Darryl Holm's 70th Birthday Meeting:

Andrew Hone – Bi-hamiltonian systems from cluster algebras

① History: What I worked on with Darryl

[4 papers 2001–04]

Camassa-Holm: $m_t + u m_x + 2u_x m = 0$, $m = u - u_{xx}$

• Bi-Hamiltonian: $m_t = B_j \dfrac{\delta H_j}{\delta m}$, $j = 1, 2$

$$B_1 = -(m D_x + D_x m), \quad H_1 = \int \tfrac{1}{2} m u \, dx$$

$$B_2 = D_x^3 - D_x, \quad H_2 = \int \tfrac{1}{2}(u u_x^2 + u^3)\, dx$$

Poisson brackets $\{F, G\}_j = \int \dfrac{\delta F}{\delta m} B_j \dfrac{\delta G}{\delta m} \, dx$

compatible: $\{\,,\}_1 + \{\,,\}_2$ satisfies Jacobi id.

• Infinite hierarchy of local symmetries (Lenard-Magri chain etc.)

$$R = B_2 B_1^{-1}; \quad m_{t_n} = R^n m_x, \quad n = 0, 1, 2, \ldots$$

Degasperis-Procesi: $m_t + u m_x + 3u_x m = 0$, $m = u - u_{xx}$

• Bi-Hamiltonian, Lax pair, conserved quantities, ..., peakons $u = g * m$, $g(x) = \tfrac{1}{2} e^{-|x|}$

$$m = 2 \sum_{j=1}^{N} p_j(t)\, \delta(x - q_j(t))$$

→ Hamiltonian system with N d.o.f. (non-canonical), integrable in the Liouville sense.

b-family: $m_t + u m_x + b u_x m$ → only one Hamiltonian structure in general.

[cf. Holm-Staley]

② Outline of talk:

- Discrete integrable systems

(P∆E) Example: Hirota's discrete KdV equation

$$V_{k+1,\ell} - V_{k,\ell+1} + \alpha\left(\frac{1}{V_{k+1,\ell+1}} - \frac{1}{V_{k,\ell}}\right) = 0$$

* zero curvature Lax pair: $\mathcal{L}_{k,\ell+1}\mathcal{M}_{k,\ell} = \mathcal{M}_{k+1,\ell}\mathcal{L}_{k,\ell}$

trav. wave.

* infinitely many symmetries

Finite dimensions

(O∆E) * infinitely many reductions to finite-dim. maps: $V_{k+L,\ell+M} = V_{k,\ell} \Rightarrow V_{k,\ell} = v_n,\ n = \ell L - kM$

$$v_{n+L+M} - v_n + \alpha\left(\frac{1}{v_{n+M}} - \frac{1}{v_{n+L}}\right) = 0 \quad \text{— } (\dagger)$$

* discrete Lax equation

→ isospectral evolution $\mathcal{L}_n^*\mathcal{M}_n^* = \mathcal{M}_n^*\mathcal{L}_{n+1}^*$

→ spectral curve $\det(\mathcal{L}_n^*(\lambda) - \mu\mathbb{1}) = 0$ generates constants of motion

Q: Poisson structure? Liouville integrability?

A: Cluster algebras

→ Bi-Hamiltonian structure / finite Lenard–Magri chain for (†).

③ Cluster algebras

A new class of commutative algebras, for which generators are produced by a recursive process called mutation. (Fomin & Zelevinsky 2000)

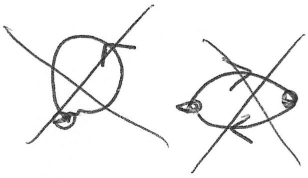
Informal definition: "A cluster algebra is a machine for generating non-trivial Laurent polynomials." (Zelevinsky)

- Appear in: Lie theory, Poisson geometry, Teichmüller theory, Solvable statistical mechanics / quantum field theory (Y-systems), discrete integrable systems, solitons, ...

Seeds: $(B, \underline{x})$, $B \in \mathrm{Mat}_N(\mathbb{Z})$, skew [coefficient-free case]

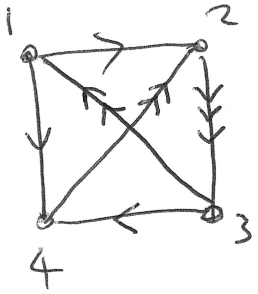
exchange matrix (B), cluster ($\underline{x}$): $\underline{x} = (x_1, \dots, x_N)$ coords.

Mutation: $B \longleftrightarrow Q$ quiver



e.g.

$$(b_{ij}) = \begin{pmatrix} 0 & 1 & -2 & 1 \\ & 0 & 3 & -2 \\ - & & 0 & 1 \\ & & & 0 \end{pmatrix} \longleftrightarrow$$



Pick node k, use $[b]_+ = \max(b, 0)$

Matrix mutation:

μ_k:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{o/wise.} \end{cases}$$

Quiver mutation:

① If $i \xrightarrow{p} k$ and $k \xrightarrow{q} j$ then add $i \xrightarrow{p \times q} j$

② Reverse arrows in/out of node k.

③ Remove any 2-cycles.

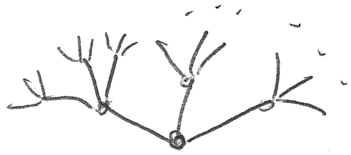
Cluster mutation:

μ_k: $\quad x_j' = x_j \quad , \quad j \neq k$

$$x_k' = \frac{1}{x_k}\left(\prod_{i:i\to k} x_i^{[b_{ik}]_+} + \prod_{i:k\to i} x_i^{[-b_{ik}]_+}\right)$$

In general, mutation generates a tree:

e.g. $N=3$



Not a single discrete dynamical system...

Defn. Cluster algebra $\mathcal{A}(B, \underline{x}) := \mathbb{Z}[\text{cluster variables}]$

Fordy-Marsh: Classification of quivers that are periodic under mutation i.e. $\mu_1(B) = \rho(B)$

$\uparrow$ cyclic perm

$\varphi = \rho^{-1}\circ\mu_1$ acts trivially on B,

but not on $\underline{x}$ $\rightsquigarrow$ cluster map $\underline{x} \longmapsto \varphi(\underline{x})$

equivalent to recurrence

$$x_n x_{n+N} = \prod_{j=2}^{N-1} x_{n+j}^{[b_{1j+1}]_+} + \prod_{j=2}^{N-1} x_{n+j}^{[-b_{1j+1}]_+}$$

Conditions on $B = (b_{ij})$:

$$b_{1j+1} = b_{jN}$$

$$b_{j+1,k+1} = b_{jk} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+$$

Properties of cluster maps: ⓪ Birational, reversible

① Volume preserving: $\Omega = \dfrac{dx_1 \wedge \ldots \wedge dx_N}{x_1 \cdots x_N}$

② Poisson brackets (sometimes trivial)

③ Presymplectic structures $\omega = \sum_{j<k} b_{jk} \dfrac{dx_j \wedge dx_k}{x_j \, x_k}$

④ Laurent property

$$x_n \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_N^{\pm 1}] \quad \forall n \in \mathbb{Z}$$

⑤ Not integrable in general...

Ex. Somos-4:

$$x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2$$

1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209,

83313, 620297, ...

Presymplectic reduction:

$$\begin{tikzcd} \mathbb{C}^N \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{C}^N \arrow[d, "\pi"] \\ \mathbb{C}^{\mathrm{rk}\, B} \arrow[r, "\hat{\varphi}"] & \mathbb{C}^{\mathrm{rk}\, B} \end{tikzcd}$$

$\mathbb{Z}^N = \mathrm{im}\, B \oplus \ker B$

Symplectic form $\hat{\omega}$ s.t. $\pi^* \hat{\omega} = \omega$.

Pick basis

$\mathrm{im}\, B = \langle \underline{v}_1, \ldots, \underline{v}_{2r} \rangle$

$(\mathrm{rk}\, B = 2r)$

$$\pi : \underline{x} \longmapsto \underline{u} = (\underline{x}^{\underline{v}_1}, \ldots, \underline{x}^{\underline{v}_{2r}})$$

e.g. Somos-4: $\mathrm{im}\, B = \langle (1,-2,1,0), (0,1,-2,1) \rangle$

$\ker B = \langle (1,1,1,1), (1,2,3,4) \rangle$

$$(u_1, u_2) = \left(\frac{x_1 x_3}{x_2^2}, \frac{x_2 x_4}{x_3^2} \right)$$

Symplectic map $\hat{\varphi}$, $\hat{\omega} = \frac{du_1 \wedge du_2}{u_1 u_2}$

e.g. Somos-4 $\quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ \dfrac{u_2+1}{u_1 u_2^2} \end{pmatrix}$

"U-system"

$\{u_1, u_2\} = u_1 u_2$

Conserved quantity:

$$H = u_1 u_2 + \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_1 u_2}$$

level sets are elliptic curves

$\Rightarrow \hat{\varphi}$ is Liouville integrable (1 d.o.f.)

(4) Bi-hamiltonian structures for integrable maps

(L,M) reduction of discrete KdV:

$$v_{n+L+M} - v_n + \alpha \left(\frac{1}{v_{n+M}} - \frac{1}{v_{n+L}} \right) = 0$$

(map in dim $L+M$)

Hirota tau function: $\quad v_n = \dfrac{\tau_n \tau_{n+L+M}}{\tau_{n+M} \tau_{n+L}}$

$\Rightarrow$ Two bilinear eqn.s

period M

$$\tau_{n+2L+M} \tau_n = \beta_n \tau_{n+L+M} \tau_{n+L} + \alpha \tau_{n+2L} \tau_{n+M}$$

$$\tau_{n+2M+L} \tau_n = \beta_n' \tau_{n+L+M} \tau_{n+M} + \alpha \tau_{n+2M} \tau_{n+L}$$

period L

$\rightsquigarrow$ Two different cluster algebras (with coefficients)

$\longrightarrow$ Two different presymplectic reductions

dim $2\left\lfloor \frac{L+M+1}{2} \right\rfloor - 2$

U-system #1 : $\underline{u}$

U-system #2 : $\underline{u}'$

same dim., same bracket!

e.g. $L = 3$, $M = 2$

u-bracket has matrix entries $\begin{pmatrix} 0 & 0 & 1 & -1 \\ & \ddots & 0 & 1 \\ - & & \ddots & 0 \\ & & & 0 \end{pmatrix}$

$\{u_j, u_{j+1}\} = 0$

i.e. $\{u_j, u_{j+2}\} = u_j u_{j+2}$ same for u_j'

$\{u_j, u_{j+3}\} = u_j u_{j+3}$

#1: $u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} = \beta_n - \alpha u_{n+2}$ ← period 2

#2: $u_n' u_{n+1}' u_{n+2}'^2 u_{n+3}' u_{n+4}' = \beta_n' \cancel{u_{n+2}} u_{n+2}' + \alpha$ ↑ period 3

dim 4 + coeffs

KdV map dim 5 $(v_0, v_1, v_2, v_3, v_4) \longmapsto \left(v_1, v_2, v_3, v_4, v_0 + \alpha\left(\frac{1}{v_3} + \frac{1}{v_2}\right)\right)$

use $v_n = u_n u_{n+1} = u_n' u_{n+1}' u_{n+2}'$

$\Rightarrow$ 2 different lifts of $\{,\}$ in 4D

#1: $\{v_0, v_1\}_1 = v_0 v_1$, $\{v_0, v_2\}_1 = v_0 v_2$,

$\{v_0, v_3\}_1 = -v_0 v_3 - \alpha$, $\{v_0, v_4\}_1 = -v_0 v_4$

#2: $\{v_0, v_1\}_2 = v_0 v_1$, $\{v_0, v_2\}_2 = v_0 v_2 - \alpha$

$\{v_0, v_3\}_2 = -v_0 v_3$, $\{v_0, v_4\}_2 = -v_0 v_4 + \frac{\alpha^2}{v_2^2}$

& $\{\cancel{v}, \}_1 + \{ , \}_2$ is a Poisson bracket

$\therefore$ compatible (bi-hamiltonian!)

Trace of monodromy $\Rightarrow$ commuting first integrals I_1, I_2, I_3

(Lenard–Magri chain)

I_1 ↑ Casimir of $\{,\}_2$ I_3 ↑ Casimir of $\{,\}_1$