# Covariant Brackets in Field Theories and Particle Dynamics

A. Ibort

ICMAT
&
Department of Mathematics
Univ. Carlos III de Madrid

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# 1. Introduction

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# 2. The Setting

## Covariant Hamiltonian First-Order Field Theories

The multisymplectic setting:

$$\pi\colon E \to M \qquad (x^\mu, u^a), \quad a = 1, \ldots, r \qquad \mu = 0, 1, \ldots, d \qquad m = 1 + d$$

$$\pi_1^0\colon J^1 E \to E \qquad (x^\mu, u^a; u^a_\mu) \qquad \mathrm{vol}_M = d^m x$$

Covariant phase space: $P(E) = \mathrm{Aff}(J^1 E)/\mathbb{R}$

$$u^a_\mu \mapsto \rho^\mu_a u^a_\mu + \rho$$

Vector bundle over $E$ modelled on $\pi^*(TM) \otimes_E VE^*$

$$\tau_1^0\colon P(E) \to E \qquad (x^\mu, u^a; \rho^\mu_a)$$

A. Ibort and A. Spivak, *Covariant Hamiltonian field theories on manifolds with boundary: Yang–Mills theories,* J. Geom. Mech. 9(1) (2017) 47–82.

# 2. The Setting (II)

Covariant phase space: $P(E) = \mathrm{Aff}(J^1E)/\mathbb{R} = J^1E^*$

$\tau_1^0 \colon P(E) \to E$ $\quad \pi^*(TM) \otimes_E VE^*$ $\quad (x^\mu, u^a; \rho_a^\mu)$

Multisymplectic model

$$M(E) = \textstyle\bigwedge_1^m E \qquad \Omega = d\Theta$$

$$(x^\mu, u^a; \rho, \rho_a^\nu)$$

$$\Theta = \rho_a^\mu \, du^a \wedge \mathrm{vol}_\mu + \rho \, \mathrm{vol}_M \qquad \mathrm{vol}_\mu = i_{\partial/\partial x^\mu} \mathrm{vol}_M$$

$$0 \to \textstyle\bigwedge_0^m E \hookrightarrow \bigwedge_1^m E \to P(E) \to 0$$

$$\rho = -H(x^\mu, u^a, \rho_a^\mu)$$

$$\Theta_H = \rho_a^\mu du^a \wedge \mathrm{vol}_\mu - H(x^\mu, u^a, \rho_a^\mu) \, \mathrm{vol}_M$$

# 2. The Setting (III)

The action

"Double sections"

$\mathcal{F}_{P(E)}$

$$\Phi\colon M \to E\,, \qquad \pi\circ\Phi = \mathrm{id}_M \qquad \Phi \in \mathcal{F}_M$$

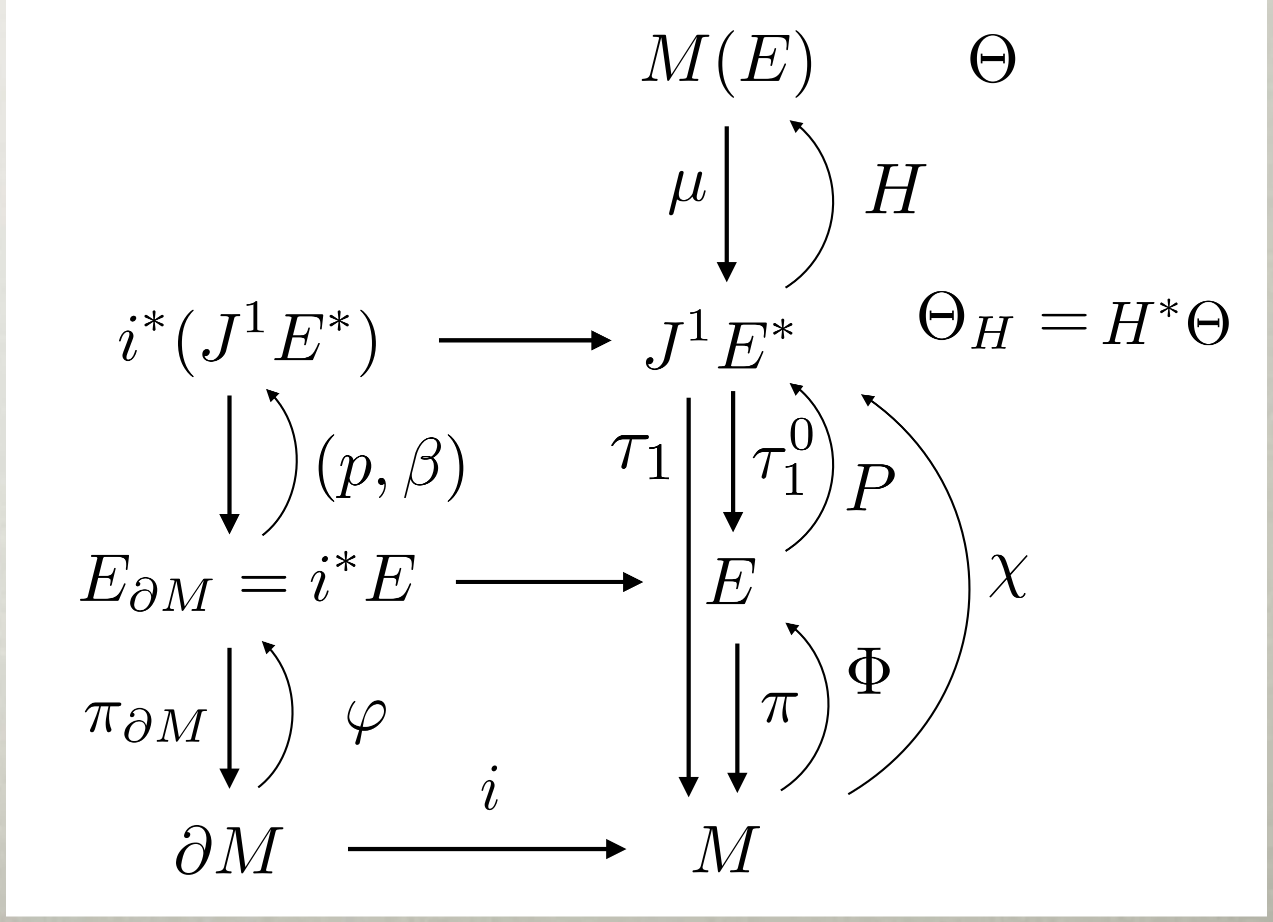
$$P\colon E \to P(E)\,, \qquad \tau_1^0 \circ P = \mathrm{id}_E$$

$$\chi\colon M \to P(E)\,, \qquad \pi\circ\tau_1^0\circ P = \mathrm{id}_M$$

$$P\circ\Phi = \chi \qquad \chi = (\Phi, P) \in \mathcal{F}_{P(E)}$$

$$S(\chi) = \int_M \chi^*\Theta_H = \int_M \left(P_a^\mu(x)\partial_\mu\Phi^a(x) - H(x,\Phi(x),P(x))\right)\mathrm{vol}_M$$

Sections, fields, forms and all that…



# 2. The Setting (IV)

Boundaries $\quad \partial M \neq \emptyset \quad x^k \quad k = 1, 2, 3$

$$\Pi \colon \mathcal{F}_{P(E)} \to T^*\mathcal{F}_{\partial M} \qquad \Pi(\Phi, P) = (\varphi, p)$$

$$\varphi = \Phi \circ i, \quad p_a = P_a^0 \circ i \qquad (\varphi, p) \in T^*\mathcal{F}_{\partial M}$$

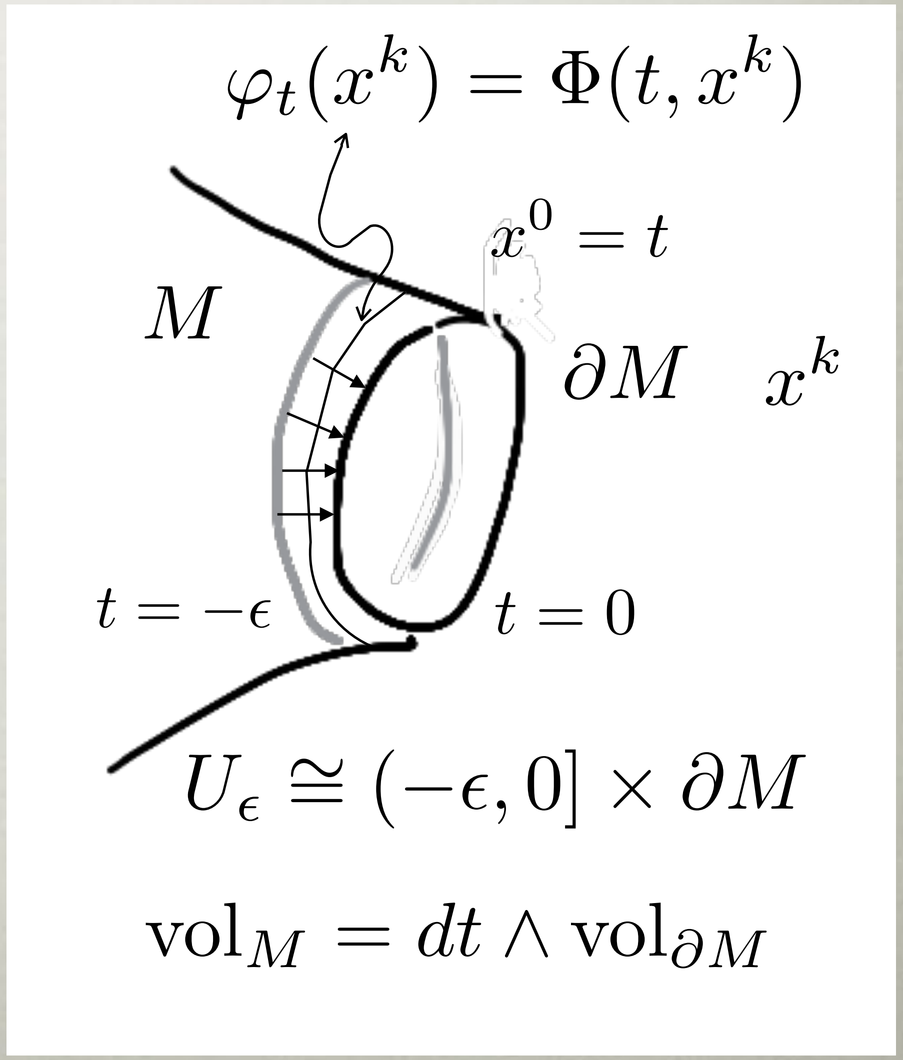
Canonical 1-form

$$\alpha_{(\varphi,p)}(\delta\varphi, \delta p) = \int_{\partial M} p_a(x)\delta\varphi^a(x)\mathrm{vol}_{\partial M}$$

$$\alpha_{\partial M} = p_a \delta\varphi^a$$

Canonical symplectic form

$$\omega_{\partial M} = -\mathrm{d}\alpha_{\partial M}$$



# 2. The Setting (V)

The fundamental formula

$$\mathrm{d}S_\chi = \mathrm{EL}_\chi + \Pi^* \alpha_\chi$$

$$\chi \in \mathcal{F}_{P(E)}$$

$$\mathrm{d}S(\chi)(U) = \int_M \chi^* \left( i_{\widetilde{U}} d\Theta_H \right) + \int_{\partial M} (\chi \circ i)^* \left( i_{\widetilde{U}} \Theta_H \right)$$

$$U = (\delta\Phi, \delta P) \in T_\chi \mathcal{F}_{P(E)}$$

The boundary term $\displaystyle \int_{\partial M} (\chi \circ i)^* \left( i_{\tilde{U}} \Theta_H \right) = (\Pi^* \alpha)_\chi (U)$

$$\Theta_H = \rho_a^\mu du^a \wedge \mathrm{vol}_\mu - H(x^\mu, u^a, \rho_a^\mu)\, \mathrm{vol}_M \qquad \tilde{U} = \delta\Phi^a \frac{\partial}{\partial u^a} + \delta P_a^\mu \frac{\partial}{\partial \rho_a^\mu}$$

# 2. The Setting (VI)

The Euler-Lagrange 1-form

$$\mathrm{EL}_\chi(U) = \int_M \chi^* \left( i_{\tilde{U}} d\Theta_H \right)$$

$$= \int_M \left[ \left( \frac{\partial \Phi^a}{\partial x^\mu} - \frac{\partial H}{\partial P^\mu_a} \right) \delta P^\mu_a + \left( \frac{\partial P^\mu_a}{\partial x^\mu} + \frac{\partial H}{\partial \Phi^a} \right) \delta \Phi^a \right] \mathrm{vol}_M$$

The space of solutions of Euler-Lagrange equations

$$\mathcal{EL}_M = \{ \chi = (\Phi, P) \mid \mathrm{EL}_\chi = 0 \}$$

$$= \{ (\Phi, P) \mid \frac{\partial \Phi^a}{\partial x^\mu} = \frac{\partial H}{\partial P^\mu_a} \, , \, \frac{\partial P^\mu_a}{\partial x^\mu} = - \frac{\partial H}{\partial \Phi^a} \}$$

# 3. The Bracket

Canonical forms on the space of fields $\mathcal{F}_{P(E)}$

0-form: action $$S(\chi) = \int_M \chi^* \Theta_H$$

1-form: Euler-Lagrange form

$$\mathrm{EL}_\chi(U) = \int_M \chi^* \left( i_{\tilde{U}} d\Theta_H \right)$$

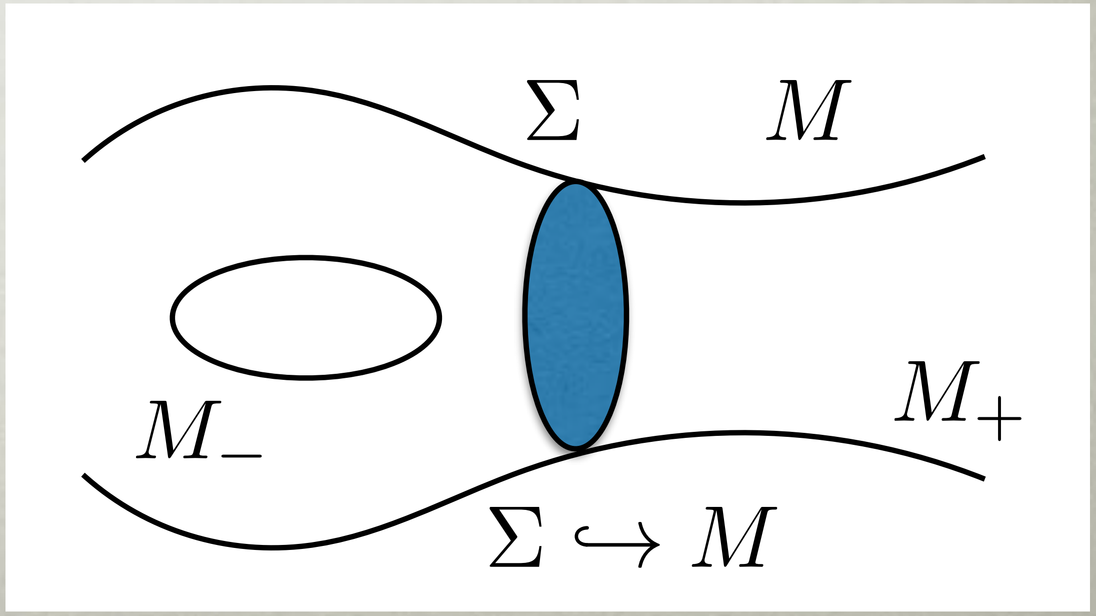
Beyond:

$$U = (\delta_U \Phi, \delta_U P) \in T_\chi \mathcal{F}_{P(E)}$$

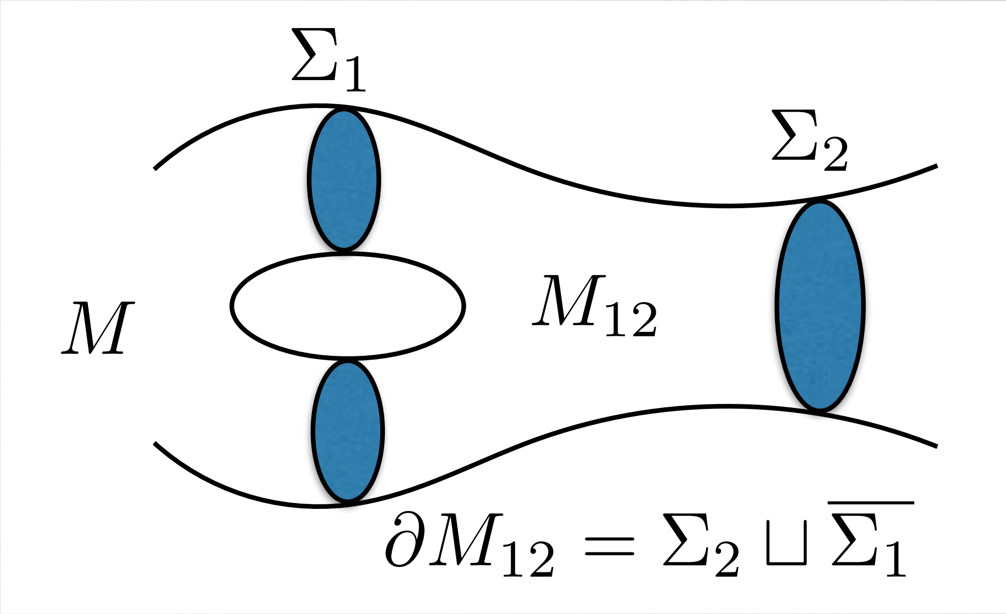
$$\mathbf{\Omega}^\Sigma_\chi(U,V) = \int_\Sigma i^*(\chi^*(i_U i_V d\Theta_H))$$

$$= \int_\Sigma (\delta_U \varphi^a \delta_V p_a - \delta_U p_a \delta_V \varphi^a) \, \mathrm{vol}_\Sigma$$

$$= \Pi^*_\Sigma \omega_\Sigma = -d \left( \Pi^*_\Sigma \alpha_\Sigma \right)$$



# 3. The Bracket (II)




$$S_{12}(\chi) = \int_{M_{12}} \chi^* \Theta_H$$

$$dS_{12} = \mathrm{EL} + \Pi^*_{\Sigma_2} \alpha_{\Sigma_2} - \Pi^*_{\Sigma_1} \alpha_{\Sigma_1}$$

$$\Pi^*_{\Sigma_1} \omega_{\Sigma_1} - \Pi^*_{\Sigma_2} \omega_{\Sigma_2} = -d\left(\Pi^*_{\Sigma_2} \alpha_{\Sigma_2} - \Pi^*_{\Sigma_1} \alpha_{\Sigma_1}\right) = d(\mathrm{EL})$$

$$\boldsymbol{\Omega}^{\Sigma_1} - \boldsymbol{\Omega}^{\Sigma_1} = \mathrm{d}(\mathrm{EL})$$

The pull-back of the 2-forms $\boldsymbol{\Omega}^{\Sigma_1}$ $\boldsymbol{\Omega}^{\Sigma_2}$

along the map $\iota\colon \mathcal{EL} \hookrightarrow \mathcal{F}_{P(E)}$ is such that

$$\iota^*(\boldsymbol{\Omega}^{\Sigma_1}) - \iota^*(\boldsymbol{\Omega}^{\Sigma_1}) = d(\iota^* \mathrm{EL}) = 0$$

Canonical closed 2-form on the space of solutions $\mathcal{EL}$

$$\boldsymbol{\Omega} = \iota^*(\boldsymbol{\Omega}^{\Sigma})$$

# 3. The Bracket (III)

In general the canonical 2-form on the space of solutions is just presymplectic $\ker \boldsymbol{\Omega} \neq \mathbf{0}$

If the canonical 2-form $\boldsymbol{\Omega}$ is symplectic, then we may define a covariant Poisson bracket on the space of solutions

$$\{F, G\} = \boldsymbol{\Omega}(X_F, X_G) \qquad i_{X_F}\boldsymbol{\Omega} = dF$$

DeWitt's formula

$$\{F_1, F_2\}(\chi) = \int_{M \times M} \frac{\delta F_1}{\delta \chi(x)} G(x, y) \frac{\delta F_2}{\delta \chi(y)} dx dy$$

$G$ is the causal Green's function of the linearisation of the equations of motion along the solution $\chi$

# 4. Jacobi brackets

$(M, \eta)$ $\quad (- + \cdots +)$ $\quad$ (Globally hyperbolic) Space-time

$x^\mu \qquad \mu = 0, 1, \ldots, d$

$$E = M \times \mathbb{R} \to \mathbb{R}$$

$\mathcal{C}_m$ Space of parametrized time-like geodesics such that

$$p_\mu p^\mu + m^2 = 0$$

$$\langle \dot\gamma, \dot\gamma \rangle = -1 \qquad L = \frac{m}{2} \eta_{\mu\nu} \dot x^\mu \dot x^\nu$$

$\mathcal{C}_m$ is a contact manifold of dimension *2m-1*

$\gamma \in \mathcal{C}_m \qquad J \in T_\gamma \mathcal{C}_m$ $\quad$ Jacobi field $\quad J'' + R(\dot\gamma, J)\dot\gamma = 0$

$\boldsymbol{\Theta}_\gamma(J) = \langle \dot\gamma, J \rangle$ $\quad$ Contact 1-form $\qquad \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\Theta}$

$\boldsymbol{\omega}_\gamma(J_1, J_2) = \langle J_1, J_2' \rangle - \langle J_2, J_1' \rangle$ $\qquad$ Reeb field $-\dot\gamma$

# 4. Jacobi brackets (II)

Theorem: The 2-form $\omega$ defined by the contact structure is the canonical covariant 2-form $\mathbf{\Omega}$ of the 1+0 field theory on $E = M \times \mathbb{R} \to \mathbb{R}$ with Lagrangian $L$

$$\ker \mathbf{\Theta} = \langle \dot{\gamma} \rangle^{\perp} \qquad i_{\dot{\gamma}} \mathbf{\Omega} = 0$$

Thus the canonical covariant 2-form of the theory defines a Jacobi bracket (not Poisson)

Jacobi manifold $\quad (\Lambda, X)\,, \qquad [\Lambda, \Lambda] = 2X \wedge \Lambda\,, \qquad \mathcal{L}_X \Lambda = 0$

$$[f, g] = \Lambda(df, dg) + fX(g) - gX(f)$$

Jacobi structure of contact manifolds

$$i_X \theta \wedge (d\theta)^n = (d\theta)^n \qquad i_\Lambda \theta \wedge d\theta^n = n\theta \wedge d\theta^{n-1}\,.$$

$$[f,\, g]\, \theta \wedge d\theta^n = (n-1) df \wedge dg \wedge \theta \wedge (d\theta)^{n-1} + (f dg - g df) \wedge d\theta^n$$

# 4. Jacobi brackets (III)

$$[F_1\,,\,F_2](\gamma) \;=\; \int ds ds' \left( \frac{\delta F_1}{\delta \gamma^\mu}(s) G^{\mu\nu}(s-s') \frac{\delta F_2}{\delta \gamma^\nu}(s') \right.$$

$$\left. +\quad F_1(s)\dot\gamma^\mu(s')\frac{\delta F_2}{\delta\gamma^\mu}(s') - F_2(s)\dot\gamma^\mu(s')\frac{\delta F_1}{\delta\gamma^\mu}(s') \right)$$

*G(s-s')* is the causal Green function of Jacobi's equation

Minkowski space-time $\mathbb{M}^m$

$$G^{\mu\nu}(s,s') = P^{\mu\nu}(s-s')\,, \quad P_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m}$$

$$k_\mu = \eta_{\mu\nu}\dot x^\nu$$

$$F = \int_{\mathbb{R}} x^\mu \delta(s-s_1) ds \qquad F(\gamma) = \gamma^\mu(s_1) \qquad F = x^\mu(s_1)$$

$$[x^\mu(s_1)\,,\,x^\nu(s_2)](\gamma) = P^{\mu\nu}(s_1-s_2) + x^\mu(s_1)k^\nu(s_2) - x^\nu(s_2)k^\mu(s_1)$$

Equal-time bracket $[x^\mu\,,\,x^\nu] = x^\mu k^\nu - x^\nu k^\mu$

Congratulations!