INFINITE DIMENSIONAL INTEGRALS AND PDEs, AND THEIR CONNECTIONS WITH STOCHASTIC & QUANTUM PHENOMENA

SERGIO ALBEVERIO

IAM + HCM

UNIVERSITY OF BONN

I
1. Finite-dim.
2. ∞ -dim.
   - probabilistic
   - oscillatory
   - higher order

II Integrals assoc. to spaces of maps & repres. th.

III Deterministic, stochastic, quantum...

I. 1 Finite dimensions

Integrals / functionals we have in mind:

$$I_\mu(f) := \langle f, \mu \rangle := \int_{\mathbb{R}^n} f(x)\, \mu(dx)$$

$$\mu(dx) := \begin{cases} e^{-\phi(x)}\, dx, \\ e^{i\phi(x)}\, dx \end{cases}$$

$f: \mathbb{R}^n \to \mathbb{C}$

$\phi: \mathbb{R}^n \to \mathbb{R}_+$ or $\mathbb{R}^n \to \mathbb{R}$

Well known constructions, under assumptions on ϕ, f:

- **Lebesgue** (probabilistic) type
- **Oscillatory** type

- ϕ s.t. μ positive σ-add. measure, $f \in L^1(\mu)$
- ϕ, f s.t. Hörmander-type cond. hold (theory of finite dimens. oscill. integrals — Fourier integral operators)

Rem:

1) Absolute integrability not requested in oscillatory case

2) Common feature: $f \to I_\mu(f)$

linear, continuous functional

Asymptotics

Replace ϕ by $\frac{1}{\varepsilon}\phi$, $\varepsilon \downarrow 0$:

Laplace method

Stationary phase method

Expansions around critical points of ϕ

Difference however for the 2 types of integrals ...

Localization principle, interplay local / global

- Connections with catastrophe theory (resolution of singularities)
- Powerful method of approximate computation : relations with moment problem, summation theory of divergent asymptotic series ...

I.2 Infinite dimensions

Want to study analogues in ∞-dimensions, with applic. to PDEs of parabolic type / Schrödinger type

Look for objects of the form:

$$:= I_\mu(f) := \langle f, \mu \rangle := \int_\Gamma f(\gamma)\, \mu(d\gamma)$$

$\gamma \in \Gamma$

Γ: infinite dim. space

μ heuristically as before

$$\mu(d\gamma) = \cdots \text{" } e^{-\Phi(\gamma)}\, d\gamma \text{ "}$$
$$\cdots \text{" } e^{:\Phi(\gamma)}\, d\gamma \text{ "}$$

Rem: "$d\gamma$" is a problem ....

I.2.1 Integrals with respect to probability measures

Assume here: μ is **probability measure** on Γ

Prototype of measure of interest: μ_W **Wiener measure** (Brownian motion measure)

$\Gamma = C_{(0)}([0,t]; \mathbb{R}^d)$

Wiener space

$\gamma \in \Gamma$: "path" (contin., typically non differ.)

Heuristically:

$$\mu_W(d\gamma) = \text{"} e^{-\phi_W(\gamma)} d\gamma \text{"}$$

with

$$\phi_W(\gamma) := \frac{1}{2}\int_0^t |\dot{\gamma}(s)|^2 ds$$

$$d\gamma = \text{"} Z^{-1} \prod_{s\in[0,t]} d\gamma(s)\text{)}$$

Z "**normalization**" (s.t. μ prob. measure on Γ)

<u>Rem</u>: μ_W proj. limit of its approximants on $P_N \Gamma \cong \mathbb{R}^{Nd}$:

$$Z_N^{-1} e^{-\phi(P_N\gamma)} d(P_N\gamma) =$$

$$= (2\pi)^{-N\frac{d}{2}} e^{-\frac{1}{2}\sum_{j=0}^{N-1} \frac{|x_{j+1}-x_j|^2}{t_{j+1}-t_j}} dx_1 \dots dx_N$$

$(t_0 \equiv 0 < t_1 < \dots < t_N, \quad x_i = \gamma(t_i) \in \mathbb{R}^d)$

• Relation with **heat eqt.**:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u - V u$$

$$u\big|_{t=0} = u_0 \quad , \quad \text{on } \mathbb{R}^d$$

solution given by: $u(t,x) =$

$$= \int_\Gamma \underbrace{\exp\left(-\int_0^t V(x+\gamma(s))ds\right) u_0(x+\gamma(t))}_{f(\gamma)} \mu_W(d\gamma)$$

$$= I_{\mu_W}(f) \quad : \quad$$ **FEYNMAN-KAC FORMULA**

(Kac '49, e.g. $V \in C_b(\mathbb{R}^d)$)

Rem: 1) $I_{\mu_W}(f)$ expectation resp. Brownian motion process $B(s)$ $(= \gamma(s))$ (started at $t=0$ in 0), transit. semigr. heat semigr. $e^{\frac{t}{2}\Delta}$

2) $\exists$ vast generalizations on $\mathbb{R}^d$ (or M^d ....) of heat eqt. $\leftrightarrow$ prob. integr. resp. μ_W

heat eqt. $\Updownarrow$ heat semigr.

prob. integr. resp. μ_W $\Updownarrow$ diffusion process $B(t)$

to parabolic PDE (with Δ replaced by diffus. oper.

$$\frac{1}{2}\langle \beta(x), \nabla_x \rangle + \frac{1}{2} \operatorname{Tr}(\sigma(x)\sigma(x)^t D_x^2))$$

Let X_t associated process: it is a diffusion process

Diffusion process X_t satisfies SDE:

$$dX_t = \beta(X_t)\,dt + \underbrace{\sigma(X_t)\,dB(t)}_{\text{noise term}}$$

(this is an example of dynam. syst. underlying "stochastization": later more on this)

Recent results:

- asymptotic small noise expansions
- transformation/reduction properties by symmetries

... Gaeta
De Vecchi, Morandi,
Ugolini, ... A.

- Also asymptotics of $I_{\mu_W}(f)$ or, more generally, $I_\mu(f)$ (**μ prob. measure on some ∞-dim. space**), with Φ replaced by $\frac{1}{\varepsilon}\Phi$, $\varepsilon \downarrow 0$

Worked out: **infinite dimensional Laplace method**

Donsker ... Ikeda ...

Recently: A + V. Steblovskaya (degeneracies of Φ included)

Applications: Study of EVs ..., heat kernels ...

I.2.2 Infinite dimensional oscillatory integrals

Feynman '48 (after Dirac '33) came up with heuristic formula for solution of quantum mechanical / quantum field theoretical problems in terms of

$$I_\mu(f) = \text{"}\int_\Gamma f(\gamma)\, \mu(d\gamma)\text{"}$$

$$\mu(d\gamma) = e^{i\Phi(\gamma)}\, d\gamma$$

Γ "space of paths"

Φ: **action functional** (associated with classical Lagrangian)

In particular: Feynman's claim for 1 quantum particle with Schrödinger eqt

$$i \frac{\partial}{\partial t}\psi = -\frac{1}{2}\Delta\psi + V\psi$$

$$\psi|_{t=0} = \psi_0 \quad , t \in \mathbb{R}, \quad \text{on } \mathbb{R}^d$$

is

$$\psi(t,x) = I_\mu(f) \quad \text{for suitable } \mu, f$$

$(x \in \mathbb{R}^d)$

Namely:

$$\Phi(\gamma) = \frac{1}{2}\int_0^t |\dot\gamma(s)|^2 ds - \int_0^t V(x+\gamma(s))\,ds$$

$$f(\gamma) = \Psi_0(x+\gamma(t))$$

$\gamma \in \Gamma = \{$"paths"$: [0,t] \to \mathbb{R}^d\}$.

Solution formula called **Feynman - Itô formula**

Rem

- By Hamilton principle critical points of ϕ yield classical orbits for Newton particle in $\mathbb{R}^d$ with force $-\nabla V$ acting on it

  Classical orbits should emerge from integral $\int_\Gamma f(\gamma)\, e^{\frac{i}{\hbar}\phi(\gamma)} d\gamma$

  ($\hbar$ small, enters in Schrödinger eqt.

  $i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2}\Delta\psi + V\psi$;

  before we had set "$\hbar = 1$")

  Heuristic application of stationary phase method should yield asymptotics in (fractional) powers of $\hbar$, in terms classical orbits ~~of~~ ("semiclassical expansion").

Mathematical implementations exist; many approaches.

$I_\mu(f)$ as **continuous linear functional**, not as integral with respect to complex σ-additive measure (i.e. μ not realized as complex measure!).

Here: approach close to the one presented for Wiener integral, which permits full program including stationary phase method.

Goes back to K. Itô, A + Høegh-Krohn, Elworthy, Truman (see references in A + S. Mazzucchi, e.g. Rev. Math. Phys. 2016; Scholarpedia...)

Main idea: projective limit from finite dim. oscillatory integrals

One result is as follows (abstract formulation, after specified for Schrödinger eqt.):

$\Gamma = \mathcal{H}$: separable real Hilbert space

$\mathcal{F}(\mathcal{H}) := \{$ Banach algebra of functions $f : \mathcal{H} \to \mathbb{C}$ which can be written as Fourier transforms of bounded variation complex measures ν_f on $\mathcal{H}\}$

P : finite dim. orth. proj. in $\mathcal{H}$

Consider the oscillatory integral

$$I_{\mu_0|P\mathcal{H}}(g_{P\mathcal{H}}) := \int_{P\mathcal{H}} g_{P\mathcal{H}}(x)\, \underbrace{\mu_0(dx)}_{\frac{e^{\frac{i}{2}|x|^2_{P\mathcal{H}}}}{(2\pi i)^{\frac{\dim P\mathcal{H}}{2}}}dx}$$

for any $g_{P\mathcal{H}} \in \mathcal{F}(P\mathcal{H})$.

Then: for any $g \in \mathcal{F}(\mathcal{H})$

$\lim_{P} I_{\mu_0|P\mathcal{H}}(g_{P\mathcal{H}})\ \exists$

(where $g_{P\mathcal{H}}$ = restriction of g to $P\mathcal{H}$)

Call it $I_{\mu_0}(g)$.

One has:

$$I_{\mu_0}(g) = \underbrace{\int_{\mathcal{H}} e^{-\frac{i}{2}|x|^2_{\mathcal{H}}}\, \nu_g(dx)}$$

integr. resp. σ-add. meas. ν_g on $\mathcal{H}$: $\hat{\nu}_g = g$

("Parseval formula")

I_{μ_0} contin. linear functional.

Moreover for any $f, W \in \mathcal{F}(\mathcal{H})$:

$I_{\mu_0}(e^{-iW}f)$ also well def. Realization of $I_{\mu}(f)$, with $\mu(d\gamma) =$ "$e^{iW(\gamma)}d\gamma$",

$\Phi(\gamma) = \frac{1}{2}|\gamma|^2_{\mathcal{H}} - W(\gamma)$

$I_\mu(f)$ contin. linear functional

Application to Schrödinger equation

Theor: The solution of the Schrödinger eqt. with potential $V \in \mathcal{F}(\mathbb{R}^d)$ and initial condition $\psi_0 \in \mathcal{F}(\mathbb{R}^d)$ $(\cap L^2(\mathbb{R}^d))$ is given by

$$\psi(t,x) = I_{\mu_0}(e^{-iW} f),$$

with
$$\begin{cases} f(\gamma) = \psi_0(x + \gamma(t)) \\ W(\gamma) = \int_0^t V(x+\gamma(s))\,ds \end{cases}$$

Rem: • $I_{\mu_0}(e^{-iW} f) = I_\mu(f)$,

with $\mu(d\gamma) = \text{"}e^{i\Phi(\gamma)}d\gamma\text{"}$,

Φ action functional

• $\exists$ several extensions: e.g. V Laplace transf; $V(x) = |x|^4 \ldots$

Recently (A + S. Mazzucchi): inclusion of electrom. pot. A (so $-\frac{1}{2}\Delta \rightsquigarrow (-\frac{1}{2}\frac{1}{i}\nabla - A)^2$)

Then $\int A(\gamma(s))\,d\gamma(s)$ appears in addition to Feynman Kac term

NATURAL INTERPRETATION OF STRATONOVICH

(Feynman - Kac - Ito for Schröd. eqt.)

• Hyperbolic systems ...

• Many other approaches. -

Asymptotics

Stationary phase method worked out, both abstractly and for case of Schrödinger eqt.

Yields in particular detailed asympt. exp. in fract. powers of $\hbar$ for Cauchy probl. for Schrödinger eqt. under C^∞ cond. get even Borel summability (for $V \in \mathcal{F}(\mathbb{R}^d)$) (J. Rezende)

In case of potential quadratic plus part in $\mathcal{F}(\mathbb{R}^d)$ one gets **trace formula (Gutzwiller Selberg trace formula)**, as asympt. series, for $\mathrm{Tr}\left(e^{-i\frac{t}{\hbar}H}\right)$ as sum of periodic orbits of corresp. classic. system (for all t except for discrete set of values)

Important for classical / quantum chaos discussions...

(Ref. A + Høegh-Krohn; Blanchard Brzezniak, Boutet de Monvel ....)

Rem: Results on Schrödinger versus heat eqt. connected via transformation

$t \rightsquigarrow -it$

For Schröd. eqt exploited: Cameron, ... Nelson.. Simon .. Doss .. Thaler ... A+ Mazzucchi

Also used for relativistic ↔ euclidean in QFTh: Ch. II

I.2.3 Integrals for higher order PDEs

Replacing $\mathcal{H}$ by suitable Banach space $\mathcal{B}$ and role of $|\cdot|_{\mathcal{H}}^2$ by other corr. norm ~~could~~ an **infinite dim. integral** approach ("Parseval type formula approach") has been applied by S. Mazzucchi (since 2015) to higher order PDEs of the form

$$\frac{\partial u}{\partial t} = (-i)^p \, \alpha \frac{\partial^p}{\partial x^p} u + Vu \quad , \text{ on } \mathbb{R},$$

$u|_{t=0} = u_0$

$\alpha \in \mathbb{C}, \quad p \in \mathbb{N}$

(i.e. $p=4$)

E.g. $\frac{\partial u}{\partial t} = -\Delta^2 u + Vu$ (Krylov '60, Hochberg '78...)

solved for $u_0 \in \mathcal{F}(\mathbb{R})$, $V \in \mathcal{F}(\mathbb{R})$ by

$I_{\tilde{\mu}_0}(e^{-W} f)$, f, W as before,

$\tilde{\mu}_0(d\gamma) =$ " $e^{\int_0^t |\dot\gamma(s)|^4 ds} \, d\gamma$ "

$B = \{$paths with $\int_0^t |\dot\gamma(s)|^4 ds < \infty\}$.

Rem: A unified presentation in A + Mazzucchi, Rev. Math. Phys. 2016

II Integrals associated with spaces of maps

Φ until now action functional on space of paths with values in $\mathbb{R}^d$

In many other problems other functionals enter in variational calculus associated to PDEs: e.g. classical fields like $\chi(t,\vec{x})$:

$$\frac{\partial^2}{\partial t^2}\chi = \Delta \chi - v'(\chi) \quad \text{(non linear KG eqt.)}$$

arises from variational principle, with action functional

$$\Phi(\gamma) = \frac{1}{2}\int |\dot{\gamma}(s,\vec{x})|^2 ds\, d\vec{x} - \frac{1}{2}\int |\nabla\gamma(s,\vec{x})|^2 ds\, d\vec{x}$$

$$- \int v(\gamma(s,\vec{x}))\, ds\, d\vec{x}, \quad s \in \mathbb{R}, \vec{x} \in \mathbb{R}^{\sigma}$$

$$\gamma \in \Gamma = \{\text{maps}: \mathbb{R}^{\sigma+1} \to \mathbb{R}\}$$

So natural to look (following Feynman) to $I_\mu(f) = \int_\Gamma f(\gamma)\,\mu(d\gamma)$.

with $\mu(d\gamma) =$ " $e^{i\Phi(\gamma)} d\gamma$ "

For $f(\gamma) = \prod_{i=1}^{n} \gamma(t_i, \vec{x}_i)$ one would get "correlation functions" of relat. QF model ("Wightman functions")

Similar as for QM $\rightsquigarrow$ heat eqt.
$t \rightsquigarrow -it$

one can look at

$$\mu(d\gamma) = \mu_E(d\gamma) = \text{"}e^{-\phi_E(\gamma)}d\gamma\text{"}$$

"Euclidean"

With

$$\phi_E(\gamma) := \frac{1}{2}\int |\dot{\gamma}|^2 ds\, d\vec{x} + \frac{1}{2}\int |\nabla_{\vec{x}}\gamma|^2 ds\, d\vec{x}$$
$$+ \int v(\gamma(s,\vec{x}))\, ds\, d\vec{x}$$

For $v = 0$ both $I_\mu(f)$ and $I_{\mu_E}(f)$ well defined:

relativ. / Euclidean free fields (Nelson)

For $v \not\equiv 0$ construction of I_μ, I_{μ_E} have big problems, unsolved for $\sigma = 4$, (for $\sigma = 0,1,2$ special cases...)

Q: Could expansion around classical fields help ?

Rem: To μ_E there are associated ∞-dim. diffusion processes (random fields): studied in various cases.

Presently eqts. for which μ_E potential invar. measure receive special attention:

$$dX_\tau = [\Delta_{\mathbb{R}^{\sigma+1}} X_\tau - v'(X_\tau)]d\tau + dB_\tau$$

space-time white noise

(Stoch. quantiz. eqt.)

Parisi ... Doering, A+ Röckner, .. Da Prato ... Gubinelli, Hairer ...

Geometric invar. ⇝ singular behaviour!

Less singular: hydrodynamics, neurobiology ... finance ..

Rem. In lower dim. geometry / topol. other types of (Euclidean) QF models studied: Yang - Mills

Chern - Simons ...

Also here problems with construction

For Chern-Simons in 3-dim:

$$\Phi(\gamma) := \frac{k}{4\pi} \int_{M^3} \gamma \wedge \gamma + \frac{2}{3} \gamma \wedge \gamma \wedge \gamma$$

$k \in \mathbb{Z}$, $\gamma \in$ {Connections 1-forms}, structure group e.g. $SU(n)$

$\rightsquigarrow$ top. invariants, from

$$I_\mu(f) \quad , \quad f(\gamma) := \prod_{j=1}^{n} Hol_{C_i}(\gamma)$$

Well def. for $M = \mathbb{R}^3$, $\mathbb{R} \times S^2$, $S^1 \times S^1 \times S^1$

A + Sengupta A. Hahn

Rem: Groups of mappings from M to compact Lie group also studied in the sense of unit. repres. given by probability measures ....

III Deterministic, stochastic, quantum ...

Geometric Mechanics

Geometry | classical
Lagrangian / Hamiltonian
Variational problems
Symmetries ...
D. Holm T. Ratiu ...

CIB Semester 2015
on Stochastic geometric mechanics

- D. Holm: Variational principles for stochastic fluid dynamics: Natural noise term

Further Contribution on "noise and dissipation in rigid body motion" (with Arnaudon, De Castro) key idea: " coadjoint motion on level sets of momentum maps ... "

Noise in PDEs: Holm's recent paper on stochastic Born-Infeld equations "

Also SPDEs community looking for "natural noises": Flandoli...

- Other stochastization approaches have roots in Bismut's "Mécanique aléatoire": stoch. Hamiltonian approach; on Poisson manifolds - Cami, Ortega
- E. Nelson ~~[illegible] work ~~ ~~[illegible]~~
  J. C. Zambrini

  stochastic processes inspired by quantum mechanics

  Schrödinger processes...

  New work by Cresson Darses and in another direction, Ch. Léonard connect both approaches...

Application to models for formation of planetary systems

Regular spacing of planets. Kepler 1595

Dynamical considerations:
protosolar nebula hypothesis:

Descartes 1644
v. Wolf 1726
Kant 1755
Lambert 1761
Laplace 1796

Titius 1766
Bode 1772

$$r_n = 4 + 3 \cdot 2^n$$

$n = -\infty$	Mercury
$= 0$	Venus
$= 1$	Earth ("reference point")
$= 2$	Mars
$(= 3$	Ceres)
$= 4$	Jupiter
$(= 5$	Uranus)
$(= 6$	Neptun)

Later refinements, e.g. Blagg-Richardson, see book by Nieto

Rem: Renewed interest because of discovery of exoplanets

Stochastic model:
A + Blanchard + Høegh-Krohn '83
(Exp. Math.)
...

If one has a diffusion with invariant measure having density with nodes and drift gradient type, then confinement phenomenon occurs (ergodic components ...)

Question: Perhaps even more general phenomenon ...

Taking "Newtonian model" with drift given by q.m. Hydrogen atom wave function (essentially Laguerre polyn.) get "explicit confinement zones" between zeros"

For $n = 7 - 10$ good agreement with Titius-Bode law

Motivation used: diffusion inspired by Nelson's stoch. mechanics

Models further studied by
A. Truman and coworkers,
interesting dynamical results:
planetesimals converging for
large times to circular /
elliptic orbits
Uses semiclassical limits
. Distinction inner / outer
. planets
In 2016 further study
connecting with Burgers
equation with rotational
term:
Shandarin - Zacharov model
of distribution of galaxies -
(also studied in A., Molchanov,
Surgailis for ~~[illegible]~~
intermittency phenomenon...)

Quantum effects and structures
become possible in the
large ...
much to further explore
between quantum
stochastic
classical ...