Hamiltonian equations of motion and the ‘extra’ conserved quantity for the Lanchester-Joukowski glider

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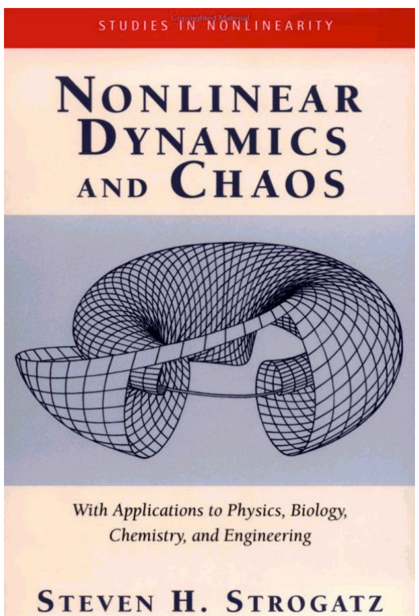

6.5.14 (Glider) Consider a glider flying at speed v at an angle θ to the horizontal. Its motion is governed approximately by the dimensionless equations

$$\dot{v} = -\sin\theta - Dv^2$$
$$v\dot{\theta} = -\cos\theta + v^2$$

where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effects of drag and lift.

a) Suppose there is no drag ($D = 0$). Show that $v^3 - 3v\cos\theta$ is a conserved quantity. Sketch the phase portrait in this case. Interpret your results physically—what does the flight path of the glider look like?

b) Investigate the case of positive drag ($D > 0$).

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Charlie
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Jane Wang

Tony Bloch

Vakhtang Poutkaradze

Melvin Leok

Dmitry Zenkov

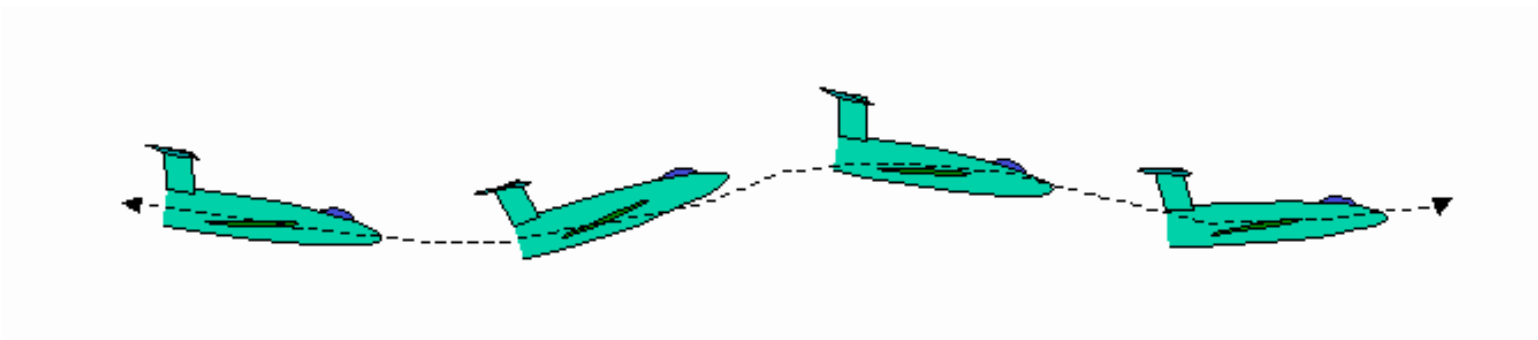
Phil Morrison

and

Darryl Dallas Holm

himself!

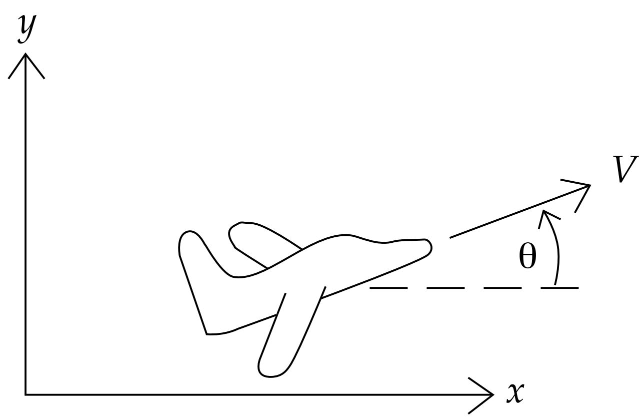
Phugoid motion of a self-propelled or gliding aircraft or underwater seacraft is one of the basic modes of flight dynamics …

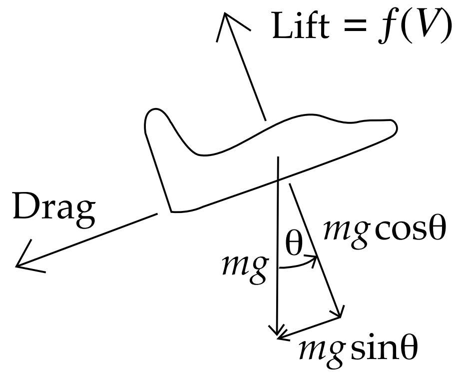


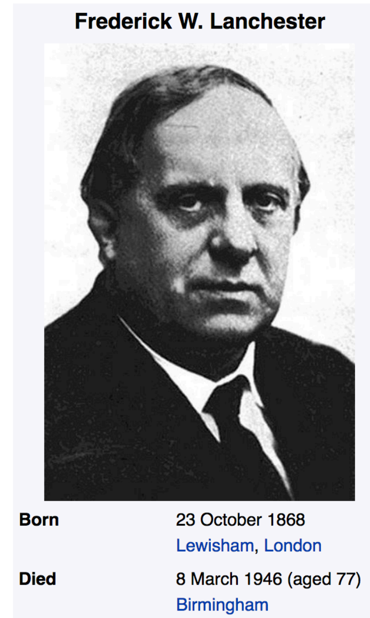
The phugoid has a constant angle of attack with varying pitch due to repeated exchange of airspeed & altitude: the vehicle periodically pitches up & climbs and subsequently pitches down & descends.

<https://www.youtube.com/watch?v=xvOnfxxaUmw>

F. W. Lanchester, *Aerial Flight: Aerodonetics* (Constable, London, 1908)







$$\dot{x} = V\cos\theta$$

$$\dot{y} = V\sin\theta$$

$$m\,d^2x/dt^2 = -f(V)\sin\theta$$

$$m\,d^2y/dt^2 = f(V)\cos\theta - mg$$

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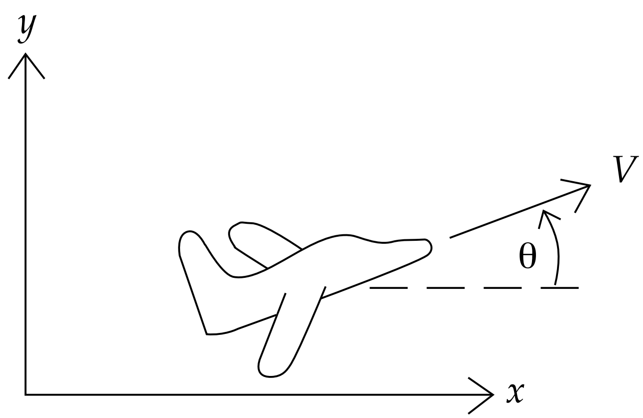

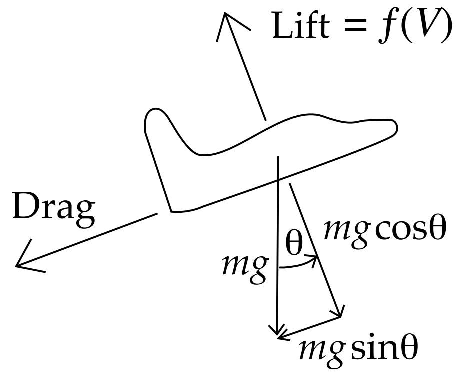

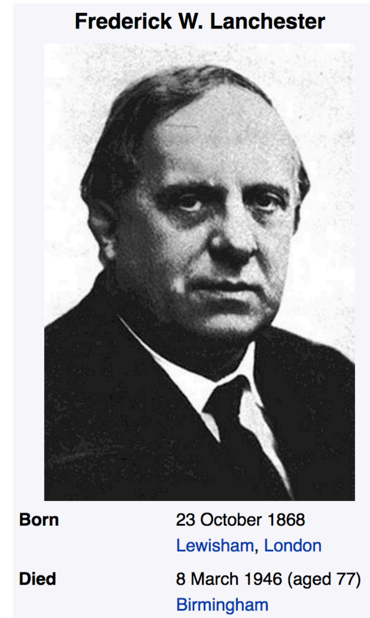

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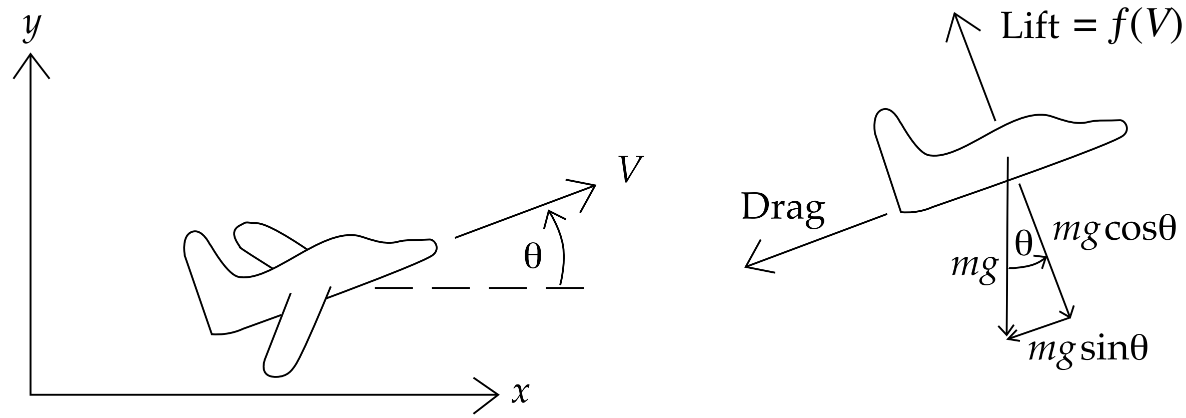
For stationary incompressible ideal irrotational flows when the circulation around the airfoil is proportional to the speed, Kutta-Jukowski $\Rightarrow f(V) \sim V^2$.

F. W. Lanchester, *Aerial Flight: Aerodonetics* (Constable, London, 1908)



Frederick W. Lanchester

Born	23 October 1868 Lewisham, London
Died	8 March 1946 (aged 77) Birmingham

The closed set of equations of motion for V and θ are

$$\dot{V} = -g \sin\theta \tag{1}$$

$$mV\dot{\theta} = f(V) - mg\cos\theta. \tag{2}$$

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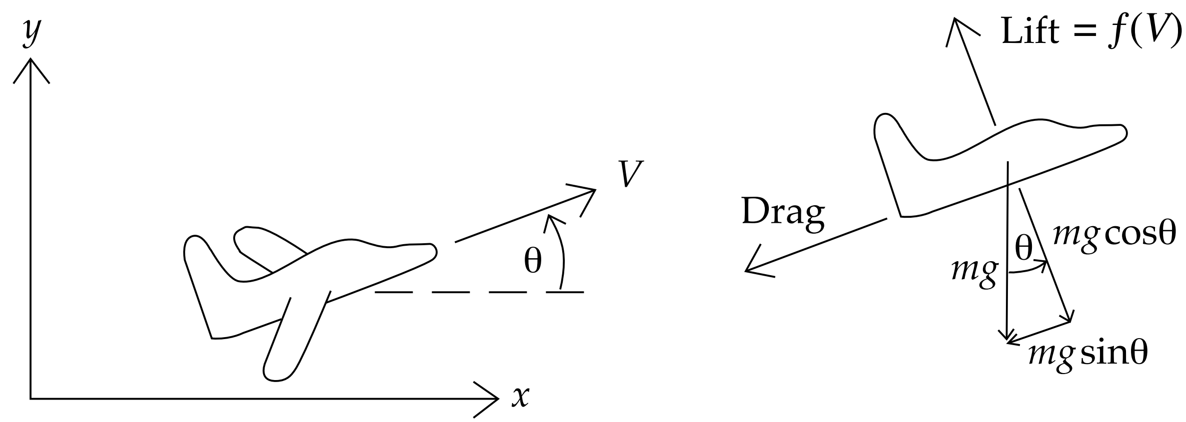

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$$\dot{V} = -g\sin\theta \quad (1)$$

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The total mechanical energy is

$$E = \frac{1}{2}mV^2 + mgy = \frac{1}{2}mV(t)^2 + mg\int^t V(t')\sin\theta(t')\,dt' \quad (3)$$

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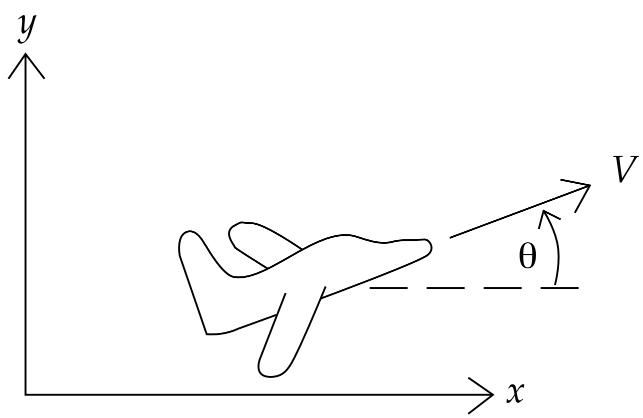

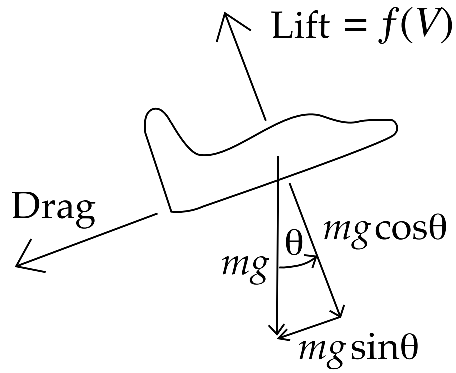

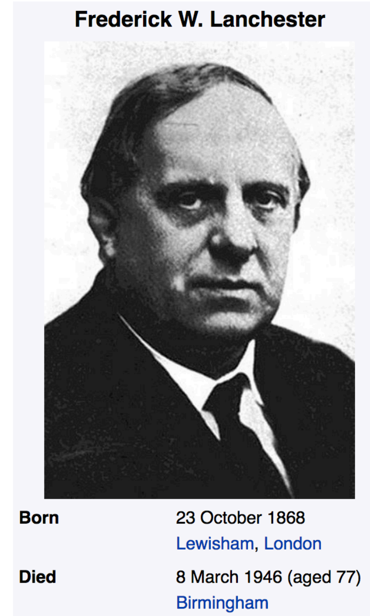

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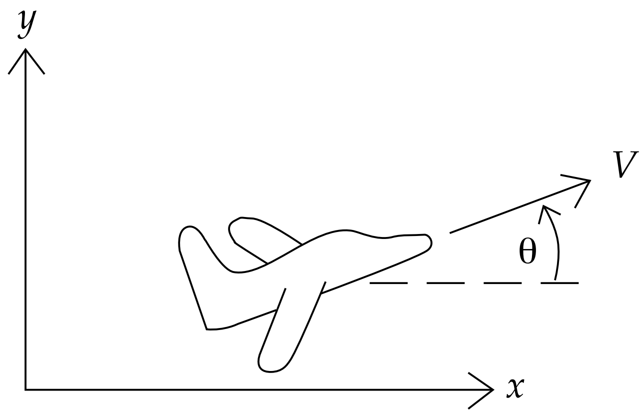
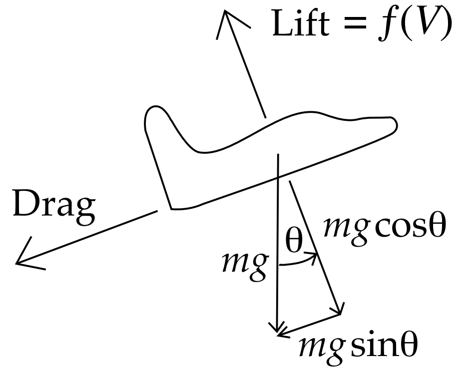
$$E = \frac{1}{2}mV^2 + mgy = \frac{1}{2}mV(t)^2 + mg\int^t V(t')\sin\theta(t')\,dt' \tag{3}$$

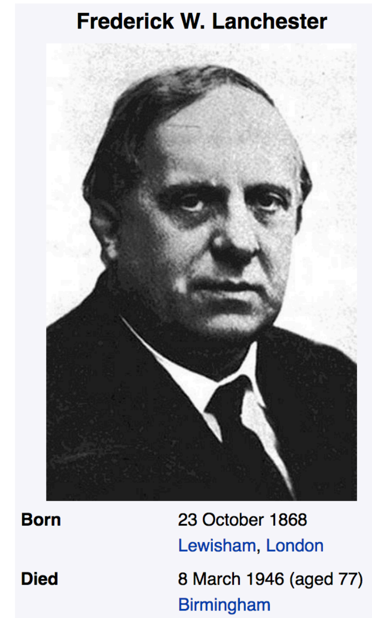
and (1) guarantees its conservation:

$$\frac{dE}{dt} = mV\left(\dot{V} + g\sin\theta\right) = 0. \tag{4}$$

F. W. Lanchester, *Aerial Flight: Aerodonetics* (Constable, London, 1908)







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$$\begin{aligned} \dot{V} &= -g \sin\theta & (1) \\ mV\dot{\theta} &= f(V) - mg\cos\theta. & (2) \end{aligned}$$

The curious 'extra' conserved quantity preserved by (1) & (2) is proportional to

$$\int^{V} f(v)\, dv \;-\; mgV\cos\theta \qquad (5)$$

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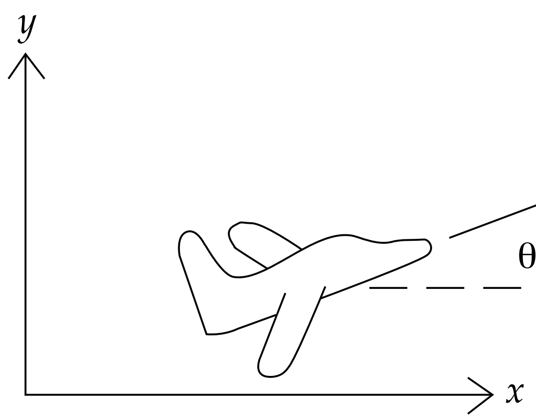

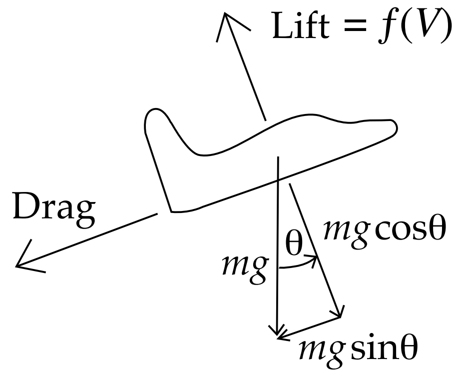

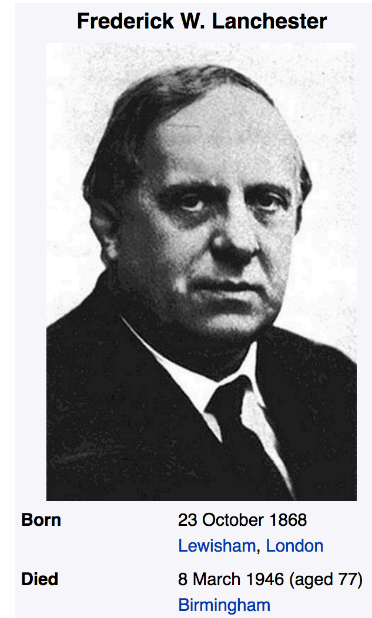

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$$= \frac{\lambda}{3}V^3 \; - \; mgV\cos\theta \qquad \text{when } f(V) = \lambda V^2$$

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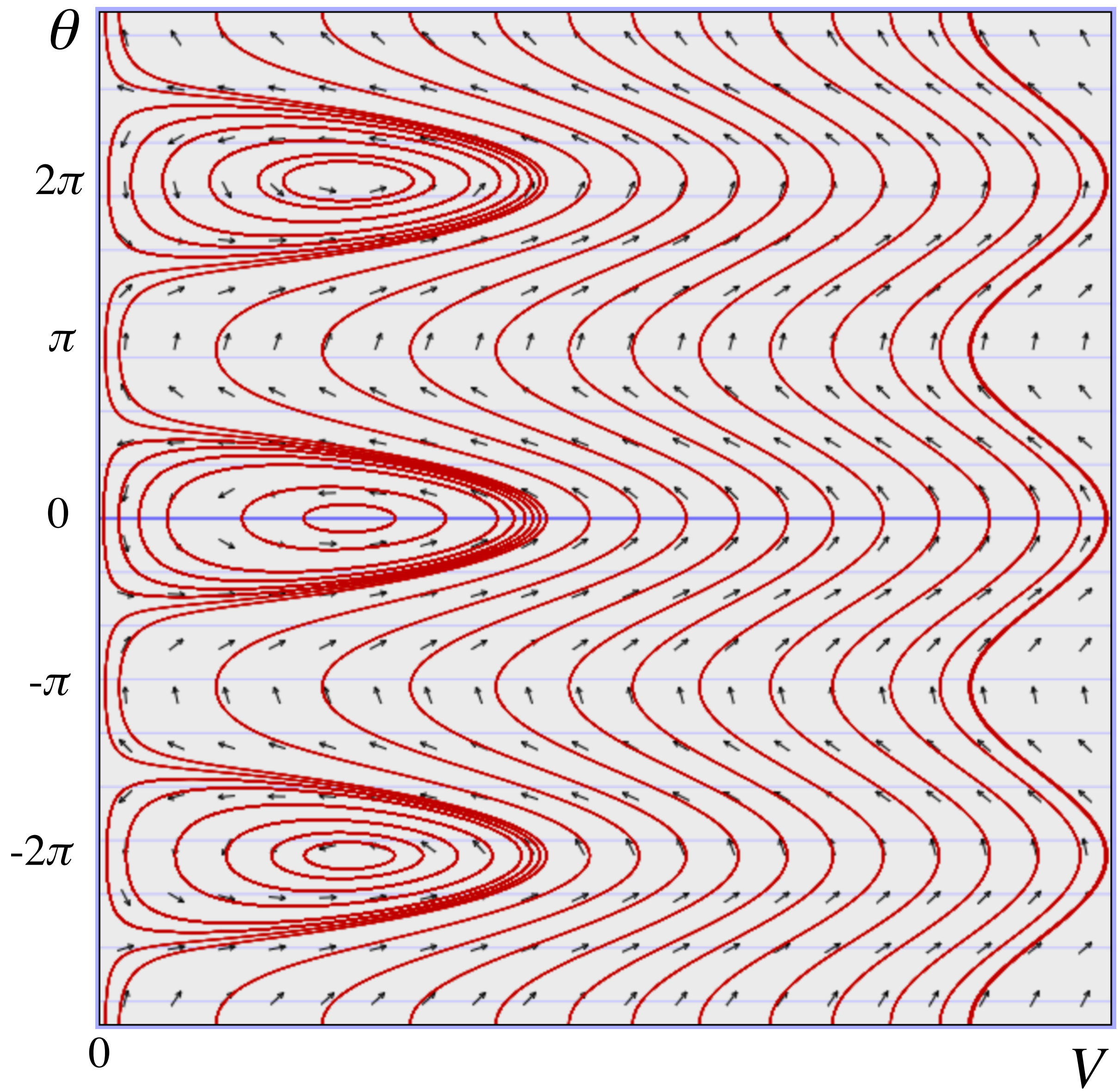

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Nonlinear Stability and Control of

Gliding Vehicles

Pradeep Bhatta

A Dissertation

Presented to the Faculty

of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance

by the Department of

Mechanical and Aerospace Engineering

(citing previous unpublished 2000 notes of Dong Chang and Naomi Leonard)

September, 2006

Point vortex-like Hamiltonian dynamics:

$$
\begin{aligned}
q &= \dot{x} = V\cos\theta \\
p &= -\dot{y} = -V\sin\theta
\end{aligned}
$$

with the conserved quantity serving directly as the Hamiltonian,

$$
\mathcal{H}_0(p,q) = \frac{1}{m}\int^{\sqrt{p^2+q^2}} f(v)\,dv \;-\; g\,q
$$

$$
= \frac{\lambda}{3m}(p^2+q^2)^{3/2} - \; g\,q \quad \text{for } f(V) = \lambda V^2
$$

so that

$$
\dot{q} = \frac{\partial \mathcal{H}_0}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}_0}{\partial q}
$$

reproduce the original Cartesian equations of motion.

… emotionally distressing dynamical variables!

$$\dot{V} = -g\sin\theta \tag{1}$$

$$mV\dot{\theta} = f(V) - mg\cos\theta \tag{2}$$

Using the dynamical variables

$$q \equiv \theta \quad \text{and} \quad p \equiv V^2$$

the extra conserved is

$$\mathcal{H}_1(p,q) = \frac{1}{m}\int^p \frac{f(r^{1/2})dr}{r^{1/2}} - 2gp^{1/2}\cos q$$

$$= \frac{2\lambda}{3m}p^{3/2} - 2gp^{1/2}\cos q \quad \text{for } f(V) = \lambda V^2$$

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\dot{V} &= -g\sin\theta & (1)\\
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$$= \frac{2\lambda}{3m}p^{3/2} - 2gp^{1/2}\cos q \quad \text{for } f(V) = \lambda V^2$$

and upon changing variables equations (1) and (2) are

$$\dot{p} = -\frac{\partial\mathcal{H}_1}{\partial q} = \{p, \mathcal{H}_1\}_1$$

$$\dot{q} = \frac{\partial\mathcal{H}_1}{\partial p} = \{q, \mathcal{H}_1\}_1$$

where $\{F,G\}_1 = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$ is the old familiar Poisson bracket.

$$\dot{V} = -g\sin\theta \tag{1}$$
$$mV\dot{\theta} = f(V) - mg\cos\theta \tag{2}$$

Darryl noted that instead of the canonical Hamiltonian structure we can reconsider the original variables

$$q \equiv \theta \quad \text{and} \quad p \equiv V$$

and the non-canonical Poisson bracket

$$\{F, G\}_2 = \left(\frac{1}{p}\frac{\partial F}{\partial q}\right)\left(\frac{\partial G}{\partial p}\right) - \left(\frac{\partial F}{\partial p}\right)\left(\frac{1}{p}\frac{\partial G}{\partial q}\right)$$

so that

$$\dot{p} = \{p, \mathcal{H}_2\}_2 \quad \text{and} \quad \dot{q} = \{q, \mathcal{H}_2\}_2$$

with the conserved quantity serving directly as the Hamiltonian,

$$\mathcal{H}_2(p,q) = \frac{1}{m}\int^p f(v)\,dv \; - \; gp\cos q$$

$$= \frac{\lambda}{3m}p^3 \; - \; g\,p\,\cos q \quad \text{for } f(V) = \lambda V^2$$

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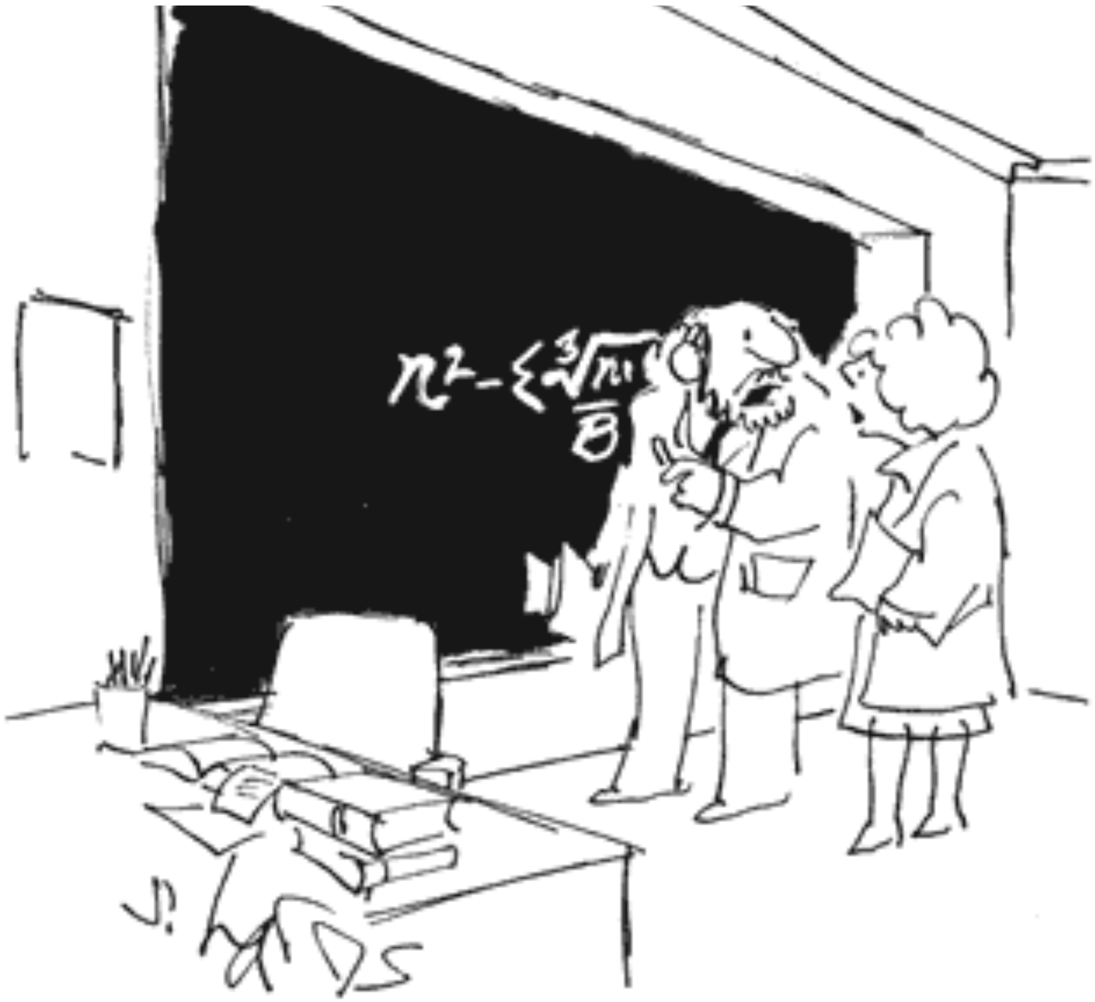
- Okay … so what? What does it all mean?
- Thermal glider ($e^{-\beta H}$)? Quantum glider (e^{iHt})?
- Question for you all: is the 'extra' conserved quantity the consequence of a symmetry?
- If so, what symmetry?
- Jane suggests that it is horizontal translation invariance ... but Darryl says "*It is what it is!*"
- Anyway maybe this is a nice elementary physically motivated model upon which to hone your tools!

Hamiltonian equations of motion and the ‘extra’ conserved quantity for the Lanchester-Joukowski glider



"THIS FORMULA IS LIKE THE CONSTITUTION. YOU CAN'T INTERPRET IT UNLESS YOU KNOW MY INTENT."

Optimal bounds and extremal trajectories for time averages in dynamical systems

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[3]*Center for the Study of Complex Systems, University of Michigan, Ann Arbor, MI 48109, USA*
(Dated: June 2, 2017)

For any quantity of interest in a system governed by ordinary differential equations, it is natural to seek the largest (or smallest) long-time average among solution trajectories. Upper bounds can be proved *a priori* using auxiliary functions, the optimal choice of which is a convex optimization. We show that the problems of finding maximal trajectories and minimal auxiliary functions are strongly dual. Thus, auxiliary functions provide arbitrarily sharp upper bounds on maximal time averages. They also provide volumes in phase space where maximal trajectories must lie. For polynomial equations, auxiliary functions can be constructed by semidefinite programming, which we illustrate using the Lorenz system.

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Introduction – For dynamical systems governed by ordinary differential equations (ODEs) whose solutions are complicated and perhaps chaotic, the primary interest is often in long-time averages of key quantities. Time averages can depend on initial conditions, so it is natural to seek the largest or smallest average among all trajectories, as well as extremal trajectories themselves. Various uses of such trajectories are described in [1]. In many nonlinear systems, however, it is prohibitively difficult to determine extremal time averages by constructing a large number of candidate trajectories, which may be dynamically unstable. It can be challenging both to compute trajectories and to determine that none have been overlooked. In this Letter we study an alternative approach that is broadly applicable and often more tractable: constructing sharp *a priori* bounds on long-time averages. We focus on upper bounds; lower bounds are analogous.

The search for an upper bound on a long-time average can be posed as a convex optimization problem [2], as described in the next section. Its solution requires no knowledge of trajectories. What is optimized is an auxiliary function defined on phase space, similar to Lyapunov functions in stability theory. We prove here that the best bound produced by solving this convex optimization problem coincides exactly with the extremal long-time average. That is, arbitrarily sharp bounds on time averages can be produced using increasingly optimal auxiliary functions. Moreover, nearly optimal auxiliary functions yield volumes in phase space where maximal and nearly maximal trajectories must reside. Whether such auxiliary functions can be computed in practice depends on the system being studied, but when the ODE and quantity of interest are polynomial, auxiliary functions can be constructed by solving semidefinite programs (SDPs) [2–4]. The resulting bounds can be arbitrarily sharp. We illustrate these methods using the Lorenz system [5].

Consider a well-posed autonomous ODE on $\mathbb{R}^d$,

$$\tfrac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}), \tag{1}$$

whose solutions are continuously differentiable in their initial conditions. To guarantee this, we assume that $\mathbf{f}(\mathbf{x})$ is continuously differentiable. Given a continuous quantity of interest $\Phi(\mathbf{x})$, we define its *long-time average* along a trajectory $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ by

$$\overline{\Phi}(\mathbf{x}_0) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Phi(\mathbf{x}(t))\,dt. \tag{2}$$

Time averages could be defined using lim inf instead; our results hold *mutatis mutandis* [6].

Let $B \subset \mathbb{R}^d$ be a closed bounded region such that trajectories beginning in B remain there. In a dissipative system B could be an absorbing set; in a conservative system B could be defined by constraints on invariants. We are interested in the maximal long-time average among all trajectories eventually remaining in B:

$$\overline{\Phi}^* = \max_{\mathbf{x}_0\in B} \overline{\Phi}(\mathbf{x}_0). \tag{3}$$

As shown below, there exist $\mathbf{x}_0$ attaining the maximum. The fundamental questions addressed here are: what is the value of $\overline{\Phi}^*$, and which trajectories attain it?

Bounds by convex optimization – Upper bounds on long-time averages can be deduced using the fact that time derivatives of bounded functions average to zero. Given any initial condition $\mathbf{x}_0$ in B and any $V(\mathbf{x})$ in the class $C^1(B)$ of continuously differentiable functions on B [7],

$$\overline{\tfrac{d}{dt}V} = \overline{\mathbf{f}\cdot\nabla V} = 0. \tag{4}$$

This generates an infinite family of functions with the same time average as Φ since for all such V

$$\overline{\Phi} = \overline{\Phi + \mathbf{f}\cdot\nabla V}. \tag{5}$$

Bounding the righthand side pointwise gives

$$\overline{\Phi}(\mathbf{x}_0) \le \max_{\mathbf{x}\in B}\{\Phi + \mathbf{f}\cdot\nabla V\} \tag{6}$$

for all initial conditions $\mathbf{x}_0 \in B$ and *auxiliary functions* $V \in C^1(B)$. Expression (6) is useful since no knowledge of trajectories is needed to evaluate the righthand side.

To obtain the optimal bound implied by (6), we minimize the righthand side over V and maximize the lefthand side over $\mathbf{x}_0$:

$$\max_{\mathbf{x}_0\in B} \overline{\Phi} \le \inf_{V\in C^1(B)} \max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V\}\,. \tag{7}$$

The minimization over auxiliary functions V in (7) is convex, although minimizers need not exist. The main mathematical result of this Letter is that the lefthand and righthand optimizations are dual variational problems, and moreover that strong duality holds, meaning that (7) can be improved to an equality:

$$\max_{\mathbf{x}_0\in B} \overline{\Phi} = \inf_{V\in C^1(B)} \max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V\}\,. \tag{8}$$

Thus, arbitrarily sharp bounds on the maximal time average $\overline{\Phi}^*$ can be obtained using increasingly optimal V.

The next section describes how nearly optimal V can also be used to locate maximal and nearly maximal trajectories in phase space. The section after illustrates these ideas using the Lorenz system, for which we have constructed nearly optimal V by solving SDPs. The final section proves the strong duality (8) and establishes the existence of maximal trajectories.

Near optimizers – An initial condition $\mathbf{x}_0^*$ and auxiliary function V^* are optimal if and only if they satisfy

$$\overline{\Phi}(\mathbf{x}_0^*) = \max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V^*\}\,. \tag{9}$$

Even if the infimum over V in (8) is not attained, there exist nearly optimal pairs. That is, for all $\epsilon > 0$ there exist $(\mathbf{x}_0, V)$ for which (6) is within ϵ of an equality:

$$0 \le \max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V\} - \overline{\Phi}(\mathbf{x}_0) \le \epsilon. \tag{10}$$

In such cases, $\max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V\}$ is within ϵ of being a sharp upper bound on $\overline{\Phi}^*$, while the trajectory starting at $\mathbf{x}_0$ achieves a time average $\overline{\Phi}$ within ϵ of $\overline{\Phi}^*$.

Nearly optimal V can be used to locate all trajectories consistent with (10). Moving the constant term inside the time average and subtracting the identity (4) gives

$$0 \le \overline{\max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V\} - (\Phi + \mathbf{f}\cdot\nabla V)} \le \epsilon \tag{11}$$

for such trajectories. The integrand in (11) is nonnegative, and the fraction of time it exceeds ϵ can be estimated. Consider the set where the integrand is no larger than $M > \epsilon$,

$$\mathcal{S}_M = \Big\{\mathbf{x}\in B : \max_{\mathbf{x}\in B} \{\Phi + \mathbf{f}\cdot\nabla V\} - (\Phi + \mathbf{f}\cdot\nabla V)(\mathbf{x}) \le M\Big\}. \tag{12}$$

Let $\mathcal{F}_M(T)$ denote the fraction of time $t\in[0,T]$ during which $\mathbf{x}(t)\in\mathcal{S}_M$. For any trajectory obeying (11), this time fraction is bounded below as

$$\liminf_{T\to\infty} \mathcal{F}_M(T) \ge 1 - \epsilon/M. \tag{13}$$

In practice, it may not be known if there exist trajectories satisfying (10) for a given V and ϵ. Still, the estimate (13) says that any such trajectories would lie in $\mathcal{S}_M$ for a fraction of time no smaller than $1-\epsilon/M$. The conclusion is strongest when $\epsilon \ll M$, but if M is too large the volume $\mathcal{S}_M$ is large and featureless, failing to distinguish nearly maximal trajectories. The result is most informative when V is nearly optimal so that there exist trajectories where $\epsilon \ll M$ with M not too large.

If a minimal V^* exists, its set $\mathcal{S}_0$ is related to maximal trajectories. Any such trajectory achieves $\epsilon = 0$ in (11). If it is a periodic orbit, for instance, it must lie in $\mathcal{S}_0$. Thus V^* is determined up to a constant on maximal orbits. More generally, V^* must satisfy

$$\Phi(\mathbf{x}) + \mathbf{f}(\mathbf{x})\cdot\nabla V^*(\mathbf{x}) = \overline{\Phi}^* \tag{14}$$

for all $\mathbf{x}\in\mathcal{S}_0$. It is tempting to conjecture that $\mathcal{S}_0$ coincides with maximal trajectories but, as described at the end of the next section, $\mathcal{S}_0$ can also contain points not on any maximal trajectory.

Nearly optimal bounds & orbits in Lorenz's system – When $\mathbf{f}(\mathbf{x})$ and $\Phi(\mathbf{x})$ are polynomials, $V(\mathbf{x})$ can be optimized computationally within a chosen polynomial ansatz by solving an SDP [2–4]. The bound $\overline{\Phi}^* \le U$ follows from (6) if $\Phi + \mathbf{f}\cdot\nabla V \le U$ for all $\mathbf{x}\in B$. A sufficient condition for this is that $U - \Phi - \mathbf{f}\cdot\nabla V$ is a sum of squares (SOS) of polynomials. The latter is equivalent to an SDP and is often computationally tractable [8, 9].

It does not follow from the strong duality result (8) that bounds computed by SOS methods can be arbitrarily sharp. This is because requiring the polynomial $U - \Phi - \mathbf{f}\cdot\nabla V$ to be SOS is generally stronger than requiring it to be nonnegative [8, 10]. Conditions under which the SOS bounding method is sharp are the subject of ongoing research. For several systems, SOS bounds have been constructed that either are sharp or appear to become sharp as the polynomial degree of V increases [3, 4].

The remainder of this section presents the results of SOS bounding computations for the Lorenz system at the standard chaotic parameters $(\beta,\sigma,r) = (8/3, 10, 28)$. We obtain nearly sharp bounds on the maximal time average of $\Phi(x,y,z) = z^4$, as well as approximations to maximal trajectories. Because there exist compact absorbing balls [5], maximization over such B in (3) is equivalent to maximization over $\mathbb{R}^d$. As reported in [4], Viswanath's periodic orbit library [11] suggests that the maximal average $\overline{z^4}^*$ is attained by the shortest periodic orbit—the black curves in Fig. 1. We have used SOS methods to construct nearly optimal $V(x,y,z)$ and accompanying upper bounds U. Similar results for various Φ in the Lorenz system appear in [4]. Here we report more precise computations for $\Phi = z^4$, obtained using the multiple precision SDP solver SDPA-GMP [12, 13]. Conversion of SOS conditions to SDPs was automated by

TABLE I. Upper bounds $\overline{z^4} \leq U$ in the Lorenz system computed using polynomial $V(x, y, z)$ of various degrees. Underlined digits agree with the value $\overline{z^4} \approx 592827.338$ attained on the shortest periodic orbit.

Degree of V	Upper bound U
4	635908.
6	$\underline{59}5152.$
8	$\underline{592}935.$
10	$\underline{592827}.568$
12	$\underline{592827.3}44$

YALIMP [14, 15], which was interfaced with the solver via mpYALMIP [16].

Table I reports upper bounds computed by solving SDPs that produce optimal V of various polynomial degrees. The bounds appear to approach perfection as the degree of V increases, reflecting the duality (8). We do not report the lengthy expressions for these V; some simpler examples appear in [4].

To demonstrate how the volumes $\mathcal{S}_M$ defined in (12) approximate maximal trajectories, we consider the polynomials V of degrees 6 and 10 that produce the bounds in Table I. For the maximum in the definition of $\mathcal{S}_M$ we use the corresponding U, which bounds it from above. In each case we find that U is within 0.1 of the true maximum over any ball B containing the attractor.

Figure 1a shows the volume $\mathcal{S}_{3000}$ for the degree-6 V, as well as the orbit that appears to maximize $\overline{z^4}$. The volume captures the rough location and shape of the orbit while omitting much of the strange attractor, but this V is not optimal enough to yield strong quantitative statements. It follows from (13) that any trajectory where $\overline{z^4}$ is within ϵ of the upper bound $U = 595152$ must lie inside $\mathcal{S}_{3000}$ for a fraction of time no less than $1-\epsilon/3000$. However, there are no trajectories on which this is close to unity; the higher-degree bounds in Table I preclude any trajectories with $\epsilon < 2324$.

The degree-10 V gives a significantly refined picture of maximal and nearly maximal trajectories for $\overline{z^4}$. Figure 1b shows the volume $\mathcal{S}_{1000}$ defined using this V. It follows from (13) that any trajectory where $\overline{z^4}$ comes within ϵ of $U = 592827.568$ must lie in $\mathcal{S}_{1000}$ for a fraction of time no less than $1-\epsilon/1000$. There exist trajectories on which this is nearly unity: on the shortest periodic orbit $\overline{z^4}$ is only $\epsilon \approx 0.23$ smaller than U. Any trajectory where $\overline{z^4}$ is so large must spend at least 99.97% of its time in $\mathcal{S}_{1000}$.

Finding maximal trajectories directly may be intractable in many systems. We propose that the next best option is to compute volumes like those in Fig. 1. However, we caution that finding points in a set $\mathcal{S}_M$ defined by (12) can itself be difficult, even for polynomials.

As the auxiliary functions producing upper bounds on $\overline{\Phi}^*$ approach optimality, the integrand in (11) approaches zero almost everywhere on maximal trajectories. This can be seen in Fig. 2a, where the integrand is plotted along the shortest periodic orbit in the Lorenz system for our polynomials V of degrees 6, 8, and 10. Along other orbits where $\overline{z^4}$ is large but not maximal, V is less strongly constrained. As an example, we consider the periodic orbit computed in [11] that winds around the two wings of the Lorenz attractor with symbol sequence $AABABB$. On this orbit $\overline{z^4}$ is smaller than the maximum by approximately 2798. The integral in (11) remains between 0 and 2798 as V approaches optimality but need not approach 0 on this orbit. In our computations it does not, as seen in Fig. 2b.

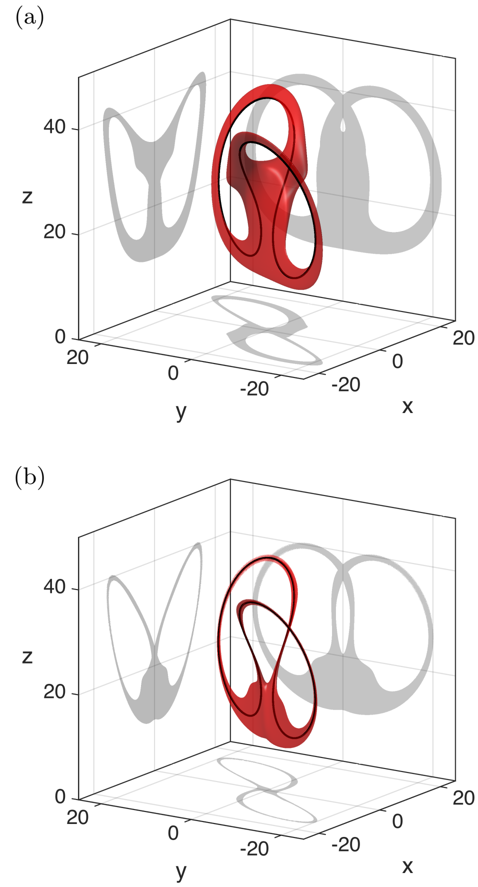

FIG. 1. (a) The volume $\mathcal{S}_{3000}$ for the optimal degree-6 polynomial V; (b) The volume $\mathcal{S}_{1000}$ for the optimal degree-10 polynomial V. Any trajectory maximizing $\overline{z^4}$ must spend at least 99.97% of its time in $\mathcal{S}_{1000}$. The black curves show the shortest periodic orbit, which appears to maximize $\overline{z^4}$.

Although the auxiliary polynomials V yielding the bounds on $\overline{z^4}$ in Table I approach optimality, they are not exactly optimal. Optimal V^* which are polynomial, however, have been constructed to prove sharp bounds on other averages in the Lorenz system, including $\overline{z}$, $\overline{z^2}$, and $\overline{z^3}$ [4, 17, 18]. These averages are maximized on the two nonzero equilibria; in each case the set $\mathcal{S}_0$ corresponding to V^* is the line through these equilibria. These $\mathcal{S}_0$ notably include points not on any maximal trajectory. In contrast, for $\overline{z^4}$ the shortest periodic orbit appears to be maximal. This conjecture could be proved by constructing a V whose $\mathcal{S}_0$ contains the shortest orbit. If such a V exists, it would necessarily be optimal. However, we expect that this orbit is non-algebraic and that no polynomial V can be optimal.

Proof of duality – To prove the strong duality (8) we require several facts from ergodic theory, which are provable by standard methods as in [19]. (See also [20, Chap. 12].) Let $\varphi_t(\mathbf{x})$ denote the flow map $\mathbf{x}(\cdot) \mapsto \mathbf{x}(\cdot + t)$ for the ODE (1). By assumption, φ_t is well-defined on B for all $t \geq 0$ and is continuously differentiable there. Let $Pr(B)$ denote the space of Borel probability measures on B. A measure $\mu \in Pr(B)$ is *invariant* with respect to φ_t if $\mu(\varphi_t^{-1}A) = \mu(A)$ for all Borel sets A and all t. Such a measure is *ergodic* if to any invariant Borel set it assigns measure either zero or one. The set of invariant probability measures on B is nonempty, convex, and weak-$*$ compact; its extreme points are ergodic.

Our proof of the duality (8) proceeds via a standard minimax template from convex analysis (see, e.g., [21]). It suffices to establish the following sequence of equalities:

$$\max_{\mathbf{x}_0 \in B} \overline{\Phi} = \max_{\substack{\mu \in \mathrm{Pr}(B) \\ \mu \text{ is invar.}}} \int \Phi \, d\mu \tag{15a}$$

$$= \sup_{\mu \in Pr(B)} \inf_{V \in C^1(B)} \int \Phi + \mathbf{f} \cdot \nabla V \, d\mu \tag{15b}$$

$$= \inf_{V \in C^1(B)} \sup_{\mu \in Pr(B)} \int \Phi + \mathbf{f} \cdot \nabla V \, d\mu \tag{15c}$$

$$= \inf_{V \in C^1(B)} \max_{\mathbf{x} \in B} \left\{ \Phi + \mathbf{f} \cdot \nabla V \right\}. \tag{15d}$$

The final equality (15d) is evident since, for each V, the supremum in (15c) is attained by a suitable Dirac measure. The remainder of this section is devoted to proving the first three equalities (15a)–(15c), along with the fact that the maximum in (15a) is attained.

We begin by proving (15a). We claim that the righthand problem appearing there is a concave relaxation of the lefthand problem, and that it attains the same maximum. To see this, note first that for each initial condition $\mathbf{x}_0$ in B there exists at least one invariant probability measure μ that attains $\overline{\Phi}(\mathbf{x}_0) = \int \Phi \, d\mu$. Thus,

$$\sup_{\mathbf{x}_0 \in B} \overline{\Phi}(\mathbf{x}_0) \leq \max_{\substack{\mu \in \mathrm{Pr}(B) \\ \mu \text{ is invar.}}} \int \Phi \, d\mu. \tag{16}$$

The righthand problem in (16) is a maximization of a continuous linear functional over a compact convex subset of $Pr(B)$, so it achieves its maximum at an extreme point μ^* [22, Chap. 13], which is an ergodic invariant measure. By Birkhoff's ergodic theorem [19],

$$\overline{\Phi}(\mathbf{x}_0) = \int \Phi \, d\mu^* = \max_{\substack{\mu \in \mathrm{Pr}(B) \\ \mu \text{ is invar.}}} \int \Phi \, d\mu \tag{17}$$

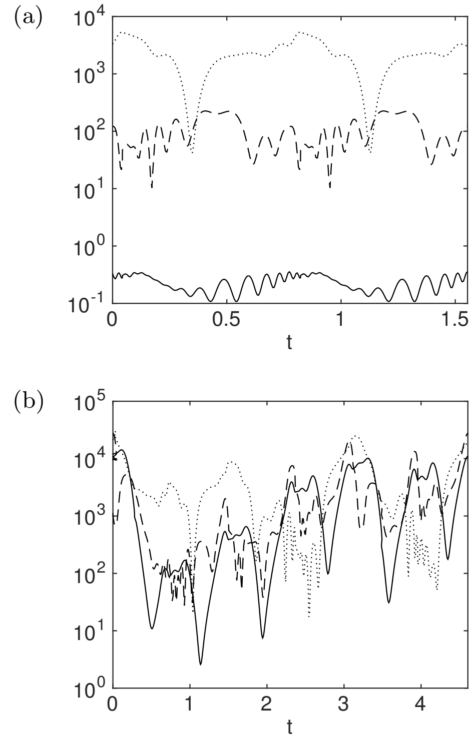

FIG. 2. The quantity $U - (\Phi + \mathbf{f} \cdot \nabla V)$ for $\Phi = z^4$ and polynomials V of degrees 6 (------), 8 (– – –), and 10 (——–), plotted along (a) the shortest periodic orbit and (b) the periodic orbit with symbol sequence $AABABB$.

for almost every $\mathbf{x}_0$ in the support of μ^*. Therefore the inequality in (16) is in fact an equality, and any such $\mathbf{x}_0$ attains the maximal time average $\overline{\Phi}^*$. This proves (15a).

To prove the second equality (15b) we require the following equivalence of Lagrangian and Eulerian notions of invariance: a Borel probability measure μ is invariant with respect to φ_t by the usual (Lagrangian) definition if and only if the vector-valued measure $\mathbf{f}\mu$ is weakly divergence-free. The latter condition, which we denote by $\operatorname{div} \mathbf{f}\mu = 0$, means that

$$\int \mathbf{f} \cdot \nabla \psi \, d\mu = 0 \tag{18}$$

for all smooth and compactly supported $\psi(\mathbf{x})$. This is an Eulerian characterization of invariance.

The fact that $\operatorname{div} \mathbf{f}\mu = 0$ is equivalent to invariance is quickly proved using the flow semigroup identity, which states that $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all t and s. It follows that

$$\frac{d}{dt} \int \psi \circ \varphi_t \, d\mu = \int \mathbf{f} \cdot \nabla(\psi \circ \varphi_t) \, d\mu \tag{19}$$

for all smooth and compactly supported ψ. If $\operatorname{div} \mathbf{f}\mu = 0$, the righthand side of (19) vanishes, so μ is invariant. Conversely, if μ is invariant then the lefthand side of (19) vanishes for all t, and at $t = 0$ we find the statement that $\mathbf{f}\mu$ is weakly divergence-free.

With the Eulerian characterization of invariance in hand, we turn to proving (15b). Depending on μ, there are two possibilities for the minimization over V in (15b):

$$\inf_{V\in C^1(B)} \int \mathbf{f}\cdot\nabla V \, d\mu = \begin{cases} 0 & \operatorname{div}\mathbf{f}\mu = 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{20}$$

Only measures for which $\operatorname{div}\mathbf{f}\mu = 0$ can give values larger than $-\infty$ in (15b). As shown above, $\operatorname{div}\mathbf{f}\mu = 0$ if and only if μ is invariant. Therefore, since there always exists at least one invariant probability measure,

$$\sup_{\mu\in Pr(B)} \inf_{V\in C^1(B)} \int \Phi + \mathbf{f}\cdot\nabla V \, d\mu = \max_{\substack{\mu\in \mathrm{Pr}(B)\\ \mu \text{ is invar.}}} \int \Phi \, d\mu. \tag{21}$$

Thus (15b) is proven. In other words,

$$\mathcal{L}(\mu, V) = \int \Phi + \mathbf{f}\cdot\nabla V \, d\mu \tag{22}$$

is a Lagrangian for the constrained maximization appearing on the righthand side of (21).

Finally, we prove the equality (15c). In terms of the Lagrangian $\mathcal{L}$, we must show that

$$\sup_{\mu\in \mathrm{Pr}(B)} \inf_{V\in C^1(B)} \mathcal{L} = \inf_{V\in C^1(B)} \sup_{\mu\in \mathrm{Pr}(B)} \mathcal{L}. \tag{23}$$

The fact that the order of inf and sup can be reversed without introducing a so-called duality gap is not trivial; it is at the heart of our proof of the strong duality (8). This reversal relies on properties of the Lagrangian $\mathcal{L}$ and the spaces $Pr(B)$ and $C^1(B)$.

The desired equality (23) can be proved using any of several abstract minimax theorems from convex analysis. Here we apply a fairly general infinite-dimensional version due to Sion [23]. We follow the notation of its statement in the introduction of [24], which contains an elementary proof. Let $X = \mathrm{Pr}(B)$ in the weak-$*$ topology. It is a compact convex subset of a linear topological space. Let $Y = C^1(B)$ in the C^1-norm topology, which is itself a linear topological space. Take $f = -\mathcal{L}$ and observe that $f(x,\cdot)$ is upper semicontinuous and quasi-concave on Y for each $x \in X$, and that $f(\cdot, y)$ is lower semicontinuous and quasi-convex on X for each $y \in Y$. Then (23) follows from a direct application of Sion's minimax theorem [24], so (15c) is proven.

This completes the proof of the equalities (15a)–(15d) and so too the proof of the strong duality (8).

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