# Hamiltonian for the zeros of the Riemann zeta function

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New Trends in Applied Geometric Mechanics
– Celebrating Darryl Holm's 70th Birthday –
ICMAT, Madrid, Spain 3–7 July 2017
(Joint work with Carl M. Bender & Markus P. Müller)


Happy Birthday Darryl !

# 1. The asymptotic law of distribution of prime numbers

In 1792, Gauss, at the age of 15, conjectured that the asymptotic law of distribution of prime numbers is

$$\pi(\Lambda) \sim \frac{\Lambda}{\log \Lambda}.$$

In 1859, Riemann published his paper *On the number of primes less than a given magnitude*, in which the zeta function

$$\zeta(z) = \sum_{n \geq 1} \frac{1}{n^z} = \prod_p \left(1 - \frac{1}{p^z}\right)^{-1}$$

played a prominent role.

## 2. Analytic number theory

Riemann's observations are based on the fact that $\zeta(z)$ can be meromorphically continued (simple pole at $z = 1$), and on the identity:

$$\mathbb{1}\{y > 1\} = \frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \frac{y^z}{z} \, \mathrm{d}z.$$

Consider to start with the counting of the integers:

$$\sum_{1 \leq n \leq \Lambda} 1 = \sum_{n \geq 1} \mathbb{1} \left\{ \frac{\Lambda}{n} > 1 \right\} = \frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \sum_{n \geq 1} \left( \frac{\Lambda}{n} \right)^z \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \zeta(z) \, \frac{\Lambda^z}{z} \, \mathrm{d}z = \Lambda - \frac{1}{2}$$

Similarly, we have

$$\psi(\Lambda) = \sum_{p\le\Lambda} \log p \;=\; \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \sum_p \log p \left(\frac{\Lambda}{p}\right)^z \frac{1}{z}\,\mathrm{d}z$$
$$= \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \sum_p \frac{\log p}{p^z}\frac{\Lambda^z}{z}\,\mathrm{d}z \approx \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \frac{\zeta'(z)}{\zeta(z)}\frac{\Lambda^z}{z}\,\mathrm{d}z$$

The derivation of the expression for $\psi(\Lambda)$ then relies on the study of the properties of $\zeta(z)$, in particular, its zeros.

The reflection formula

$$\zeta(z) = 2^z\pi^{z-1}\sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$$

shows that the zeta function vanishes *trivially* for $z = -2n$ $(n = 1, 2, \ldots)$.

Riemann conjectured (the Riemann Hypothesis) that the *nontrivial* zeros of $\zeta(z)$ all lie on the straight line

$$\Re(z) = \frac{1}{2}.$$

## 3. The Hilbert-Pólya conjecture

Assuming that the Riemann hypothesis holds true, and writing

$$z_n = \frac{1}{2} + \mathrm{i}E_n,$$

the (real) numbers $\{E_n\}$ should correspond to the eigenvalues of a Hermitian operator (the so-called Riemann operator).

## 4. The Berry-Keating conjecture

In 1989, Berry and Keating conjectured that the Riemann operator should be given by a quantisation of the classical Hamiltonian

$$H = xp.$$

A lot of efforts have been made by various authors to find such Hamiltonian, but without success until now.

# 5. Outline of the talk

We consider the 'Hamiltonian' operator

$$\hat{H} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} \left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) \left(\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}\right),$$

which reduces to the classical Hamiltonian function $H = 2xp$.

It will be shown that with the boundary condition

$$\psi(0) = 0$$

the eigenvalues $\{E_n\}$ of $\hat{H}$ satisfy the property that $\left\{\frac{1}{2}(1 - \mathrm{i}E_n)\right\}$ are the zeros of the Riemann zeta function.

The Riemann hypothesis follows if all eigenvalues of $\hat{H}$ are real.

Using the pseudo-Hermiticity of $\hat{H}$, a heuristic analysis will be presented that suggests that this is indeed the case.

# 6. The shift operator and its inverse

Defining

$$\hat{\Delta} \equiv \mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}},$$

in units $\hbar = 1$ we have

$$\hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$$

so that

$$\hat{\Delta} f(x) = f(x) - f(x-1).$$

As for $\hat{\Delta}^{-1}$ we have

$$\hat{\Delta}^{-1} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} = \frac{\mathbb{1}}{\mathrm{i}\hat{p}} \frac{-\mathrm{i}\hat{p}}{\mathrm{e}^{-\mathrm{i}\hat{p}} - \mathbb{1}} = \frac{\mathbb{1}}{\mathrm{i}\hat{p}} \sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!}.$$

In particular, if $f(x) \to 0$ sufficiently fast, then we have

$$\hat{\Delta}^{-1} f(x) = -\sum_{k=1}^{\infty} f(k+x).$$

# 7. Uniqueness of $\hat{\Delta}\psi$

We multiply the eigenvalue equation

$$\hat{H}\psi = E\psi$$

on the left by $\hat{\Delta}$.

Recall that

$$\hat{H} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}.$$

This gives a first-order linear differential equation

$$(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}\psi = -\mathrm{i}\left(2x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)\hat{\Delta}\psi = E\,\hat{\Delta}\psi$$

for the function $\hat{\Delta}\psi$, whose solution is unique and is given by

$$\hat{\Delta}\psi = x^{-z}$$

up to a multiplicative constant.

Therefore,

$$\psi(x) = \hat{\Delta}^{-1}x^{-z}.$$

## 8. Eigenstates and eigenvalues

The eigenstates of $\hat{H}$ are given by the Hurwitz zeta function

$$\psi_z(x) = -\zeta(z, x+1)$$

on the positive half line $\mathbb{R}^+$, with eigenvalues

$$E = \mathrm{i}(2z-1).$$

To see this, observe that, up to an additive constant,

$$\begin{aligned}\hat{\Delta}^{-1}x^{-z} &= \frac{1}{\mathrm{i}\hat{p}}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} x^{-z} \\ &= \frac{1}{\mathrm{i}\hat{p}}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} (\mathrm{i}\hat{p}) \frac{x^{1-z}}{1-z} \\ &= \frac{1}{1-z}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} x^{1-z}.\end{aligned}$$

Because $\mathrm{i}\hat{p} = \partial_x$ and

$$\partial_x^n \, x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\, x^{\mu-n},$$

setting $\mu = 1 - z$ we find

$$\hat{\Delta}^{-1} x^{-z} = \frac{\Gamma(2-z)}{1-z} \sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!} \frac{x^{1-z-n}}{\Gamma(2-z-n)},$$

but we have $\Gamma(2-z) = (1-z)\Gamma(1-z)$ and

$$\frac{1}{\Gamma(2-z-n)} = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}u \, \mathrm{e}^u \, u^{n+z-2},$$

so

$$\begin{aligned} \hat{\Delta}^{-1} x^{-z} &= \frac{\Gamma(1-z)}{2\pi\mathrm{i}} x^{1-z} \int_C \mathrm{d}u \, \mathrm{e}^u \, u^{z-2} \sum_{n=0}^{\infty} B_n \frac{(-u/x)^n}{n!} \\ &= \frac{\Gamma(1-z)}{2\pi\mathrm{i}} x^{1-z} \int_C \mathrm{d}u \, \mathrm{e}^u u^{z-2} \frac{-u/x}{\mathrm{e}^{-u/x} - 1} \\ &= \frac{\Gamma(1-z)}{2\pi\mathrm{i}} x^{-z} \int_C \mathrm{d}u \, \frac{\mathrm{e}^u u^{z-1}}{1 - \mathrm{e}^{-u/x}}. \end{aligned}$$

Now we scale the integration variable according to $u/x = t$ and obtain

$$\hat{\Delta}^{-1} x^{-z} = \frac{\Gamma(1-z)}{2\pi\mathrm{i}} \int_C \mathrm{d}u \, \frac{\mathrm{e}^{xt} t^{z-1}}{1 - \mathrm{e}^{-t}} = -\zeta(z, x+1).$$

As for the eigenvalues, we have

$$\hat{H}\psi_z(x) = \hat{\Delta}^{-1}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right)\hat{\Delta}\hat{\Delta}^{-1}x^{-z} = \mathrm{i}(2z-1)\psi_z(x).$$

Note that for $\Re(z) > 1$ we have

$$-\zeta(z, x+1) = \frac{\Gamma(1-z)}{2\pi\mathrm{i}}\int_C \mathrm{d}u\,\frac{\mathrm{e}^{xt}t^{z-1}}{1-\mathrm{e}^{-t}} = -\sum_{k=1}^{\infty}\frac{1}{(x+k)^z}.$$

## 9. The boundary condition

We now impose the boundary condition that $\psi_n(0) = 0$ for all $n$.

Because $\zeta(z,1) = \zeta(z)$, this implies that $z$ can only be discrete zeros of the Riemann zeta function: If $z = \frac{1}{2}(1 - \mathrm{i}E)$ then $\mathrm{i}(2z-1) = E$.

Can $z$ be a trivial zero?

For the trivial zeros $z = -2n$, $n = 1, 2, \ldots$, we have

$$\psi_z(x) = -\frac{1}{2n+1} B_{2n+1}(x+1),$$

where $B_n(x)$ is a Bernoulli polynomial $\Longrightarrow |\psi_z(x)|$ grows like $x^{2n+1}$ as $x \to \infty$.

For the nontrivial zeros $\psi_z(x)$ oscillates and grows sublinearly.

Thus, for the trivial zeros $\hat{\Delta}\psi_z(x)$ blows up and for the nontrivial zeros $\hat{\Delta}\psi_z(x)$ goes to zero as $x \to \infty$.

## 10. Relation to pseudo-Hermiticity

If we consider the space-time (PT) inversion on the canonically transformed variables $(\hat{x}, \hat{p}) \to (\hat{p}, -\hat{x})$ so that $\mathcal{PT} : (\hat{x}, \hat{p}, \mathrm{i}) \longrightarrow (\hat{x}, -\hat{p}, -\mathrm{i})$, then we find that $\mathrm{i}\hat{H}$ is PT symmetric.

However, since $\mathcal{PT}\psi_n(x) = \psi_{-n}(x)$, the PT symmetry is broken for all $z_n \in \mathbb{C}$.

"Eigenvalues of $\mathrm{i}\hat{H}$ are purely imaginary" $\Rightarrow$ "The Riemann hypothesis holds"

To proceed, assume that $\hat{p}^\dagger$ is symmetric and that

$$\hat{H}^\dagger = (\mathbb{1} - \mathrm{e}^{\mathrm{i}\hat{p}})\,(\hat{x}\hat{p} + \hat{p}\hat{x})\,\frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{\mathrm{i}\hat{p}}}\,.$$

Then if we define the operator $\hat{\eta}$ according to

$$\hat{\eta} = \sin^2 \tfrac{1}{2}\hat{p},$$

which is nonnegative, bounded, and Hermitian under our assumption, we get

$$\hat{H}^\dagger = \hat{\eta}\hat{H}\hat{\eta}^{-1}.$$

Thus, our Hamiltonian $\hat{H}$ is pseudo-Hermitian:

$$\hat{\rho}\hat{H}\hat{\rho}^{-1} = \hat{h},$$

where

$$\hat{\rho}^{\dagger}\hat{\rho} = \hat{\eta} = \sin^2 \tfrac{1}{2}\hat{p}.$$

## 11. Quantisation condition for the Berry-Keating Hamiltonian

Recall that $\hat{\rho}^\dagger \hat{\rho} = \hat{\eta} = \sin^2 \frac{1}{2}\hat{p}$.

Choosing $\hat{\rho} = \hat{\Delta}$ we have the Berry-Keating Hamiltonian

$$\hat{h}^{BK} = \hat{x}\,\hat{p} + \hat{p}\,\hat{x},$$

with eigenstates and eigenvalues

$$\phi_z^{\rm BK}(x) = x^{-z} \quad \text{and} \quad E = {\rm i}(2z-1).$$

The boundary condition $\psi(0) = 0$ then translates into the quantisation condition for the Berry-Keating Hamiltonian, either as

$$\lim_{x\to 0} \left[\phi_z^{\rm BK}(x) - \zeta(z, x-1)\right] = 0$$

or alternatively as

$$\lim_{x\to 1} \phi_z^{\rm BK}(x) = -\lim_{x\to 1} \zeta(z, x+1).$$

## 12. Biorthogonal systems

The eigenstates $\{\tilde{\psi}_n(x)\}$ of

$$\hat{H}^\dagger = (\mathbb{1} - e^{i\hat{p}})\,(\hat{x}\hat{p} + \hat{p}\hat{x})\,\frac{\mathbb{1}}{\mathbb{1} - e^{i\hat{p}}}$$

are given by

$$\tilde{\psi}_n(x) = x^{-z_n} - (x+1)^{-z_n}.$$

Using $\{\tilde{\psi}_n(x)\}$, we introduce an inner product as follows.

For any

$$\psi(x) = \sum_n c_n \psi_n(x),$$

where $\sum_n |c_n|^2 < \infty$, we define its associated state by

$$\tilde{\psi}(x) = \sum_n c_n \tilde{\psi}_n(x).$$

[Ref: Brody, *Biorthogonal quantum mechanics*. J. Phys. A**47**, 035305 (2014)]

The inner product of a pair of such functions $\psi(x)$ and $\varphi(x)$ is then defined by

$$\langle \varphi, \psi \rangle = \langle \tilde{\varphi} | \psi \rangle \equiv \int \overline{\tilde{\varphi}(x)} \psi(x) \mathrm{d}x.$$

Alternatively stated, since $\tilde{\varphi}(x) = \hat{\eta}\varphi(x)$, we have $\langle \varphi, \psi \rangle = \langle \varphi | \hat{\eta} | \psi \rangle$.

Because

$$\tilde{\psi}_n(x) = \hat{\Delta}^\dagger \hat{\Delta} \psi_n(x) = \hat{\Delta}^\dagger \hat{\Delta} \hat{\Delta}^{-1} x^{-z_n} = \hat{\Delta}^\dagger x^{-z_n},$$

we find

$$\langle \tilde{\psi}_m | \psi_n \rangle = \int_0^\infty \mathrm{d}x \, x^{-1+\mathrm{i}(E_n - \bar{E}_m)/2}.$$

It follows that if the Riemann hypothesis is correct, then for $m \neq n$ we have

$$\langle \tilde{\psi}_m | \psi_n \rangle = 0$$

for the nontrivial zeros, whereas $\langle \tilde{\psi}_m | \psi_n \rangle = \infty$ for the trivial zeros.

It also follows from $\langle \tilde{\psi}_n | \psi_n \rangle \neq 0$ that the eigenvalues are nondegenerate if RH holds true; conversely, for any nontrivial zero $z_n$ such that $\Re(z_n) \neq \frac{1}{2}$, the eigenstates are degenerate.

In terms of the inner product introduced above, under the assumption on the Hermiticity of $\hat{p}$ we find, using $\hat{\Delta}^\dagger \hat{\Delta} = \hat{\eta}$, that

$$\begin{aligned} \langle \hat{H}\varphi, \psi \rangle &= \int_0^\infty \mathrm{d}x \, \bar{\varphi}(x) \hat{\Delta}^\dagger (\hat{x}\,\hat{p} + \hat{p}\,\hat{x})(\hat{\Delta}^\dagger)^{-1} \hat{\Delta}^\dagger \hat{\Delta} \psi(x) \\ &= \int_0^\infty \mathrm{d}x \, \bar{\varphi}(x) \hat{\Delta}^\dagger (\hat{x}\,\hat{p} + \hat{p}\,\hat{x}) \hat{\Delta} \psi(x) \\ &= \int_0^\infty \mathrm{d}x \, \bar{\varphi}(x) \hat{\Delta}^\dagger \hat{\Delta} \hat{\Delta}^{-1} (\hat{x}\,\hat{p} + \hat{p}\,\hat{x}) \hat{\Delta} \psi(x) \\ &= \langle \varphi, \hat{H}\psi \rangle. \end{aligned}$$

Hence under this assumption, $\hat{H}$ is Hermitian (symmetric).

We also find that on the inner product space $\langle \cdot, \cdot \rangle$, if we demand $\hat{p}^\dagger = \hat{p}$, then it is necessary and sufficient that $\psi(0) = 0$.

## 13. Fourier representation

One might ask: Why the Hamiltonian

$$\hat{H} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} \left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) \left(\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}\right) \ ?$$

In the momentum space the eigenfunction can be written

$$\hat{\psi}(p) = \Gamma(1-z) \left[ \frac{(-\mathrm{i}p)^{z-1}}{1-\mathrm{e}^{\mathrm{i}p}} - \mathrm{i}(2\pi)^{z-1} \left( \sum_{k=1}^{\infty} \frac{k^{z-1}}{p+2\pi k} - (-1)^z \sum_{k=1}^{\infty} \frac{k^{z-1}}{p-2\pi k} \right) \right].$$

This provides an integral representation for the Riemann zeta function

$$\int_{\mathrm{i}\epsilon-\infty}^{\mathrm{i}\epsilon+\infty} \hat{\psi}(p)\, \mathrm{d}p = \zeta(z).$$

Identifying the differential equation satisfied by $\hat{\psi}(p)$ in the momentum space is an open question.

## 14. Relation to quantum mechanics

A possible way of making a connection to quantum theory is to introduce a regularisation scheme, for example, by letting $x \in [\Lambda^{-1}, \Lambda]$, renormalising the states according to

$$\psi_n(x) \to (\ln \Lambda)^{-1/2} \psi_n(x),$$

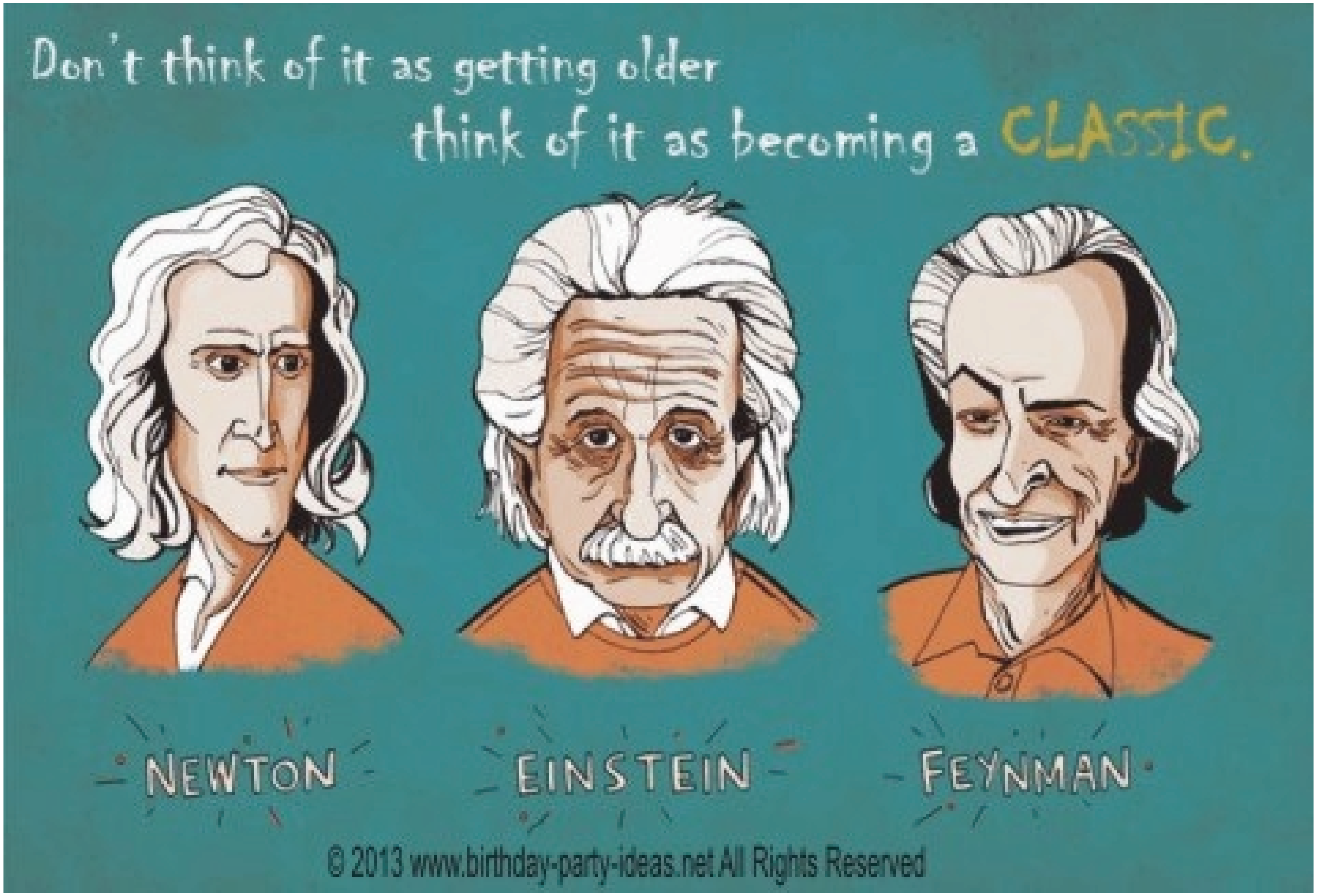
and then taking the limit $\Lambda \to \infty$.

Interestingly, the expectation value of the position operator $\hat{\rho}^{-1}\hat{x}\hat{\rho}$ in the state $\psi_n(x)$ for any $n$ in the renormalised theory is

$$\frac{\Lambda}{\ln \Lambda},$$

which for large $\Lambda$ gives the leading term in the counting of prime numbers smaller than $\Lambda$

**Reference**: Bender, C.M., Brody, D.C. & Müller, M.P. "Hamiltonian for the zeros of the Riemann zeta function" Physical Review Letters, **118**, 130201 (2017) .




Happy Birthday again Darryl!