Making gauge symmetries into Casimirs in non-holonomic systems

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Background

- Non-holonomic systems are not Hamiltonian. However, van der Schaft and Mashcke (1994) introduced an 'almost' Poisson bracket (no Jacobi identity) for which the dynamics is 'almost' Hamiltonian.
- There are several symmetric non-holonomic systems which appear to be integrable. Duistermaat (2004) writes, *Although the [Chaplygin sphere] is integrable in every sense of the word, it neither is a hamiltonian system, nor is the integrability an immediate consequence of the symmetries.*
- Borisov and Mamaev (2002) find a different Possion bracket for the Chaplygin sphere for which the reduced equations are Hamiltonian (explaining integrability).
- LGN shows that vdS-M and B-M Poisson brackets are related by a 'deformation'
- Ideas extended by LGN & Balseiro (2012), showing how vdS-M Poisson brackets can be modified in some examples.

Question

In Hamiltonian systems with symmetry, any G-invariant conserved quantity arising from Noether's theorem passes down to the orbit space as a Casimir of the canonical Poisson structure.

Does this hold for nonholonomic systems?

Basically, YES!!!

But not in such generality

and anyway, ... Casimirs with respect to which Poisson structure?

Nonholonomic systems

- Configuration space Q
- Lagrangian $\mathcal{L} = \text{kinetic} - \text{potential}$ (assumed to be regular)
- Linear constraint: sub-bundle (distribution) $D \subset TQ$

Thus phase space is $q \in Q,\ \dot{q} \in D_q$; that is, phase space is D.

Leads to the Lagrange-D'Alembert equations of motion:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}}{\partial \dot{q}^i} &= R_i(q, \dot{q}) \\ R_i \dot{q}^i &= 0 \end{aligned}$$

where $\mathbf{R} = R_i dq^i$ is the (unknown) reaction force,
(linear form at each point of TQ annihilating D)

• Solving for R_i leaves a first order ODE on D.

Symmetry

A group G acting on Q lifts naturally to an action on TQ.
To be a symmetry, it should

- preserve Lagrangian $\mathcal{L}$
- preserve constraint distribution D.

In that case the dynamics on D is equivariant, and the dynamics passes down to the orbit space D/G.

Horizontal symmetries: For an infinitesimal generator of the action lying everywhere in the constraint distribution there is an analogue of Noether's theorem stating that the corresponding (linear) momentum is conserved.

'Unfortunately' this is not very common (eg, rolling sphere)

Gauge symmetry

Choose a non-zero element of $D_q \cap \mathfrak{g} \cdot q$, depending smoothly on q

— write corresponding (non-constant) element of $\mathfrak{g}$ as ξ_q and put

$$Z(q) = \xi_q(q).$$

This is a **gauge symmetry** (provided it still preserves $\mathcal{L}$ and D), and corresponding momentum p is conserved.

If this is equivariant then p is G-invariant, so passes down to orbit space D/G.

Question : is it a Casimir of (some) dynamical Poisson structure?

Hamiltonian version

Poisson brackets are naturally on D^* not D, so let D^* be any sub-bundle of T^*Q for which pairing $D \times D^* \to \mathbb{R}$ is non-degenerate.

Use coordinates q^i, π_α on D^* ($i = 1, \ldots, n$, $\alpha = 1, \ldots, r$)

Have Legendre transform as usual — defines Hamiltonian $H(q, \pi)$, and ODE (vector field) on D^*.

Can define a 'non-holonomic' almost Poisson structure on D^* for which vector field is Hamiltonian vector field of H:

$$X_H = \{H, -\}_{nh}.$$

That is,

$$\begin{cases} \dot{q}^i & = & \{H, q^i\}_{nh} \\ \dot{\pi}_\alpha & = & \{H, \pi_\alpha\}_{nh} \end{cases}$$

But it's not unique !!

Regularity

Assume G acts freely

or at least restrict to part of Q where it does act freely (but see later)

and

D is of constant rank ($\dim D_q$ is constant)

Moving frame approach

Choose *equivariant* vector fields which span constraint distribution D:

$$\{X_1, \ldots, X_r\} \;=\; \{Z_1, \ldots, Z_\ell; X_{\ell+1}, \ldots, X_r\}$$

where Z_α are gauge symmetries (requires *free* action!). Define functions on Q by,

$$C_{\alpha\beta\gamma} \;=\; \langle [X_\alpha, X_\beta], X_\gamma \rangle \, .$$

The vdS-M nh almost-Poisson structure on D^* is given by

$$\{q^i, q^j\}_{nh} = 0, \quad \{q^i, \pi_\alpha\}_{nh} = \rho^i_\alpha, \quad \{\pi_\alpha, \pi_\beta\}_{nh} = C^\gamma_{\alpha\beta}\pi_\gamma$$

where

- $X_\alpha(q) = \rho^i_\alpha(q)\frac{\partial}{\partial q^i}$ (moving frames)
- $C^\gamma_{\alpha\beta} = g^{\delta\gamma} C_{\alpha\beta\delta}$ (g = metric tensor = kinetic energy)

Characterization of gauge symmetries

Let X_α be a horizontal vector field which is tangent to group orbits.

That is, $X_\alpha(q) \in D_q \cap \mathfrak{g} \cdot q$

Then we show,

Theorem 1 (LGN & JM)

X_α is a gauge symmetry iff,

$$\forall \beta, \gamma, \qquad C_{\alpha\beta\gamma} = -C_{\alpha\gamma\beta}.$$

That is, iff

$$\langle [X_\alpha, X_\beta], X_\gamma \rangle = -\langle [X_\alpha, X_\gamma], X_\beta \rangle$$

Gauge 3-form Λ

Let $\{\mu^\alpha\}$ be 1-forms on Q spanning D^* and dual to X_α. Choose a 3-form on Q by

$$\Lambda = B_{\alpha\beta\gamma}\, \mu^\alpha \wedge \mu^\beta \wedge \mu^\gamma$$

where

- $B_{\alpha\beta\gamma}$ are G-invariant functions on Q (and skew-symmetric in the indices),
- for $\alpha = 1, \ldots, \ell$, $B_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}$

(skew-symmetry guaranteed for these last coefficients)

Main point: Using Λ we can modify the nh-Poisson bracket in such a way that the momenta associated to the gauge symmetries Z_α become Casimirs on D^*/G. Relies on 'gauge transformation' of Poisson brackets introduced by Ševera & Weinstein (2001):

Modifying Poisson brackets

After Ševera-Weinstein (2001) . . . Let P be an (almost) Poisson manifold

Ingredient: a 2-form B

Graph of the (almost) Poisson bracket at each point:

$$\Gamma = \Big\{(\mathbf{u}, \alpha) \in TP \oplus T^*P \mid \mathbf{u} = \{\alpha, -\}\Big\}$$

Shift this as follows:

$$\Gamma_B = \Big\{(\mathbf{u}, \alpha) \in TP \oplus T^*P \mid (\mathbf{u}, \alpha + i_{\mathbf{u}}B) \in \Gamma\Big\}$$

Under some conditions on B this is an (almost) Poisson structure.

In our scenario . . .

Define B by contracting the form Λ with the Hamiltonian vector field. Then use above gauge transformation of $\{-,-\}_{nh}$ to produce

$$\{-,-\}_{nh}^{\Lambda}$$

(NB: condition on B for this to be almost Poisson is always satisfied)

Theorem 2 (LGN & JM)

For any collection $\{Z_\alpha\}$ of equivariant gauge symmetries, this G-invariant almost Poisson structure has the following properties,

- the dynamics is (almost) Hamiltonian,
- the p_α become Casimirs on D^*/G $(\alpha = 1, \ldots, \ell)$

Hamiltonization

- Because every almost-Poisson structure on a surface is in fact Poisson (cf. every 2-form on a surface is closed), this explains some Hamiltonization results when dimension of $\{p_\alpha = \mathrm{const}\}$ in D^*/G is 2.

- If this dimension is > 2 then in some examples in spite of being not Poisson, the characteristic distribution is integrable, and this allows replacement of $\{-,-\}$ by

$$\phi(x)\{-,-\}$$

  which *is* Poisson, and the ϕ-term can be interpreted as change of time parameter.
- Can use energy-Casimir methods to study stability of relative equilibria.

Global consideration

Construction of Λ requires choice of equivariant moving frame — not global in general!

However, using the flexibility in the coefficients of Λ one can show, if the action is free,

Theorem 2 (LGN & JM)

Given a cover of Q by G-invariant open sets there is a choice of coefficients in each open set of the cover such that the different Λ coincide on the intersections, thus defining a global 3-form.

Finally, in examples the global 3-form extends to points where the action is not free.

Bibliography

- L.C. García-Naranjo & J. Montaldi, *Gauge momenta as Casimirs of nonholonomic systems*. ArXiv: 1610.05618

Congratulations Darryl