius of tube $R(s,t)$ contributing to elastic energy, $A = \pi R^2$, $Q = A|\boldsymbol{\Gamma}|$
3. Remove the incompressibility condition
4. Equations for density and entropy

$$\xi_t + \partial_s \xi u = 0\,, \quad S_t + u\partial_s S = 0\,, \quad \xi := \rho Q\,.$$

5. Symmetry reduced Lagrangian

$$\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u, \xi, S, R, \dot{R}) = \ell_0 - \xi e\,, \quad \xi := \rho Q$$

6. Perform variations to obtain equations of motion

Equations of motion

$$
\begin{cases}
(\partial_t + \boldsymbol{\omega}\times)\dfrac{\partial \ell_0}{\partial \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \dfrac{\partial \ell_0}{\partial \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\partial \ell_0}{\partial \boldsymbol{\Omega}} + \cancel{p\dfrac{\partial Q}{\partial \boldsymbol{\Omega}}}\right) \\
\qquad + \boldsymbol{\Gamma} \times \left(\dfrac{\partial \ell_0}{\partial \boldsymbol{\Gamma}} + \cancel{p\dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}}\right) = 0 \\
(\partial_t + \boldsymbol{\omega}\times)\dfrac{\partial \ell_0}{\partial \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\partial \ell_0}{\partial \boldsymbol{\Gamma}} + p\dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\right) = 0 \\
\partial_t \dfrac{\partial \ell_0}{\partial u} + u\partial_s \dfrac{\partial \ell_0}{\partial u} + 2\dfrac{\partial \ell_0}{\partial u}\partial_s u = \xi \partial_s \dfrac{\partial \ell_0}{\partial \xi} - Q\partial_s p \\
\partial_t \dfrac{\partial \ell_0}{\partial \dot{R}} - \partial_s^2 \dfrac{\partial \ell_0}{\partial R''} + \partial_s \dfrac{\partial \ell_0}{\partial R'} - \dfrac{\partial \ell_0}{\partial R} - p\dfrac{\partial Q}{\partial R} = 0 \\
\partial_t \boldsymbol{\Omega} = \boldsymbol{\Omega} \times \boldsymbol{\omega} + \partial_s \boldsymbol{\omega}, \qquad \partial_t \boldsymbol{\Gamma} + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \partial_s \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \boldsymbol{\gamma} \\
\partial_t \xi + \partial_s(\xi u) = 0, \qquad \partial_t S + u\partial_s S = 0
\end{cases}
$$

If $Q = \pi R^2 |\boldsymbol{\Gamma}|$ then some terms ~~cancel~~.

Rankine-Hugoniot conditions

Define $[f]$ to be the jump of f across the shock. Then, assume that the tube is continuous so $[\gamma]=0$, $[\mathbf{\Gamma}]=0$ *etc.* to obtain

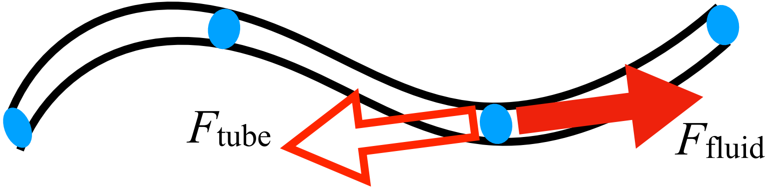
$$c[\rho]=[\rho u] \quad \text{(mass)}$$
$$c\left[\rho u\right]=\left[\rho u^2+\frac{1}{|\mathbf{\Gamma}|^2}p\right] \quad \text{(momentum)}$$
$$c\left[E\right]=\left[\frac{1}{2}\rho\left|\gamma+\mathbf{\Gamma}u\right|^2+\frac{p}{|\mathbf{\Gamma}|^2}\mathbf{\Gamma}\cdot(\gamma+\mathbf{\Gamma}u)+\rho u e\right] \quad \text{(energy)}$$

Compare with R-H conditions for straight tube: $\mathbf{\Gamma}=\mathbf{E}_1$, $\gamma=0$:

$$c[\rho]=[\rho u] \quad \text{(mass)}$$
$$c[\rho u]=[\rho u^2+p] \quad \text{(momentum)}$$
$$c[E]=\left[\left(\frac{1}{2}\rho u^2+\rho e+p\right)u\right] \quad \text{(energy)}$$

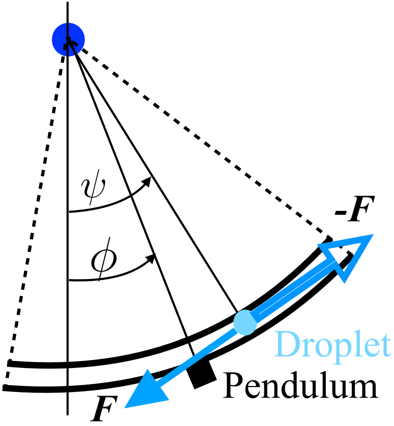
(FGB, VP, in preparation)

On the role of friction in the tube conveying fluid



In spatial frame, there are equal and opposite forces acting on the tube from the fluid, and fluid from the tube.
Let us study a simplified model where friction dominates the motion of the fluid – Darcy's law

A simple problem: pendulum with a viscous droplet



Spatial frame: Deviation of pendulum of mass M from vertical is ϕ, deviation of droplet of mass m from vertical is ψ; length of pendulum L.

Dynamics of the pendulum with droplet I

Lagrangian:

$$L_0 = \frac{1}{2}ML^2\dot{\phi}^2 + MgL\cos\phi + \frac{1}{2}mL^2\dot{\psi}^2 + mgL\cos\psi$$

Choose time scale $T = \sqrt{L/g}$, rescale Lagrangian by MgL to obtain

$$L = \frac{1}{2}\dot{\phi}^2 + \cos\phi + \epsilon\left(\frac{1}{2}\dot{\psi}^2 + \cos\psi\right), \quad \epsilon := \frac{m}{M}$$

Darcy's law: Assume that friction dominates the motion of fluid. Darcy's law reads

$$\text{Relative velocity} = K \times \text{gravity force} \qquad \Rightarrow \dot{\psi} - \dot{\phi} = -\alpha\sin\psi$$

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Nonholonomic constraint! Use Lagrange-d'Alembert's method

$$\delta\int L dt = 0 \quad \text{on variations satisfying} \quad \delta\psi - \delta\phi = 0$$

$$\text{Equations of motion}: \begin{cases} \ddot{\phi} + \sin\phi + \epsilon\left(\ddot{\psi} + \sin\psi\right) = 0 \\ \dot{\psi} - \dot{\phi} = -\alpha\sin\psi \end{cases} \tag{10}$$

Energy behavior on solutions

Define total energy $E = \frac{1}{2}\left(\dot{\phi}^2 + \epsilon\dot{\psi}^2\right) - (\cos\phi + \epsilon\cos\psi)$.
Then energy evolves according to

$$\dot{E} = (\ddot{\phi} + \sin\phi)\dot{\phi} + \epsilon(\ddot{\psi} + \sin\psi)\dot{\psi} = \epsilon\left(\ddot{\psi} + \sin\psi\right)\left(\dot{\psi} - \dot{\phi}\right)$$

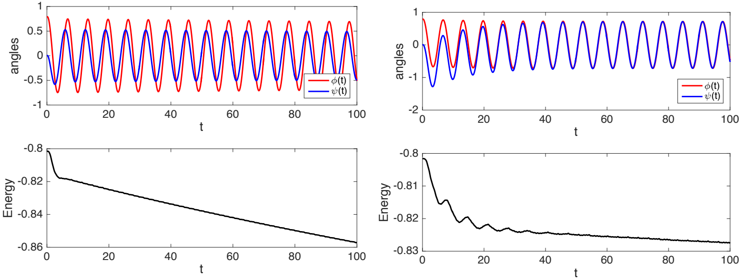

Figure: Top: solutions $\phi(t)$ and $\psi(t)$. Bottom: Energy $E(t)$.

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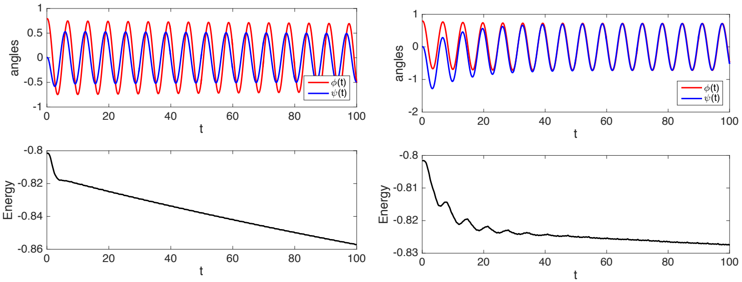

Figure: Top: solutions $\phi(t)$ and $\psi(t)$. Bottom: Energy $E(t)$.

The answer is wrong! The energy cannot increase, since all the friction forces are internal

Another approach

Kozlov (1980's-90's) and follow-up papers:

It is not sufficient to define the Lagrangian and the constraint. One needs to know what the physics is to derive the equations of motion.

Lagrange-d'Alembert's for dynamics with non-conservative forces

$$\int L\mathrm{d}t = \int F_{\text{body}}\delta\phi + F_{\text{fluid}}\delta\psi = \int A\left(\dot{\phi} - \dot{\psi}\right)(\delta\phi - \delta\psi)\,\mathrm{d}t$$

Equations of motion: $$\begin{cases} \ddot{\phi} + \sin\phi = -A(\dot{\phi} - \dot{\psi}) \\ \epsilon\left(\ddot{\psi} + \sin\psi\right) = A(\dot{\phi} - \dot{\psi}) \end{cases}$$

Then, energy evolves as

$$\dot{E} = \left(\ddot{\phi} + \sin\phi\right)\dot{\phi} + \epsilon\left(\ddot{\psi} + \sin\psi\right)\dot{\psi} = -A\left(\dot{\psi} - \dot{\phi}\right)^2 \leq 0$$

Moreover, $\dot{E} = 0$ iff $\dot{\phi} = \dot{\psi}$ (synchronization)

Results of simulations

$$z = \phi - \psi \Rightarrow z'' + A\frac{1+\epsilon}{\epsilon}z' + 2\cos(\frac{\phi+\psi}{2})\sin(\frac{z}{2}) = 0$$

For small ϕ and ψ, the state $z_* = 0$ is linearly stable

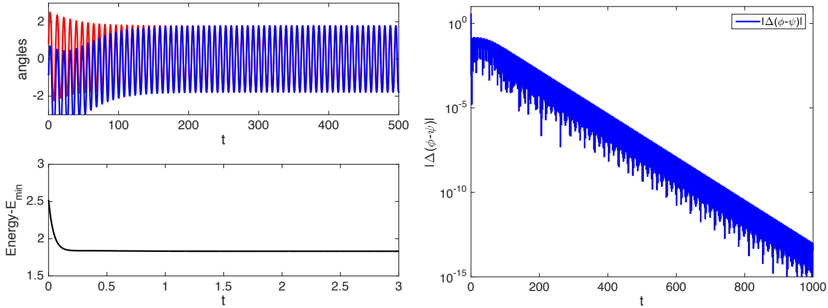

Figure: Left: solutions for $\epsilon = 0.1$ and $A = 1$, Right: $|\phi - \psi|$ vs t.

Solutions converge to $\phi = \psi$ ('constraint manifold') after initial decay.

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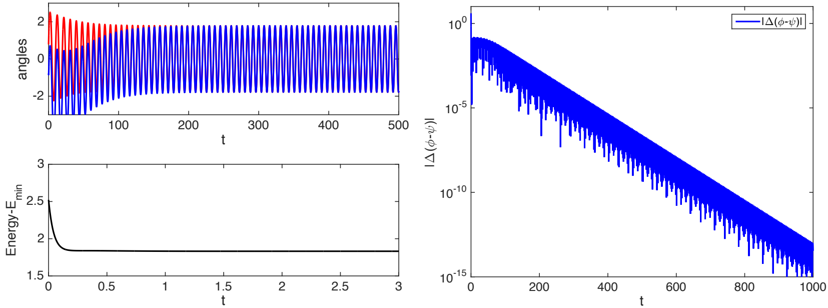

Figure: Left: solutions for $\epsilon = 0.1$ and $A = 1$, Right: $|\phi - \psi|$ vs t.

Solutions converge to $\phi = \psi$ ('constraint manifold') after initial decay. The constraint is holonomic from the dynamics chosen by the system as $t \to \infty$. That is the 'dynamic' Darcy's law.

Darcy's law, energy behavior, and generalizations

Let us introduce another potential force on the droplet to augment Darcy's law

$$L = \frac{1}{2}\dot{\phi}^2 + \cos\phi + \epsilon\left(\frac{1}{2}\dot{\psi}^2 + S\cos\psi\right)\,, \quad \epsilon := \frac{m}{M}, \quad S \neq 1 > 0$$

We obtain the equations of motion

$$\begin{cases} \ddot{\phi} + \sin\phi = -A(\dot{\phi} - \dot{\psi}) \\ \epsilon\left(\ddot{\psi} + S\sin\psi\right) = A(\dot{\phi} - \dot{\psi}) \end{cases}$$

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No convergence to constraint manifold! $\psi = \psi = 0$ is asymptotically stable, and all solutions $\to 0$ as $t \to \infty$

Behavior of solutions

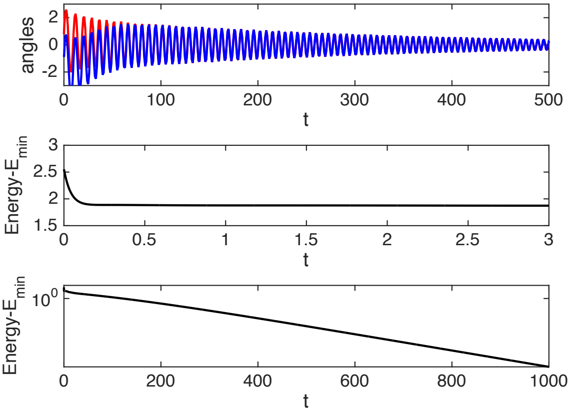

Figure: Simulations of equations with $S = 2$, $A = 1$ and $\epsilon = 0.1$

Fast decay to 'slow manifold'; slow decay to 0.

Need to consider time scales as well (order of limits, large but finite times)

Digression: an even simpler problem

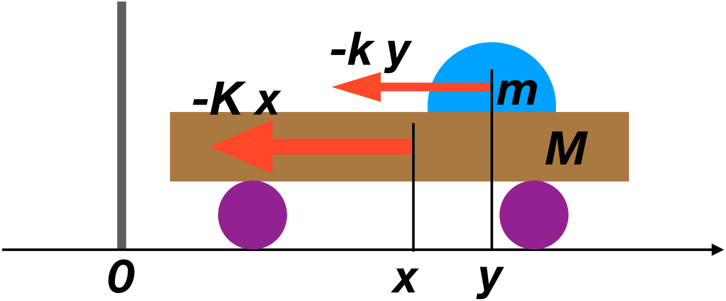

Figure: Droplet on a moving cart

Linear equations of motion:

$$\begin{cases} \ddot{x} + x = -A\left(\dot{x} - \dot{y}\right) \\ \epsilon\left(\ddot{y} + Sy\right) = A\left(\dot{x} - \dot{y}\right) \end{cases} \tag{11}$$

Asymptotically stable for $S \neq 1$, all solutions $\to 0$

Stable at $S = 1$: convergence to $x = y$ (synchronization).

Coming back to the pendulum: body frame

Body variables (capitals) are defined

$$\Phi = \phi \,, \quad \Psi = \psi - \phi \quad \Rightarrow \quad \phi = \Phi, \quad \psi = \Psi + \Phi \tag{12}$$

Spatial Lagrangian transforms into the body Lagrangian as

$$L_B = \left(\frac{1}{2} \dot{\Phi}^2 + \cos \Phi \right) + \epsilon \left(\frac{1}{2} \left(\dot{\Psi} + \dot{\Phi} \right)^2 + \cos(\Phi + \Psi) \right) \tag{13}$$

Transformation of forces using L-d'A external forces

$F_{\mathrm{f,sp}}$ and $F_{\mathrm{s,sp}}$ are forces acting on the fluid and the solid in spatial frame. Then body frame forces are computed as

$$\delta \int L \mathrm{d}t = \int F_{\mathrm{f,sp}} \delta\psi + F_{\mathrm{s,sp}} \delta\phi \mathrm{d}t = \int \underbrace{\cancel{\left(F_{\mathrm{f,sp}} + F_{\mathrm{s,sp}} \right)}}_{\text{body}} \delta\Phi + \underbrace{F_{\mathrm{f,sp}}}_{\text{fluid}} \delta\Psi \tag{14}$$

$$\text{Equations of motion}: \begin{cases} \ddot{\Phi} + \sin \Phi = A \dot{\Psi} \\ \epsilon \left(\ddot{\Psi} + \ddot{\Phi} + \sin(\Psi + \Phi) \right) = -A \dot{\Psi} \end{cases} \tag{15}$$

Darcy's law $u_{\text{rel}} = \mu(\nabla p + \mathbf{f})$ (spatial frame)
However, μ depends on the local properties of the fluid – must be in the body frame.

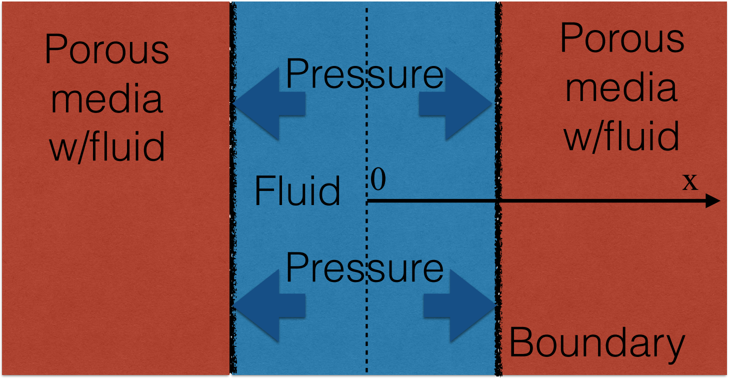

Figure: One-dimensional porous media: opening the gap

Variables and variations

1. Motion of porous media $x = \psi(X,t)$ – embeddings in $\mathbb{R}^1$
2. $A = \varphi(X,t)$ is the Lagrangian motion of fluid particles starting at X
3. $f(X,t)$ is porosity with conservation law $Q_0 \circ \varphi^{-1}(X,t)\partial_X \varphi^{-1}(X,t) = Q(X,t)$, and $Q = f(\psi_x)\psi_X$
4. Relative fluid velocity $U = \dot{\varphi} \circ \varphi^{-1}(X,t)$
5. Absolute fluid velocity
$$u(X,t) = \frac{\partial x_f}{\partial t} \circ \varphi^{-1}(X,t) = \psi_t + U\psi_X \,.$$
6. Variations in U are computed as $\delta U = \eta_t + U\partial_X \eta - \eta \partial_X U$, with $\eta = \delta\varphi \circ \varphi^{-1}$
7. Lagrangian $L = L(\psi, \psi_X, U)$
8. Spatial friction $F_{\text{fluid,s}} = -K(u - \dot{\psi})$, $F_{\text{media,s}} = K(u - \dot{\psi})$

Variational principle

1. Taking variations as follows

$$\delta \int L(\psi_t, \psi_X, U) - P\big(Q_0 \circ \varphi^{-1}(X,t)\partial_X \varphi^{-1}(X,t) - Q(X,t)\big)\mathrm{d}X\mathrm{d}t$$

$$= \int F_{\text{fluid,b}}\, \eta + F_{\text{media, b}}\, \delta\psi \,\mathrm{d}X\mathrm{d}t$$

2. Can be generalized to 3D and arbitrary metrics using D/Dt, DIV and ∇ operators (see Marsden & Hughes, and also FGB's talk)

3. Equations of motion (cf. MacMinn et al, 2016 in spatial frame and spatial Darcy's law):

$$\begin{cases} \partial_t \dfrac{\partial L}{\partial U} + U\partial_X \dfrac{\partial L}{\partial U} + 2\dfrac{\partial L}{\partial U}\partial_X U = -Q\dfrac{\partial P}{\partial X} - \mu U\,, \quad Q := f(\psi_X)\psi_X \\ \partial_t \dfrac{\partial L}{\partial \psi_t} + \partial_X \dfrac{\partial L}{\partial \psi_X} - \partial_X\left(P\dfrac{\partial Q}{\partial \psi_X}\right) = \cancel{F_{pm}} \\ Q_t + \partial_X(QU) = 0 \end{cases}$$

$$E := \int \left(\psi_t \frac{\partial L}{\partial \psi_t} + U\frac{\partial L}{\partial U} - L\right)\mathrm{d}X \quad \Rightarrow \quad \dot{E} = -\int \mu U^2 \mathrm{d}X \le 0$$

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4. Procedure can also be repeated in spatial frame. Why?

Conclusions and future work

1. Variational methods lead to consistent equations for fluid-structure interactions problem
2. Fluid conservation leads to holonomic constraints, viscous forces lead to constraints on 'inertial manifold' (non-holonomic?)
3. One needs to be careful defining limits and computing Darcy's law
4. ? How do we compute Darcy's law without solving the complete problem
5. ? Darcy's law as non-holonomic constraint?
6. ? Dynamic porosity/permeability
7. ? When should we use elastic body frame vs spatial frame?
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Happy birthday, Darryl!