Cubic vs. minimal time splines on the sphere

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New Trends in Applied Geometric Mechanics

Celebrating Darryl Holm's 70th birthday

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Happy birthday, Darryl!

References

• About simple variational splines
from the Hamiltonian viewpoint
JGM, 9:3, 257-290, 2017 doi:10.3934/jgm.2017011

• Minimal time splines on the sphere
São Paulo Journal of Mathematical Sciences)
special number in honor to Waldyr Oliva (to appear, 2017)

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... my colleagues in Brazil ...

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Motivation: space engineering rendez-vous maneuvres

Warning: spoilers next

The Martian


Get it on Blu-ray & Digital HD
MATT DAMON
THE
MARTIAN
BRING HIM HOME

Hidden figures

Go-no go from elliptic to parabolic

"If she says the numbers are good,
I am ready to go" (says John Glenn)

Mary Jackson

Katherine Johnson

Dorothy Vaughan

John Glenn

Simple variational splines: optimal control with state space TQ

- State equation $\nabla_{\dot\gamma(t)}\dot\gamma(t) = u \in TQ$
- Minimize some functional of $\gamma(t)$, $\dot\gamma(t)$, $u(t)$.
- Boundary conditions: initial and final vectors.

The Lagrangian approach was first studied by Andrew Lewis and Richard Murray in the mid 1990's.

Online appendix of Lewis and Bullo's book has a section.

Recent results by M. Barbero-Liñán.

Many people worked (and still work) in the theme since the 1990's.

Noakes, Heinzinger and Paden

P. Crouch, Fatima Silva-Leite (and her group)

For higher order splines

Balmaz/Holm/Meier/Ratiu/Vialard

T. Bloch, L. Colombo, D. Martin, ...

(sorry for many omissions)

Simple variational splines

A smooth time-parametrized curve $\gamma(t)$ connecting two prescribed tangent vectors in TQ where Q is a Riemannian manifold.

Cubic, or L^2 splines

$$L = \int_0^T |u|^2 \, dt$$

Time minimal, or L^∞ splines

Connect the end vectors in minimum time, under the constraint of acceleration norm $\leq A$.

Research proposal:

time minimal splines in Diff (controlling EPDiff/LDDMM)

Recent papers used cubic splines in computational anatomy.

$$
\begin{array}{ll}
\underset{u}{\text{minimize}} & \frac{1}{2}\int_0^1 \|u(t)\|_{\mathfrak{g}}^2 dt \\
\text{subject to control} & u(t) - \dot{m}(t) - \mathrm{ad}^*_{v(t)}\, m(t) = 0, \\
\text{subject to right action} & \dot{g}(t) = v \circ g(t), \\
\text{subject to image evolution} & I(t) = I(0) \circ g^{-1}(t), \\
\text{subject to momenta duality} & v(t) = K \star m(t).
\end{array}
$$

N. Singh. M. Niethammer, Splines for Diffeomorphic Image Regression. MICCAI 2014. Lecture Notes in Computer Science, vol 8674.

N. Singh, F.-X. Vialard and M. Niethammer, Splines for diffeomorphisms, Medical Image Analysis, 25 (2015), 56 –71.

We argue (following L. Noakes) that time-minimal may have advantages over cubic splines.

• **Pauley and Noakes showed that cubic splines behave badly in manifolds of negative curvature** - *the scalar velocity diverges in finite time.* **With bounded acceleration the issue disappears.**

• **The time minimal problem is always** *accessible* **no matter how small A is chosen. A. Weinstein used in his thesis an interesting construction: nearly dense curves with bounded geodesic curvature.**

M. Pauley and L. Noakes, Cubics and negative curvature. Differential Geometry and its Applications 30, Issue 6 (2012) 694-701.

A. Weinstein, The cut locus and conjugate locus of a riemannian manifold, Annals, 87 (1968), 2941.

The ODEs for time minimal splines

$$\begin{aligned}
\dot{x} &= v \\
\nabla_{\dot{x}} v &= A\,\alpha/|\alpha| \\
\nabla_{\dot{x}} \alpha &= -p \\
\nabla_{\dot{x}} p &= R(A\alpha/|\alpha|, v)v
\end{aligned}$$

Focus of the talk:
some observations on S^2 splines

- Cubic splines on the sphere: revisiting the special solutions in Darryl's and associates paper on Invariant Variational Problems (Gay-Balmaz, Holm, Meier, Ratiu, Vialard)

- These special solutions also exist in the time minimal problem

- Speculations about the dynamics in $T^*(TS^2)$

Yet another figure eight!!

Gay-Balmaz, Holm, Meier, Ratiu, Vialard

Invariant Higher-order Variational Problems II
JNLS, 22:4553597, 2012 (IHOVP2)

Invariant Higher-order Variational Problems
CMP, 309, 413458, 2012

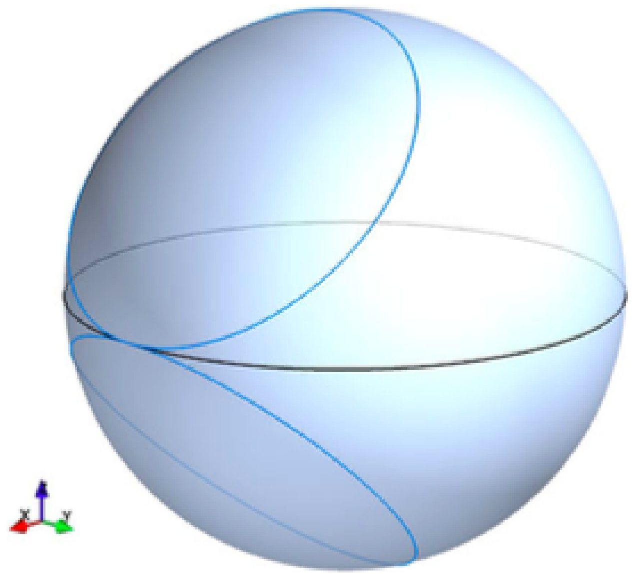


Fig. 1 Ballistic curves and cubics on the sphere. For given initial position and velocity, there are two types of trajectory that are, simultaneously, projections of geodesics on the rotation group (ballistic curves) *and* Riemannian cubics. The two trajectories are shown for initial position $(1, 0, 0)$ and initial velocity parallel to the y-axis. In *black* a unit-speed trajectory along the equator corresponding to the projection of a horizontal geodesic on the rotation group. The *blue curves* are the circular unit-speed trajectories of radius $\frac{1}{\sqrt{2}}$ described in Corollary 6.7 and Remark 6.8 (Color figure online)

The blue circles forming the tilted figure eight have $\kappa_g = 1$.

We will present here another view on these special cubic splines

• **The figure eight solutions form a center manifold of dimension 4:** $C \subset T^*(TS^2)$**.**

• **2-dimensional stable and unstable manifolds** $W_u(C), W_s(C)$**, with** *loxodromic* **eigenvalues**

$$(v/r)\,\sqrt{2}\ \sqrt[4]{3}\ \left(\ \pm\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}\ \ \pm\ \ \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}\ i\ \right)$$

r **is the radius of the sphere, and the parameter** v **is the linear velocity along the trajectories.**

For time-minimal splines: also loxodromic

$$\lambda = \mu \left(\pm\sqrt{(\sqrt{2}-1)/2} \pm i\sqrt{(\sqrt{2}+1)/2} \right)$$

• $\mu = \sqrt{2A/r}$ **is the radius of the momentum sphere that contains the reduced system equilibrium.** A **is the maximal acceleration allowed.**

• **The velocity in the circles is** $v = \sqrt{Ar}$**.**
Fix the corresponding energy level: the phase space has dimension 7. The center manifold is parametrized by $T^1(S^2) \equiv SO(3)$**.**

The dimension count is $\mathrm{dim} C = 5 + 5 - 7 = 3.$

The figure eights and the equators are organizing centers for the dynamics of both problems

Loxodromic eigenvalues and nonintegrability make a good combination to produce spline curves.

A poetic analogy: Joy of life fountain, Hyde park

(pretend its a rotating spinkler)

Warm-up: numerical experiment using BOCOP*

$q = r\,(\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ \sin\phi)$ spherical coordinates

$$
\begin{aligned}
\nabla_{\dot q}\dot q \quad (:= &\quad \left(r\cos\phi\,\ddot\theta - 2r\sin\phi\,\dot\theta\,\dot\phi\right)e_\theta + \left(r\ddot\phi + r\cos\phi\sin\phi\,\dot\theta^2\right)e_\phi) \\
= &\quad \bar u_1\,e_\theta + \bar u_2\,e_\phi = u_1\,\mathbf{t} + \mathbf{u_2}\,\mathbf{n}
\end{aligned}
$$

State equations

$$
\begin{aligned}
\dot\theta &= v_\theta \\
\dot\phi &= v_\phi \\
\dot v_\theta &= 2\tan\phi\ v_\theta\,v_\phi + \bar u_1/(r\cos\phi) \\
\dot v_\phi &= -\cos\phi\sin\phi\,v_\theta^2 + \bar u_2/r\ .
\end{aligned}
$$

*** BOCOP implements Pontryagin's method to optimal control problems**

(F. Bonnan's group at INRIA, www.bocop.org)

Decompose the acceleration in terms of the tangent vector and normal in the surface

$$\bar{u}_1\, e_\theta + \bar{u}_2\, e_\phi = u_1\, \mathbf{t} + \mathbf{u_2}\, \mathbf{n}$$

$$\mathbf{t} = \frac{v_\theta \cos\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}}\, e_\theta + \frac{v_\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}}\, e_\phi$$

$$\mathbf{n} = -\frac{v_\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}}\, e_\theta + \frac{v_\theta \cos\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}}\, e_\phi$$

$$\bar{u}_1 = u_1 \frac{v_\theta \cos\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}} - u_2 \frac{v_\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}}$$

$$\bar{u}_2 = u_1 \frac{v_\theta \cos\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}} + u_2 \frac{v_\theta \cos\phi}{\sqrt{v_\theta^2 \cos^2\phi + v_\phi^2}}$$

For the time minimal problem the implicit equation solver in BOCOP adjusts the *four unknown momenta* $(p_\theta, p_\phi, p_{v_\theta}, p_{v_\phi})$ **at the initial time and finds** *the time interval* T **leading to the four coordinates of the end velocity vector*.**

Due to the SO(3) symmetry, in the simulations the initial and final positions can be taken at the equator ($\phi = 0$), and the initial longitude also set at $\theta^o = 0$.

Thus the data to be chosen are θ^f and the initial and final values of the velocities v_θ, v_ϕ.

The implicit solver is a shooting method to reach $\theta^f, v_\theta^f, v_\phi^f$ in an unknown time T from the initial values $\theta^o = \phi^o = 0$, v_θ^o, v_ϕ^o.

*At first sight there are 5 unknowns to 4 implicit equations, but the momenta p_{v_θ}, p_{v_ϕ} act under a scale invariance so they behave as just one unknown.

A challenge to the audience

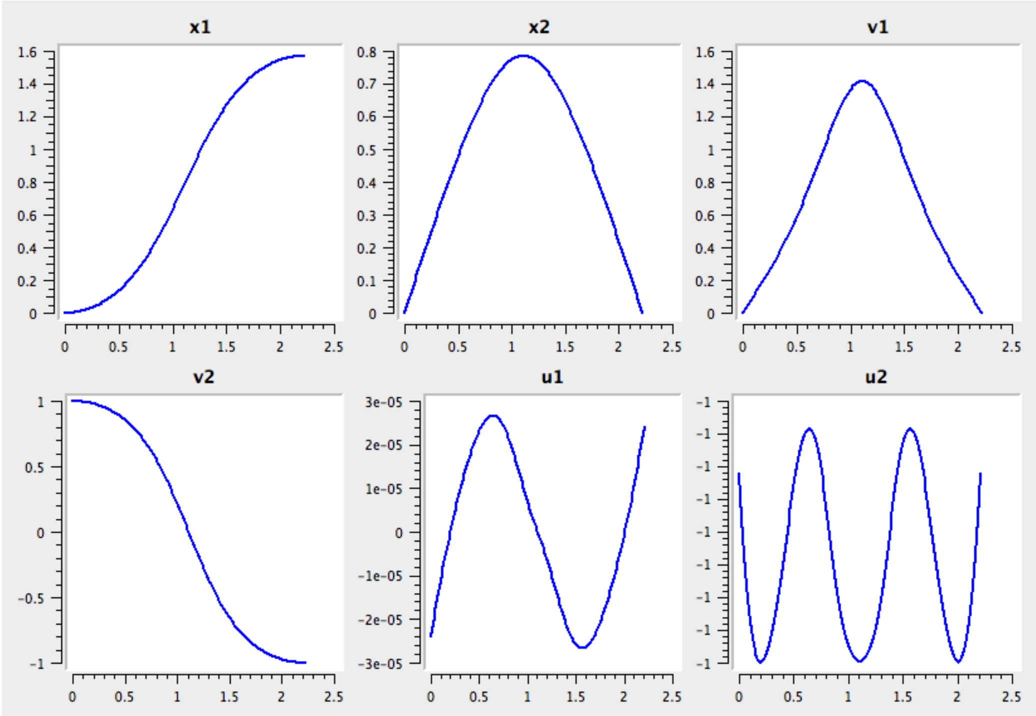
$$\theta^o = \phi_o = \phi_f = 0,\ \theta_f = \pi/2,$$

$$\dot{\theta}_o = 0,\ \dot{\theta}_f = 0,\ \dot{\phi}_o = +1,\ \dot{\phi}_f = -1\,.$$

Objective value
f(x*) = 2.221489e+00

The correct value is 2.22144146908... What it is?
A beer or chocolate to whoever guesses during the talk

Hint: These boundary conditions correspond to unit tangent vectors at the endpoints of a semicircle with $k_g = 1$ inside the unit sphere.



$x_1 = \theta$, $x_2 = \phi$, $v_1 = \dot{\theta}$, $v_2 = \dot{\phi}$;

u_1, u_2 are tangential and normal accelerations, respectively.

Note the scale of the vertical axis. Small numerical error.

Educated guess: $u_1 \equiv 0$, $u_2 \equiv 1$.

Added after the talk:

Peter Lynch, Cornelia Vizman and François-Xavier Vialard guessed $\pi/\sqrt{2}$ correctly.

They got their prizes.

Part I. Some theory

State equation

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = u \in TQ$$

Minimize some functional of $\gamma(t)$, $\dot{\gamma}(t)$, $u(t)$**.**

Our methodology: Pontryagin's principle to get a Hamiltonian system in $T^*(TQ)$

• Introduce split coordinates using a connection so that the u-family of Hamiltonians is as simple as possible.

• The symplectic form will be noncanonical, with curvature terms.

This approach can be extended to higher order variational problems, addressed in IHOVP I, II by Balmaz, Holm, Meier, Ratiu, Vialard.

Control problems on anchored vector bundles

State space = $A \ni (x,a)$ a vector or affine bundle $q : A \to Q$ with a connection ∇. Anchor: $\rho : A \to TQ$.

State equations: $\dot{x} = \rho(a),\ \nabla_{\dot{x}} a = u$

For $u = 0$, geodesic equations relative to (ρ, ∇).

Examples:

i) $A = TQ$ and $\rho = id$

ii) Control of nonholonomic systems (Bloch, Colombo, ...)

iii) Control on (almost) algebroids (Martinez, Marrero, ...)

An useful observation for landmark splines

Mario Michelli: landmarks geodesics are best described in terms of the cometric.

Control on TQ with a Levi-Civita connection can be recast, via the dual connection, to a control problem with state space $A = T^*Q$.

Just a glimpse for $A = TQ$ (see our JGM paper for details)

$(\tilde{p}_i, \tilde{\alpha}_j, \tilde{v}^k, \tilde{x}^k)$ **canonical coordinates in** $T^*(TQ)$ **relative to** $(\tilde{v}^k, \tilde{x}^k)$ **on** TQ

$(p_i, \alpha_j, v^k, x^k)$ **be coordinates on** $T^*Q \oplus_{TQ} T^*Q$

Using the connection in TQ **gets invariantly**

$$\tilde{p}_i = p_i + \Gamma^k_{ij} v^j \alpha_k,\ \tilde{\alpha}_j = \alpha_j, \tilde{v}^k = v^k,\ \tilde{x}^k = x^k.$$

Canonical 1-form in $T^*(TQ)$ **writes as**

$$\theta = p_i dx^i + \alpha_a (dv^a + \Gamma^a_{ib} v^b dx^i),$$

Symplectic structure in split variables

$$\begin{aligned}\Omega_{\nabla}|_{(x,v,p,\alpha)} &= dx^i \wedge dp_i + (dv^a + \Gamma^a_{ib} v^b dx^i) \wedge (d\alpha_a - \Gamma^c_{ja}\alpha_c dx^j) \\ &\quad -\frac{1}{2} R^b_{ija} v^a \alpha_b dx^i \wedge dx^j,\end{aligned}$$

$R \in \Omega^2(M, End(TQ))$ **is the Riemannian curvature tensor of ∇ and the Christoffel symbols are those of the dual connection*** $\tilde{\nabla}$ **on** $T^*Q \to Q$**.**

*The Christoffel symbols of $\tilde{\nabla}$ are minus the transpose of those of ∇, $\tilde{\nabla}_{\partial_{x^i}} dx^j = -\Gamma^j_{ik} dx^k$.

Hamiltonian vectorfield

$$
\begin{aligned}
\dot{x}^i &= \partial_{p_i} H \\
\dot{p}_i &= -\partial_{x^i} H + \Gamma^b_{ia}(v^a \partial_{v^b} H - \alpha_b \partial_{\alpha_a} H) + R^b_{ija} v^a \alpha_b \dot{x}^j
\end{aligned}
$$

$$
\dot{v}^a + \Gamma^a_{ib} \dot{x}^i v^b = \partial_{\alpha_a} H
$$

$$
\dot{\alpha}^a - \Gamma^b_{ia} \dot{x}^i \alpha_b = -\partial_{v^a} H.
$$

The equations simplify for functionals depending on the metric (a nice cancellation occurs, see the JGM paper)

Cost functionals depending on metric g

- **For cubic splines,**

$$H^{\mathsf{cubic}} := H_{*,\nabla} = \frac{1}{2\beta}\, g^{-1}(\alpha,\alpha) + \langle p, v\rangle \ , \quad u_* = \alpha^\sharp/\beta,$$

where $g(\alpha^\sharp, v) = \alpha(v)$**.**

- **Time minimal:**

$$H^{\mathsf{tmin}} := H_{*,\nabla} = -1 + A\sqrt{g^{-1}(\alpha,\alpha)} + \langle p, v\rangle \ , \ u_* = A\, \alpha^\sharp/|\alpha^\sharp|.$$

Recovering $\nabla_{\dot{x}}^{(3)}\dot{x} = -R(\nabla_{\dot{x}}\dot{x}, \dot{x})\dot{x}$ for cubic splines

• For cubic splines, $u_* = \alpha^\sharp$ (take $\beta = 1$).

Differentiate $\nabla_{\dot{x}}\dot{x} = u_* = \alpha^\sharp$ covariantly twice and use the equations of motion for α and p.

We recover the equations found by Crouch and Leite and Noakes, Heinzinger, Paden.

• For the time minimal problem, the system cannot be cast as a single equation of third order.

$$\begin{aligned} \dot{x} &= \mathbf{v}, \;\; \nabla_{\dot{x}}\mathbf{v} = A\,\alpha^\sharp/|\alpha^\sharp| \\ \nabla_{\dot{x}}\alpha^\sharp &= -p^\sharp, \nabla_{\dot{x}}p^\sharp = R(A\alpha^\sharp/|\alpha^\sharp|, \mathbf{v})\mathbf{v}. \end{aligned}$$

Part II (remaining of the talk)

• •We present the reduced equations for $Q = SO(2)$.

Reconstruction of the curve $\gamma(t)$ is achieved by a time dependent linear system of ODEs for the orthogonal matrix $R(t)$ whose first column is the unit tangent vector of the curve and whose last column is the unit normal vector to the sphere.

• We find special analytical solutions, that are organizing centers for the dynamics: precisely the solutions in IHOVP2.

• Simulations show chaotic behavior

Reduction of $SO(3)$ symmetry

Fom eight variables

$\theta, \phi, v_\theta, v_\phi,$ (states) $p_\theta, p_\phi, p_{v_\theta}, p_{v_\phi}$ (costates) in $T^*(TS^2)$

to five variables

$$(a, v, M_1, M_2, M_3).$$

Ⓥ is the scalar velocity, conjugated to costate ⓐ and (M_1, M_2, M_3) are costate variables such that

$$\{M_i, M_j\} = \epsilon_{ijk} M_k \,.$$

Casimir:

$$\mu^2 = M_1^2 + M_2^2 + M_3^2 \ .$$

Poisson map (taking $r = 1$)

The Poisson map from unreduced variables $(x,\, \mathbf{v},\, p,\, \alpha)$, where $x \in S^2$ and $\mathbf{v},\, \mathbf{p},\, \alpha \perp \mathbf{x}$ to the reduced (a, v, M_1, M_2, M_3) is

$$
\begin{aligned}
a &= \alpha \cdot \mathbf{v}/v \ , \ \ v = |\mathbf{v}| \\
M_1 &= \mathrm{det}(p, \mathbf{v}/v, x) \\
M_2 &= \ p \cdot \mathbf{v}/v \\
M_3 &= \mathrm{det}(\alpha,\, x,\, \mathbf{v})
\end{aligned}
$$

See the JGM paper for the derivation.

Reduced Equations, time minimal on $S^2(r)$

$$\dot{v} = aA/\sqrt{a^2 + M_3^2/v^2}$$

$$\dot{a} = -M_2/r + \frac{AM_3^2}{v^3\sqrt{a^2 + M_3^2/v^2}}$$

$$\dot{M} = \det \begin{pmatrix} i & j & k \\ M_1 & M_2 & M_3 \\ 0 & v/r & \frac{AM_3}{v^2\sqrt{a^2+M_3^2/v^2}} \end{pmatrix}$$

$v = 0$ **is a regularizable singularity**

(unreduction and the various symmetries).

Reduced Equations, cubic splines

$(\min \int_0^T \beta |u(t)|^2 \, dt)$

$$\begin{aligned}
\dot{v} &= a/\beta \\
\dot{a} &= -M_2/r + M_3^2/(\beta\, v^3) \\
\dot{M} &= \det \begin{pmatrix} i & j & k \\ M_1 & M_2 & M_3 \\ 0 & v/r & M_3/(\beta\, v^2) \end{pmatrix}
\end{aligned}$$

Both have Casimir $\mu^2 = M_1^2 + M_2^2 + M_3^2$

Before showing results about the dynamics of these reduced ODEs and the corresponding reconstructed trajectories in $S^2(r)$ I outline the derivation.

(In retrospect, I think this idea for reduction was already in a presentation by Krishna)

Darboux frame

For a closed smooth convex surface $\Sigma \subset \mathbb{R}^3$, the Gauss map induces a diffeomorphism between

$$T\Sigma - 0 \;\equiv\; \mathbb{R}^+ \times SO(3)$$

$$\mathbf{v}_q \leftrightarrow (v, R)$$

Here $v = ||\mathbf{v}_q|| > 0$, $\mathbf{v}_q = v\, e_1$. $R \in SO(3)$ **as follows.**

Gauss: $q \in \Sigma \mapsto e_3(q)$ **(external normal).**

Take $e_2 = e_3 \times e_1$, $R = \text{columns}(e_1, e_2, e_3)$.

$R(t)$ **is the Darboux frame of a curve** $\gamma(t)$ **in** Σ**.**

Darboux formulas and reconstruction

$$\begin{aligned} e_1' &= \kappa_g\, e_2 + \kappa_n\, e_3 \\ e_2' &= -\kappa_g\, e_1 + \tau_g\, e_3 \\ e_3' &= -\kappa_n\, e_1 - \tau_g\, e_2 \qquad ({}'= d/ds) \end{aligned}$$

$\kappa_g =$ **geodesic curvature**
$\kappa_n =$ **normal curvature**
$\tau_g =$ **geodesic torsion.**

Reconstruction equations:

$$\dot{R} = R\,X \ , \ \ X = v \begin{pmatrix} 0 & -\kappa_g & -\kappa_n \\ \kappa_g & 0 & -\tau_g \\ \kappa_n & \tau_g & 0 \end{pmatrix}.$$

Controls: $u = (u_1, u_2)$

$$\nabla_{\dot\gamma}\dot\gamma = u_1\, e_1 + u_2\, e_2, \quad u_1 = \dot v\ , \ \ u_2 = v^2 \kappa_g$$

The normal curvature κ_n cannot be a control. It is determined by the constraining force.

[In fact, taking derivatives in the ambient space,

$$\ddot\gamma = \dot v\, e_1 + v^2\, e_1' = u_1\, e_1 + v^2(\kappa_g\, e_2 + \kappa_n\, e_3) = \nabla_{\dot\gamma}\,\dot\gamma + v^2 \kappa_n\, e_3$$

with

$$\kappa_n = (e_1', e_3) = -(e_3', e_1) := B(e_1, e_1)$$

where B is the second fundamental form of the surface.]

Geodesic torsion is also intrinsic

Darboux found the interesting formula

$$\tau_g = \tau_g(e_1) = (\kappa_1 - \kappa_2) \sin\phi \cos\phi$$

ϕ **is the angle between the unit tangent vector** e_1 **to the curve and a principal direction on the surface.**

The geodesic torsion vanishes identically on any spherical curve.

Optimal control problems in TQ
Q two dimensional convex surface

Cubic splines min $\int_0^T (\beta/2)\,(u_1^2 + u_2^2)\,dt$, **fixed** T

Time minimal: min $\int_0^T dt$, **free** T

State equations: $\dot{v} = u_1$, $\dot{R} = R\,X$

$$X = \begin{pmatrix} 0 & -u_2/v & -v\,B(e_1,e_1) \\ u_2/v & 0 & -v\,\tau_g(e_1) \\ v\,B(e_1,e_1) & v\,\tau_g(e_1) & 0 \end{pmatrix} .$$

with prescribed initial and end vectors.

For the sphere $S^2(r)$: $\tau_g \equiv 0, B \equiv -1/r$

$$X = \begin{pmatrix} 0 & -u_2/v & v/r \\ u_2/v & 0 & 0 \\ v/r & 0 & 0 \end{pmatrix}$$

Usual identification:

$$X \equiv \Omega = (0\,,\,v/r\,,\,u_2/v)$$

Introduce costates (a, M)

$$a \leftrightarrow v\;,\quad M = (M_1, M_2, M_3) \leftrightarrow \Omega = (\Omega_1, \Omega_2, \Omega_3)$$

with commutation relations

$$\{a, v\} = 1,\ \{M_i, M_j\} = \epsilon_{ijk} M_k\ .$$

Optimal controls by very simple static optimizations

- **Time minimal Hamiltonian u-family:**

$$H = -1 + a \cdot u_1 + M_2\, v/r + M_3 \cdot u_2/v \; . \qquad (1)$$

Maximize (1) subject to $u_1^2 + u_2^2 \leq A^2$.

- **Cubic splines Hamiltonian u-family:**

$$H = -(\beta/2)\,(u_1^2 + u_2^2) + a \cdot u_1 + M_2\, v/r + M_3 \cdot u_2/v \; . \quad (2)$$

Maximize (2) without restrictions on u_1, u_2**.**

so ....

Optimal controls and Hamiltonians

- **For time minimal**

$$u_1^* = A\, a/\sqrt{a^2 + M_3^2/v^2}\ , \quad u_2^* = A\, M_3/(v\sqrt{a^2 + M_3^2/v^2})\ .$$

$$H_* = -1 + A\sqrt{a^2 + M_3^2/v^2} + M_2 v/r$$

- **For cubic splines**

$$u_1^* = a/\beta,\, u_2^* = M_3/(\beta v)$$

$$H_* = \frac{1}{2\beta}\left(a^2 + (M_3/v)^2\right) + M_2\, v/r\ .$$

Non uniformly run geodesics are splines (cubic or time minimal) for any metric.

They have only tangential acceleration, there is no normal acceleration. The trajectory runs as t^3 for cubic splines, and as t^2 for time minimal. In the latter, however, there is in general a bang-bang phenomenon (that happens only once): a sudden jump in the acceleration from positive to negative.

Linearization around these solutions is hopeless.

For $Q = S^2$ we have the equators.

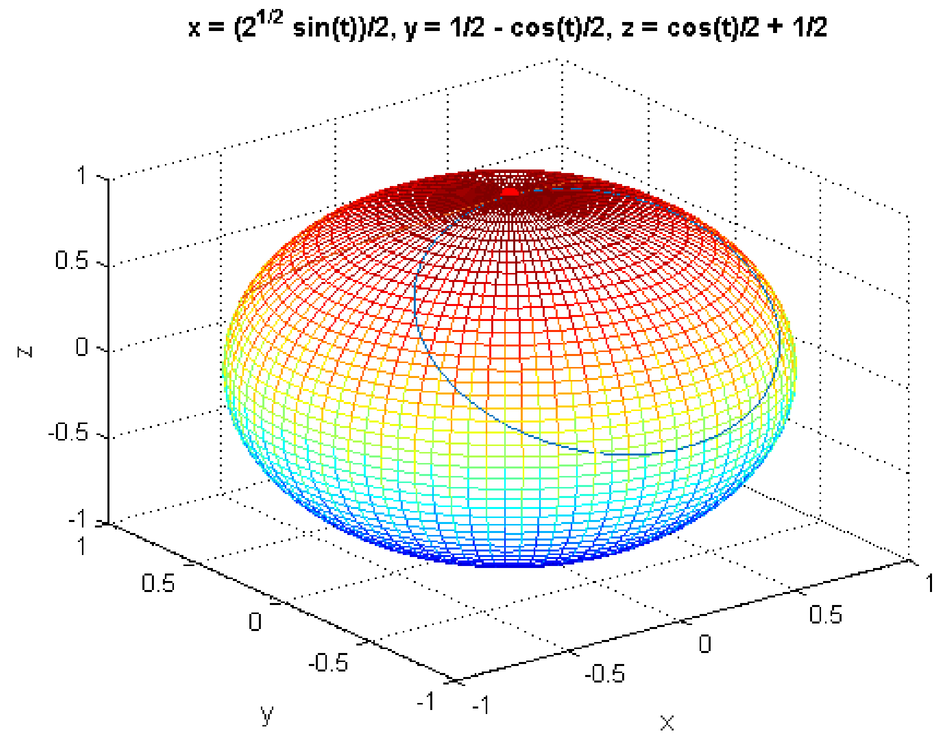
Figure eights: they run linearly in time.

Along these trajectories, the tangential acceleration vanishes.

For cubic splines, they were already given in IHOVP2.

These figure-eight trajectories in the sphere also exist in the time minimal splines problem.

We now show that the figure eights are relative equilibria: correspond to reduced system fixed points (both for cubic and time minimal splines) and these are of *loxodromic* **type.**


x = (2^{1/2} sin(t))/2, y = 1/2 - cos(t)/2, z = cos(t)/2 + 1/2
1
0.5
0
-0.5
-1
z
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x

Cubic splines: reduced system fixed points

Parametrize by $v \in \mathbb{R}_+$**,** $\mu = \sqrt{2}\,\beta v^3/r$**.**

$$a = 0,\ \ M_1 = 0,\ \ M_2 = \beta\frac{v^3}{r},\ \ M_3 = \pm\beta\frac{v^3}{r}$$

Since $u_2^* = M_3/(\beta v) = \kappa_g\, v^2$ **and** $M_3 = \pm\beta\frac{v^3}{r}$**, we get**

$$|\kappa_g| = \frac{1}{r}\ .$$

[On the sphere of radius r**, the parallel of latitude** θ **has geodesic curvature** $\kappa_g = \tan\theta/r$**. Hence** $\theta = \pi/4$**.]**

Reconstruction

The reconstructed curves in S^2 with $R(0) = I$ are two *orthogonal touching circles* **making a 45° angle with the equatorial plane.**

They are given by

$$\gamma(t) = r\left(\frac{\sqrt{2}}{2}\sin\alpha,\ \pm\frac{1}{2}(1-\cos\alpha),\ \frac{1}{2}(1+\cos\alpha)\right)$$

with

$$\alpha = \frac{v}{r}\sqrt{2}\,t.$$

Another proof: $u_2^* = \frac{M_3}{\beta v}$, $M_3 = \pm\frac{\beta v^3}{r} \;\Rightarrow\; u_2^* = \pm\frac{v^2}{r}$.

$$\dot{R} = RX_* \quad \textbf{with} \quad X_* = \begin{pmatrix} 0 & \mp v/r & v/r \\ \pm v/r & 0 & 0 \\ -v/r & 0 & 0 \end{pmatrix}$$

R(t) = rotations with angular velocity $\omega = \sqrt{2}\, v/r$ **about**

$$(u_x, u_y, u_z) = (0\,,\, \frac{\sqrt{2}}{2}\,,\, \pm\frac{\sqrt{2}}{2}).$$

Recall that for an unit vector (u_x, u_y, u_z) **the rotation matrix** $R(\alpha)$ **with** $R(0) = I$ **is given by**

$$\begin{bmatrix} \cos\alpha + u_x^2(1-\cos\alpha) & u_x u_y(1-\cos\alpha) - u_z\sin\alpha & u_x u_z(1-\cos\alpha) + u_y\sin\alpha \\ u_x u_y(1-\cos\alpha) - u_z\sin\alpha & \cos\alpha + u_y^2(1-\cos\alpha) & u_z u_y(1-\cos\alpha) - u_x\sin\alpha \\ u_z u_x(1-\cos\alpha) - u_y\sin\alpha & u_z u_y(1-\cos\alpha) + u_x\sin\alpha & \cos\alpha + u_z^2(1-\cos\alpha) \end{bmatrix}$$

Reconstructed solution: third column of $R(\alpha)$.

We could allow $v < 0$ also, so we can describe both twin circles in both directions. We have therefore *four* **solutions, each twin pair starting at the north pole $(0, 0, r)$ with velocity vector $(v, 0, 0)$.**

Count variables: the family those parametrized circles, under the $SO(3)$ action, forms a 4-dimensional invariant manifold for the dynamics in $T^*(TS^2)$.

The fixed points are focus-focus singularities

Spherical coordinates on the momentum sphere

$$\mathbf{M} = \mu\,(\,\cos\phi\,\cos\theta\,,\,\sin\phi\,,\,\cos\phi\,\sin\theta\,)\ .$$

Restrict to the symplectic manifold

$$M_\mu := T^*\mathbb{R}_+ \times S^2_\mu$$

where S^2_μ is the momentum sphere of radius $|\mu|$ (and recall that $T^*\mathbb{R}_+ = \{(v,a)\,:\,v>0\}$).

We will get an interesting Hamiltonian system...

The fixed points are focus-focus singularities, ctd

Let $z = \sin\phi$. The symplectic form on M_μ becomes

$$\Omega_{M_\mu} = da \wedge dv + \mu \cos\phi \, d\phi \wedge d\theta = da \wedge dv + \mu \, dz \wedge d\theta$$

and the reduced optimal Hamiltonian is

$$\begin{aligned} H_*^{\mathrm{red}} &= \frac{1}{2\beta} a^2 + \frac{\mu^2}{2\beta} \frac{(\cos\phi \, \sin\theta)^2}{v^2} + \mu \sin\phi \, (v/r) \\ &= \frac{1}{2\beta} a^2 + \frac{\mu^2}{2\beta} (1 - z^2) \, (\sin\theta)^2 / v^2 + \mu \, z \, v/r \, . \end{aligned}$$

Equilibria

$$a_o = 0 \ , \ \ v_o^3 = \pm \left(\frac{\mu\, r}{\beta}\right) \sqrt{2}/2$$

$$\theta_o = \pi/2 \text{ or } 3\pi/2 \ , \ \ z_0 = \pm\sqrt{2}/2$$

with energy

$$h^* = (3/2)\,\beta\,(v^4/r^2).$$

Take v or μ as parameter, together with r, β.

It turns out that the matrix that linearizes the Hamiltonian system is the same for all equilibria.

Linearization

$$A = \begin{pmatrix} 0 & -\frac{3\,\beta\,v^2}{r^2} & -\frac{3\,\sqrt{2}\,\beta\,v^3}{r^2} & 0 \\ \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}\,v}{2\,r} \\ 0 & \frac{3}{r} & -\frac{\sqrt{2}\,v}{r} & 0 \end{pmatrix}$$

Furthermore, its characteristic polynomial does not depend on β:

$$p = \lambda^4 + \frac{4\,v^2}{r^2}\,\lambda^2 + \frac{12\,v^4}{r^4}.$$

Eigenvalues are loxodromic (focus-focus type)

$$(v/r)\,\sqrt{2}\ \sqrt[4]{3}\ \left(\pm\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}\ \ \pm\ \ \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}\ i \right)$$

In T^*TS^2, the union for all $v \neq 0$ of these circles with $\kappa_g = 1/r$ forms a center manifold C.

dim C = 4

In the reduced space we have local unstable and stable (spiralling) manifolds of dimension two.

They lift to W^s_C, W^u_C, which then are 6-dimensional stable and unstable manifolds inside T^*TS^2.

This dimension count is coherent:

$$\mathrm{dim} C = 6 + 6 - 8 = 4\,.$$

Several global dynamical question can now be posed: on the unreduced system, take initial conditions near the focus-focus equilibrium. What happens with their solutions and with the corresponding unreduced solutions?

Global behavior of W^u and W^s is in order. Do they intersect transversally?

Equators are in the 'periphery' of phase space

With $z = \sin\phi$. Then $a, v, \theta \in \mathbb{R}\,, \quad |z| \leq 1$.

$$\dot{v} \;=\; a/\beta\;, \qquad \dot{a} = \mu\left(-z/r + \frac{\mu}{\beta}(1-z^2)\,\frac{(\sin\theta)^2}{v^3}\right),$$

$$\dot{\theta} \;=\; \frac{v}{r} - \frac{\mu}{\beta}\,\frac{z\,(\sin\theta)^2}{v^2}\;, \qquad \dot{z} = \frac{\mu}{\beta}\sin\theta\,\cos\theta\,\frac{(z-1)(z+1)}{v^2}\;.$$

The horizontal lines $z = \pm 1$ are invariant.

They corresponds to $M_1 = M_3 = 0, M_2 = \pm\mu$.

Hence: reconstruction at $z = \pm 1$ yields equators.

Reconstruction at $z = \pm 1$ yields the equators

The coordinate a runs uniformly in time (from left to right at $z = -1$ and from right to left at $z = +1$)

$$a(t) = -\text{sign}(z)\mu t/r + a_o.$$

As we expect, v is quadratic on time, with leading term $-\text{sign}(z)\mu t^2/(2r\beta)$.

As for θ, for $|t|$ sufficiently large the second term in the equation for $\dot{\theta}$ can be dropped out. Thus for such large $|t|$ we have $\theta(t) \sim -\text{sign}(z)\mu t^3/(6r\beta)$.

Of interest to symplectic topologists?

This means that except possibly at intermediate times, the horizontal invariant θ lines in the plane (θ, z) for $z = \pm 1$ run in opposite ways.

Poincaré-Birkhoff theorem should be applicable.

Time minimal: reduced system fixed points

$$
\begin{aligned}
\dot{v} &= aA/\sqrt{a^2 + M_3^2/v^2} \\
\dot{a} &= -M_2/r + \frac{AM_3^2}{v^3\sqrt{a^2 + M_3^2/v^2}} \\
\dot{M} &= \det\begin{pmatrix} i & j & k \\ M_1 & M_2 & M_3 \\ 0 & v/r & \frac{AM_3}{v^2\sqrt{a^2+M_3^2/v^2}} \end{pmatrix}
\end{aligned}
$$

Equilibria live in the Casimir sphere $\mu = \sqrt{2A/r}$

$$a = 0 \ , \ v = \pm\sqrt{Ar}$$

$$\begin{aligned} M &= \mu\,(0, \sqrt{2}/2, \pm\sqrt{2}/2) \text{ if } v > 0 \\ M &= \mu\,(0, -\sqrt{2}/2, \mp\sqrt{2}/2) \text{ if } v < 0 \end{aligned}$$

The reconstructed $R(t)$ is the product of $R(0)$ by rotation around the unit vector

$$(0, \mathrm{sign}(v)/\sqrt{2}, \mathrm{sign}(M_3)/\sqrt{2})\,.$$

with angular velocity

$$\omega = \sqrt{2A/r}\,.$$

Who got the prize?

Take $A = r = 1$ **then we get the same parametrized curve of the cubic splines problem, with** $v = 1$**.**

$$\gamma(t) = \left(\frac{\sqrt{2}}{2}\sin\alpha,\ \pm\frac{1}{2}(1-\cos\alpha),\ \frac{1}{2}(1+\cos\alpha)\right),$$

with

$$\alpha = \sqrt{2}\,t$$

For the end point $\alpha = \pi$ **we get**

$$\boxed{T = \pi/\sqrt{2}}$$

Symplectic description of the reduced system

$$H = \mu\, zv/r\, + A\sqrt{a^2 + \mu^2(1 - z^2)\,(\sin\theta)^2/v^2}$$

$$\Omega = da \wedge dv + \mu dz \wedge d\theta, \;\; dz = \cos\phi\, d\phi$$

with $-1 \leq z \leq 1,\; \theta \in \Re \bmod 2\pi$.

Equations of motion in variables (a, v, z, θ)

$$\begin{aligned}
\dot{a} &= -H_v = -\mu z/r + \frac{\mu^2 A(1-z^2)(\sin\theta)^2}{v^3\sqrt{P}} \\
\dot{v} &= H_a = Aa/\sqrt{P} \\
\mu\dot{z} &= -H_\theta = -\mu^2\, A\,(1-z^2)\sin\theta\cos\theta\,/(v^2\sqrt{P}) \\
\mu\dot{\theta} &= H_z = \mu v/r - \mu^2\, A\, z\,(\sin\theta)^2\,/\,(v^2\sqrt{P})
\end{aligned}$$

where $P = a^2 + \mu^2(1-z^2)\,(\sin\theta)^2/v^2$.

The equilibria are

$$\begin{aligned}
a &= 0\ ,\ v = \pm\sqrt{Ar} \\
v > 0 \quad &: \qquad \theta = \pm\pi/2\ , \phi = \pi/4\ \ (z = \sqrt{2}/2) \\
v < 0 \quad &: \qquad \theta = \pm\pi/2\ , \phi = -\pi/4\ \ (z = -\sqrt{2}/2)
\end{aligned}$$

Linearization at the four equiilibria ($\mu = \sqrt{2A/r}$)

These matrices are all equal. In the order (a, v, z, θ):

$$L = \begin{bmatrix} 0 & 2/r^2 & 2\sqrt{2Ar} & 0 \\ -Ar & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{A/(2r)} \\ 0 & -2/r & 2\sqrt{2}\sqrt{A/r} & 0 \end{bmatrix}$$

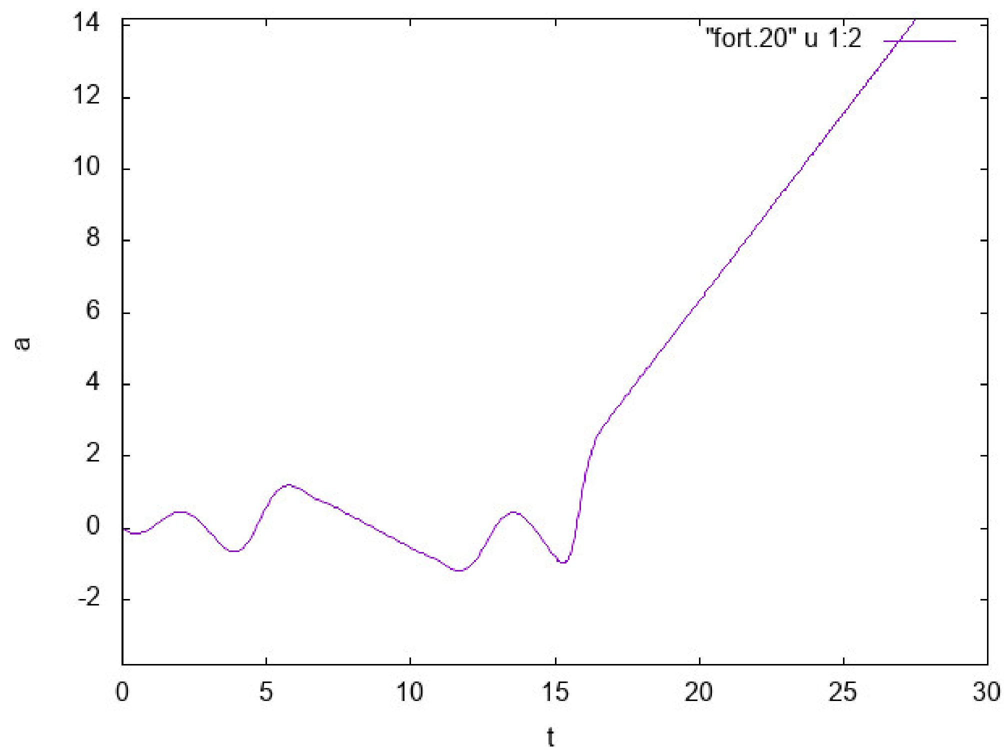
The characteristic polynomial is

$$p(\lambda) = \lambda^4 + 4\,\frac{A}{r}\,\lambda^2 + 8\,\frac{A^2}{r^2}$$

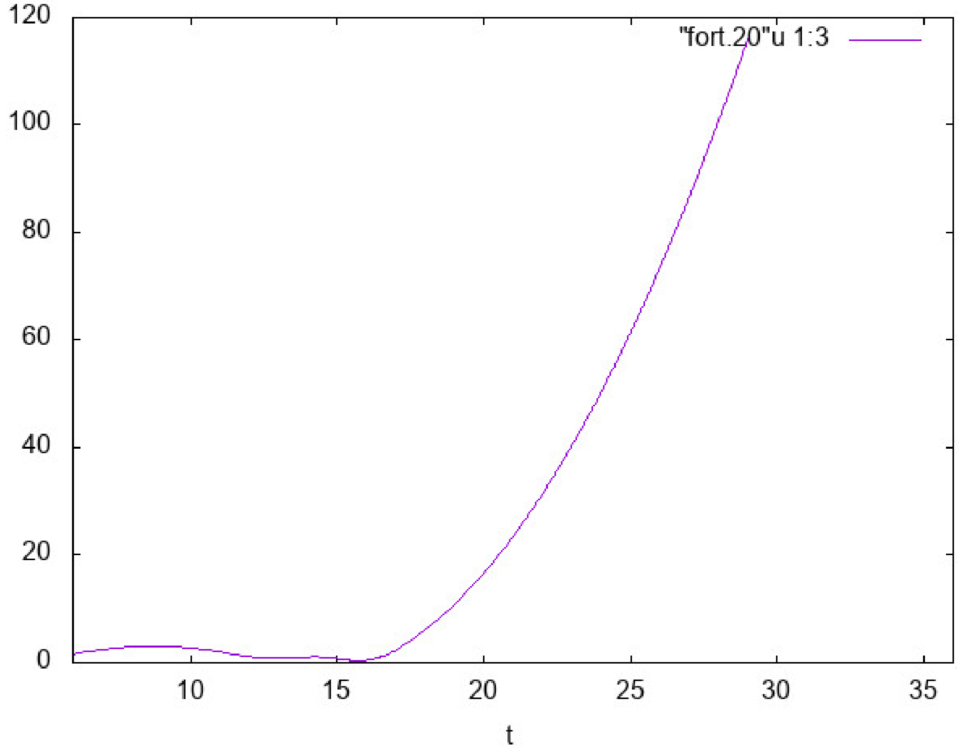
The eigenvalues are loxodromic:

$$\lambda = \mu\left(\pm\sqrt{(\sqrt{2}-1)/2} \pm i\,\sqrt{(\sqrt{2}+1)/2}\right)$$

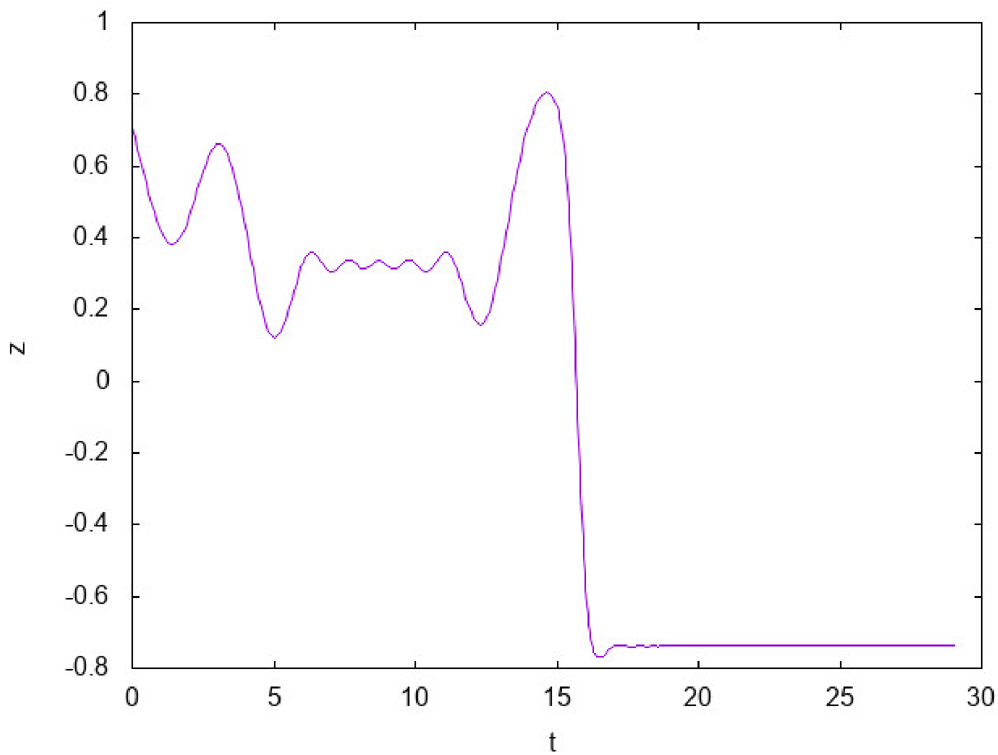
Gallery, time optimal problem



Coordinate $a(t)$ of a solution emanating near the equilibrium. Parameters $r = A = 1$. Note the near linear evolution of $a(t)$ for larger values of t, with slope near 1.

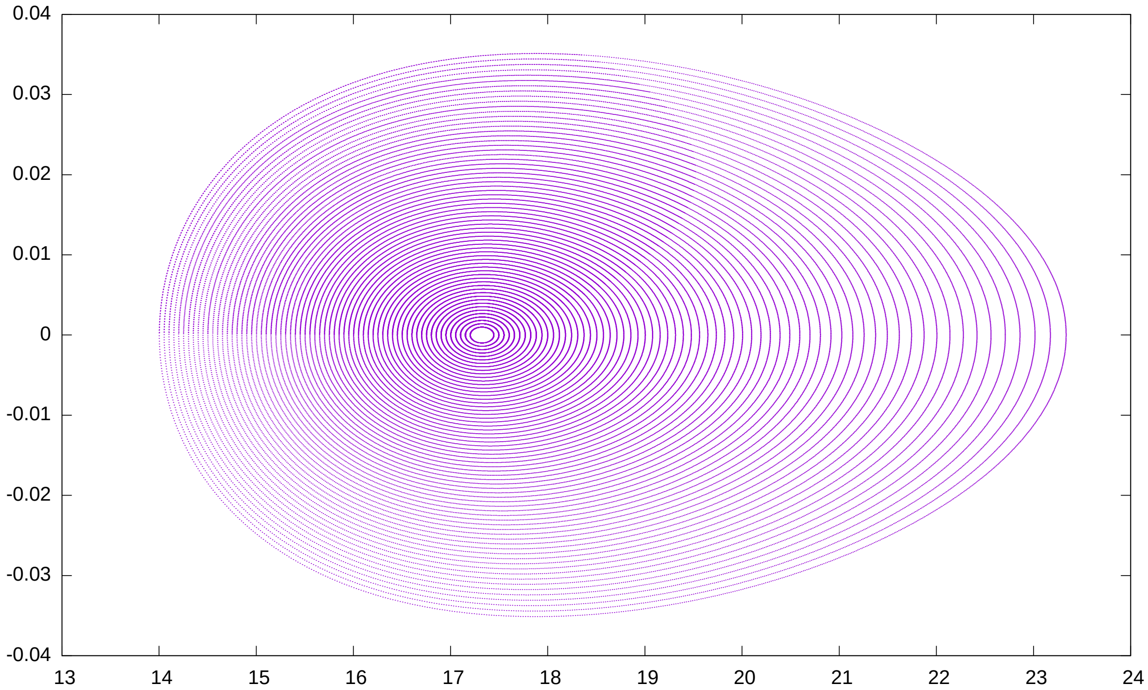


Coordinate $v(t)$. At $t \sim 16$ further work is needed to see if v reaches zero. Note the quadratic evolution for larger t.

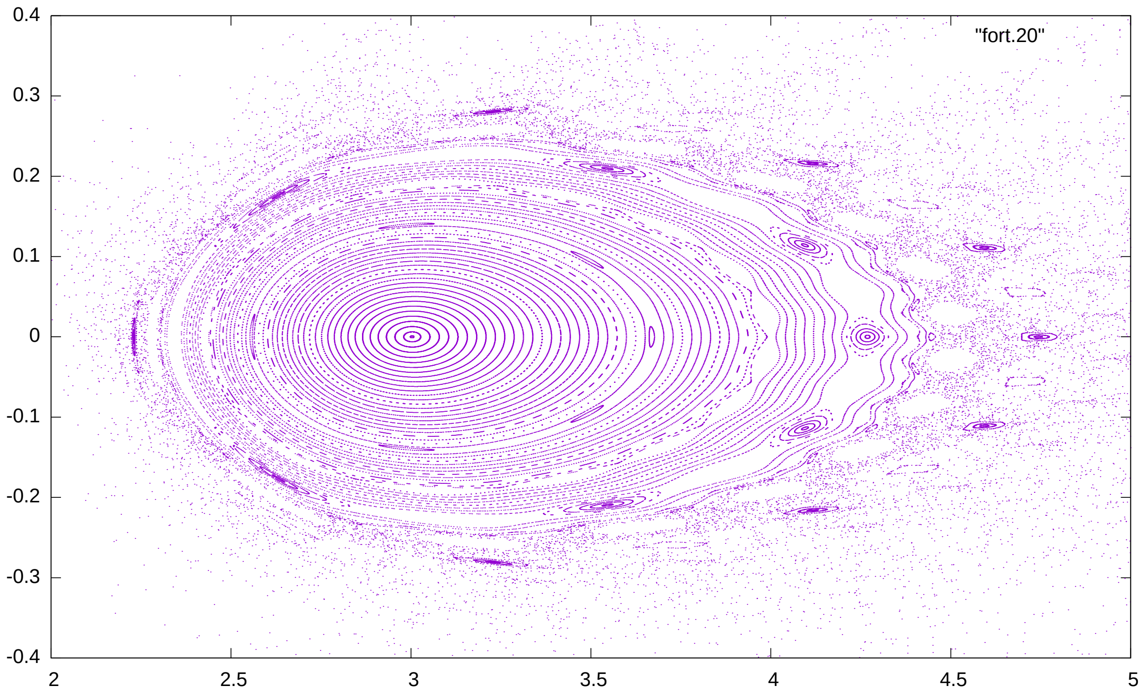


Note the dramatic change in sign of z **around** $t \sim 16$**. For larger** t **it seems to stabilize** ***short*** **of** $z = -1$**.**

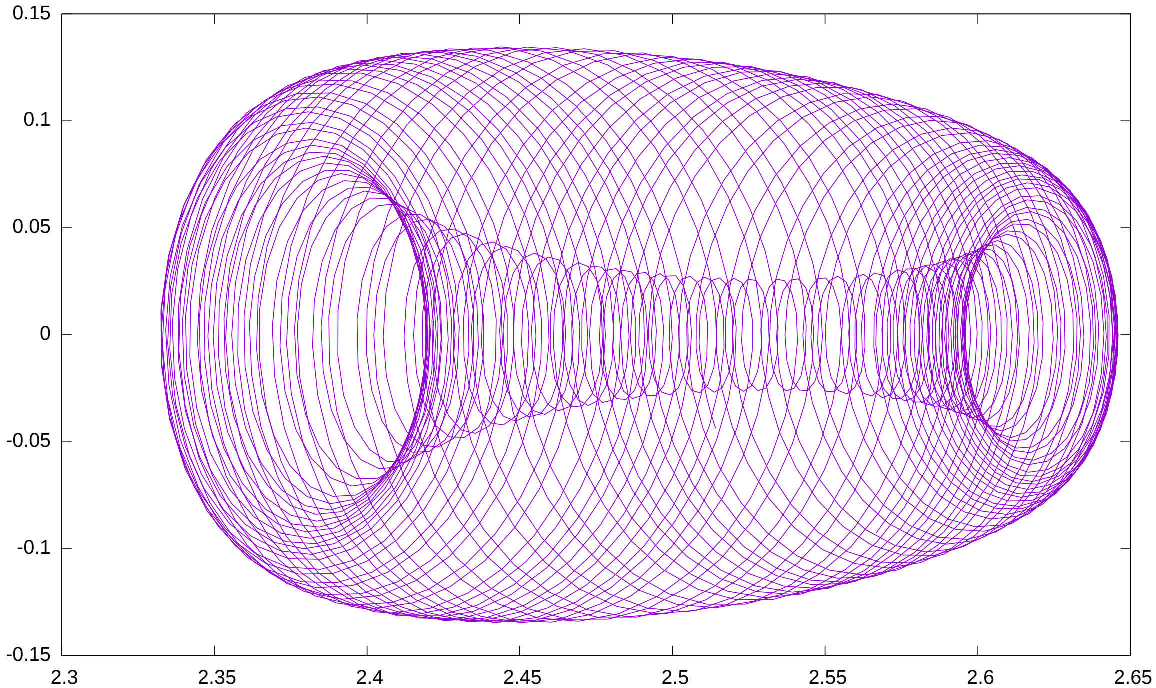
Gallery: cubic splines



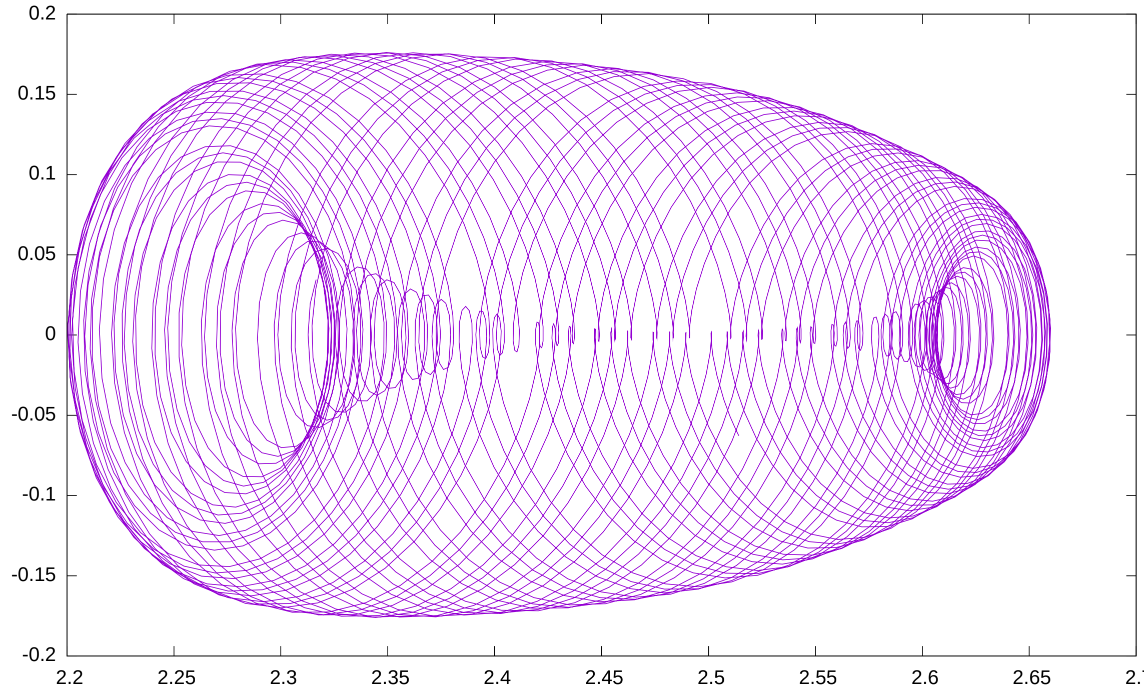
Energy $h = 0.01$. Regular trajectories.



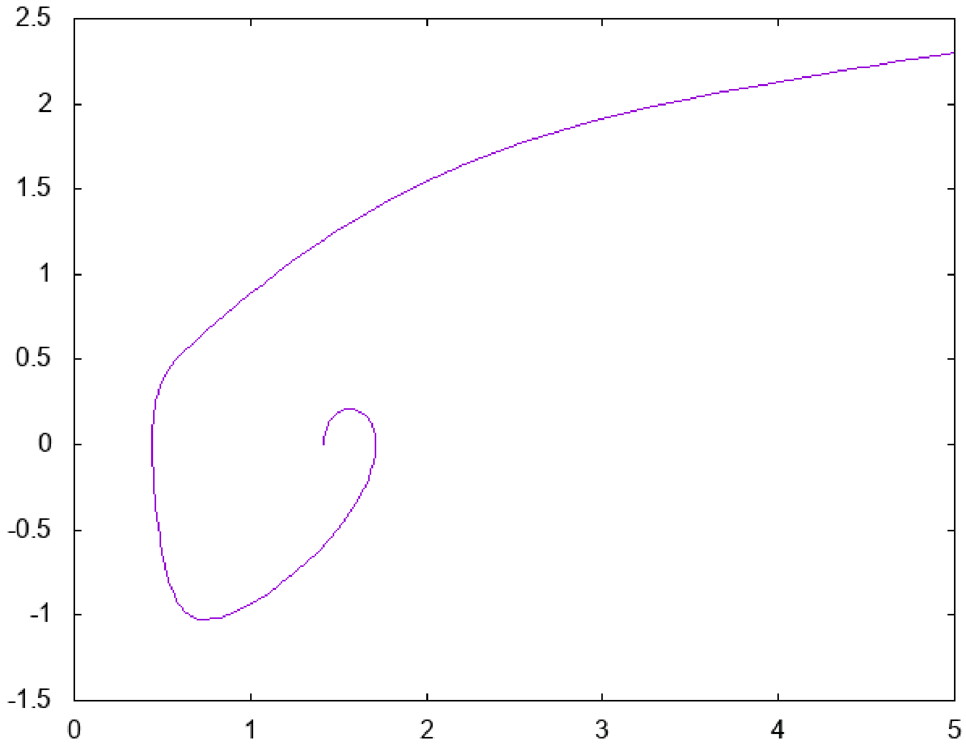
Energy $h = 0.332412099$. There is a large chaotic zone, with escaping trajectories.



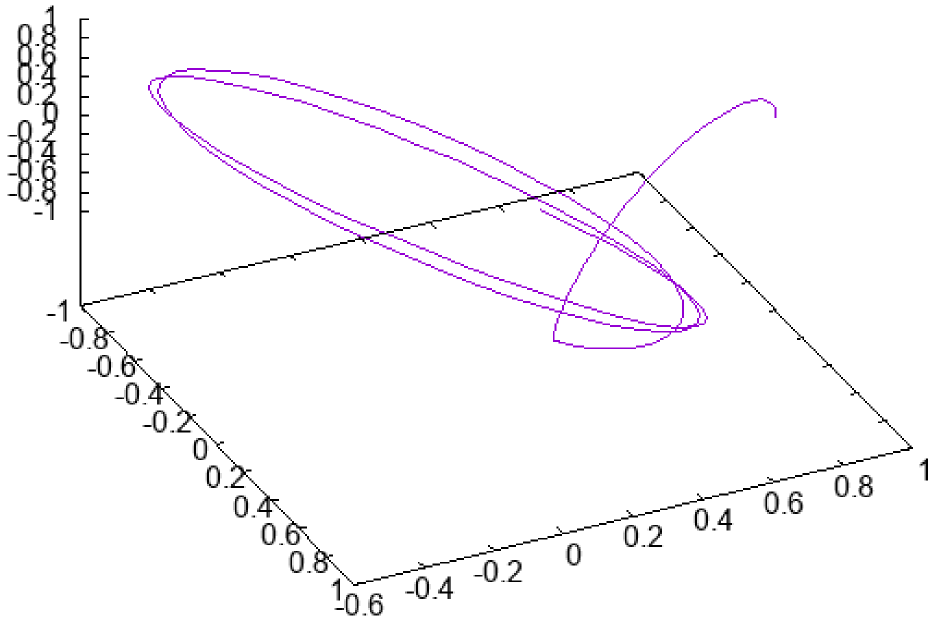
Invariant torus, seen on a Lagrangian projection in the plane (a, z). Energy $h = 0.49494873$. $\beta = 1, \mu = r = 2$



Invariant torus, seen on a Lagrangian projection in the plane (a, z). Energy $h = 0.522397316$. $\beta = 1, \mu = r = 2$



Reduced trajectory emanating from the unstable equilibrium, projected in the (v, a) plane. v is growing quadratically with respect to a.



Corresponding reconstructed trajectory in the physical sphere. It approaches (a neighborhood of) an equator. It stays there or returns to a vicinity of the reduced equilibrium?

Thanks for the attention.

Darryl, keep up the good work!!