Spectral Synthesis for Operators and Systems: Old Problems and Recent Advances

Anton Baranov (St. Petersburg State University)

Spectral synthesis is the possibility of the reconstruction of any invariant subspace of a linear operator from generalized eigenvectors that it contains. Another version of the spectral synthesis problem is the reconstruction of a vector in a Banach space from its Fourier series with respect to some complete and minimal system. These problems (which go back to J. Wermer and H. Hamburger) were studied in the 1970s by N. Nikolski and A. Markus who constructed examples of compact operators with complete sets of eigenvectors for which the synthesis failed.

It was a long-standing problem in the nonharmonic Fourier analysis whether any complete and minimal system of exponentials in $L^2(-\pi, \pi)$ has the spectral synthesis property. Namely, given a complete and minimal system of exponentials $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ with the biorthogonal system $\{g_\lambda\}$, is it true that any function $f \in L^2(-\pi, \pi)$ belongs to the closed linear span of its ‘harmonics’ $(f, g_\lambda)e^{i\lambda t}$? In 2013 this problem was solved in the negative by Baranov, Belov and Borichev. At the same time it was shown that the spectral synthesis for exponential systems always holds up to one-dimensional defect. Later similar questions were studied for systems of reproducing kernels in de Branges Hilbert spaces of entire functions and in Fock spaces (the exponential systems correspond to the classical Paley–Wiener space). In the de Branges space setting the problem can be related to the spectral theory of rank one perturbations of compact selfadjoint operators.

In the short course we plan to discuss:

- Spectral synthesis for exponential systems.
- Spectral synthesis in de Branges spaces with applications to the spectral theory of rank one perturbations.

The course will be based on recent joint works with Yurii Belov, Alexander Borichev, and Dmitry Yakubovich.
Ellipses, Blaschke Products, and the Numerical Range

Pamela Gorkin (Bucknell University)

The numerical operator $T$ on a Hilbert space is defined by

$$W(T) = \{(Tx, x) : \|x\| = 1\}.$$  

The set $W(T)$ contains the eigenvalues of an operator and often contains much more information about the operator than the eigenvalues do. The set $W(T)$ is interesting not only because of the information it contains about the operator, but also because of its geometry and function theoretic connections. This course includes an introduction to the numerical range in the setting of operators that are compressions of the shift operator to certain finite dimensional spaces. Even for low dimensions, complete descriptions of the numerical range of an operator are difficult to obtain. We will discuss some attempts at these descriptions as well as several open questions in this area.

Commutators, Calderon-Zygmund Operators and BMO

Brett Wick (Washington University)

An important theorem in harmonic analysis connects the commutator of multiplication by a function and Calderon-Zygmund operators and the functions of bounded mean oscillation. And as a dual statement, it connects the Hardy space with a certain “factorization” of Lebesgue spaces. During these lectures we will give proofs of these theorems using tools from dyadic harmonic analysis. Connections with analytic function theory will be provided and extensions to the case of weighted estimates and to the case of multi-parameter harmonic analysis will also be discussed.
Plenary Lectures

Conjugations and asymmetric truncated Toeplitz operators

Cristina Cámara (Instituto Superior Técnico de Lisboa)

Truncated Toeplitz operators on a model space are complex symmetric with respect to a natural conjugation on that model space. In this talk we discuss the relations between this and other conjugations with asymmetric truncated Toeplitz operators and we show that some of those relations can be expressed in terms of Hankel operators.

This talk is based on joint work with Kamila Klis-Garlicka and Marek Ptak.

Multipliers for representations of nilpotent Lie groups

José E. Galé (Universidad de Zaragoza)

$L^p$ boundedness properties of multipliers associated with the Weyl transform on the Heisenberg group were studied by G. Mauceri in 1980. The Weyl transform was generalized to nilpotent Lie groups by N. V. Pedersen in 1994. In this talk, multipliers related to representations via the Pedersen transform will be introduced, and it will be shown how to extend Mauceri’s results to this setting. The method followed relies on transference using abstract harmonic analysis.

This is joint work with D. Beltita and I. Beltita.

Schroedinger operators with complex potentials: transition from spectra to pseudospectra

David Krejcirik (Czech Technical University in Prague)

We shall explain how recent developments in physics have lead to new challenging mathematical problems in spectral analysis of non-self-adjoint operators of Schroedinger type. An emphasis will be put on unexpected wild properties of the operators that we suggest correctly describing through the concept of pseudospectra.
Multipliers between Toeplitz kernels
Jonathan R. Partington (University of Leeds)

Multipliers between kernels of Toeplitz operators are characterised in terms of test functions (so-called maximal vectors for the kernels); these maximal vectors may easily be parametrised in terms of inner and outer factorizations. Immediate applications to model spaces are derived. The case of surjective multipliers is also analysed.

This is joint work with Cristina Câmara (Lisbon)

Rigidity of non-compact composition and Volterra operators on $H^p$
Hans-Olav Tylli (University of Helsinki)

I will describe recent work on the structural rigidity of composition operators $C_\phi$ and Volterra operators $T_g$ on the Hardy spaces $H^p$ for $p \neq 2$. Here
\[ f \mapsto C_\phi(f) = f \circ \phi, \quad f \mapsto T_g(f)(z) = \int_0^z f(w)g'(w)dw, \]
where the analytic map $\phi : \mathbb{D} \to \mathbb{D}$, respectively $g \in \text{BMOA}$ are the fixed symbols. The main results imply that non-compact $C_\phi$ and $T_g$ both have quite a restricted range of linear qualitative behaviour compared to that of arbitrary bounded operators on $H^p$ (for $p \neq 2$), which is determined by restrictions to subspaces $M \subset H^p$ that are isomorphic to $\ell^p$ or $\ell^2$.

The talk is based on joint works with Jussi Laitila, Santeri Miihkinen, Pekka Nieminen and Eero Saksman.

Completeness of rank one perturbations of compact normal operators with lacunary spectrum
Dmitry Yakubovich (Universidad Autónoma de Madrid)

Let $A$ be a compact normal operator on a Hilbert space $H$ with a certain lacunarity condition on the spectrum (which means, in particular, that its eigenvalues $\{\lambda_n\}$ go to zero exponentially fast), and let $L$ be its rank one perturbation, defined by $Lx = Ax + \langle x, b \rangle a$, $x \in H$, where $a, b$ are vectors in $H$. Generically, $L$ is non-normal. We can expand
\[ A = \sum_{n \in \mathbb{N}} \lambda_n P_n, \]
where \( P_n \) are orthogonal spectral projections corresponding to \( \lambda_n \). We show that either the moment equalities

\[
\sum_n \lambda_n^{-k} \langle P_n a, b \rangle = -\delta_{1k}
\]

hold for all \( k = 1, 2, \ldots \) (so that in particular, all these moments exist) or the linear span of root vectors of \( L \), corresponding to non-zero eigenvalues, is of finite codimension in \( H \). Here \( \delta_{jk} \) stands for the discrete delta function. In contrast to classical results, we do not assume that the perturbation is weak. We also prove that, in a certain sense, the above result is sharp. Namely, for non-lacunary spectra, it can happen that some moments fail to exist and nevertheless the root vectors of \( L \) span a subspace of infinite codimension in \( H \).

This is a joint work with Anton Baranov and complements the lectures he will give on this conference.
Universality of composition operators

Céleriés, Benjamin (Université Claude Bernard Lyon 1)

In 1987, Nordgren, Rosenthal and Wintrobe proved that, if \( \phi(z) = \frac{z+1/2}{1+1/2z} \), then \( C_\phi - Id \) is universal in the sense of Rota on the classical Hardy space \( H^2(\mathbb{D}) \). In 2016, Cowen and Gallardo gave another proof using the embedding of \( C_\phi \) into a semigroup \( (C_{\phi_t})_{t \geq 0} \) and the similarity of \( (C_{\phi_t})_{t \geq 0} \) to a semigroup of Toeplitz operators.

We use this approach to study the similarity of \( (C_\phi - Id) \) on weighted Hardy spaces.

Uniqueness of entire functions sharing a small function with linear differential polynomials

Das, Shubhashish (University of Kalyani)

We consider the situation when an entire function shares a small function with linear differential polynomials. Our result improves a result of H. Zhong.

The above title of paper with author Indrajit Lahiri and Shubhashish Das has recently accepted in the journal “Annali della Scuola Normale Superiore di Pisa, Classe di Scienze”.

Growth rates of frequently hypercyclic harmonic functions

Gilmore, Clifford (University of Helsinki)

The notion of frequent hypercyclicity stems from ergodic theory and was introduced by Bayart and Grivaux (2004). Many natural bounded linear operators are frequently hypercyclic, for instance the differentiation operator on the space of entire functions.

We consider the partial differential operator acting on the space of harmonic functions on \( \mathbb{R}^n \) and we identify sharp growth rates, in terms of the \( L^2 \)-norm on spheres, of its frequently hypercyclic vectors. This answers a question posed by Blasco et al. (2010).

This is joint work with Eero Saksman and Hans-Olav Tylli.
Algebras of convolution type operators with PSO data on weighted Lebesgue spaces

Karlovich, Yuri (Universidad Autónoma del Estado de Morelos)

The Banach algebra $A_{p,w}$ generated by all multiplication operators by piecewise slowly oscillating functions and by all convolution operators with piecewise slowly oscillating symbols being Fourier multipliers on the weighted Lebesgue space $L^p(\mathbb{R}, w)$, where $p \in (1, \infty)$ and $w$ is a Muckenhoupt weight, is studied. A Fredholm symbol calculus for the algebra $A_{p,w}$ is constructed and a Fredholm criterion for the operators $A \in A_{p,w}$ is established under some restriction on weights $w$.

Application of the Cauchy transforms on Carleson curves to some classical problems in abstract evolution equations

Król, Sebastian (Nicolaus Copernicus University)

I will present an applications of the theory of Cauchy singular integral operators on curves and the Carleson embedding theorem to some classical questions in abstract evolution equations.

In particular, we will discuss the similarity problem for cosine function generators on Hilbert spaces: a variant of McIntosh's characterisation of the boundedness of the holomorphic functional calculus, estimations of a square function type and dilations to a multiplication operator.

Moreover, I will show new characterisation of cosine function generators (by means of complex inversion formulas) and apply them to answer in the affirmative a question by Fattorini on the growth bounds of perturbed cosine functions.

Holomorphic functions with universal approximation properties

Manolaki, Myrto, (University of South Florida)

In 1996 Nestoridis showed that the Taylor series of most holomorphic functions on the unit disc $\mathbb{D}$ are “universally overconvergent” outside $\mathbb{D}$; that is, by considering suitable subsequences of their partial sums, it is possible to approximate every polynomial on every compact set in $\mathbb{C} \setminus \mathbb{D}$ that has connected complement. The theory of universal Taylor series has been
significantly enriched during the past two decades, yet there are fundamental questions which still remain open. In this talk we will discuss some of these questions as well as some recent results about functions with universal approximation properties on subsets of the unit circle.

A harmonic maps approach to fluid flows

Martín, María (University of Eastern Finland)

Fluid dynamics is a highly complex and interesting branch of mathematical research for a variety of reasons. The complexity of the governing equations even in the context of a perfect (inviscid and incompressible) fluid is well-known, and is evidenced strongly by the extreme dearth of explicit solutions which number among a handful, such as Gerstner’s wave, Kirchhoff’s elliptical vortex, and the Ptolemaic vortices.

It has been observed that all those explicit solutions mentioned have labeling maps in the Lagrangian setting which are harmonic at all times, an insight which has been exploited and employed recently in a number of mathematical contexts in the study of fluid flows.

In this talk, we will explain how to address this interesting aspect of explicit fluid flows. In particular, in the main result of our research, a novel complex analysis approach is used to achieve a full characterization of fluid flows which have harmonic labeling maps, an issue that has been addressed by some significant recent publications, but in an incomplete fashion.

This is a joint work with Olivia Constantin.

A generalized Hilbert operator acting on conformally invariant spaces

Merchán, Noel (Universidad de Málaga)

If $\mu$ is a positive Borel measure on the interval $[0, 1)$ we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0, 1, 2, \ldots$, $\mu_n$ denotes the moment of order $n$ of $\mu$. This matrix induces formally the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n$$
on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $\mathbb{D}$. This paper is devoted to study the operators $H_\mu$ acting on certain conformally invariant spaces of analytic functions on the disc.

**Generalized Cesàro operators on Sobolev-Lebesgue sequence spaces**

Miana, Pedro J. (Universidad de Zaragoza)

In this talk, we present a complete spectral research of generalized Cesàro operators on Sobolev-Lebesgue sequence spaces. The main idea is to subordinate such operators to suitable $C_0$-semigroups on these sequence spaces. We introduce that family of sequence spaces using the fractional finite differences and we prove some structural properties similar to classical Lebesgue sequence spaces. In order to show the main results about fractional finite differences, we state equalities involving sums of quotients of Euler’s Gamma functions. Finally, we display some graphical representations of the spectra of generalized Cesàro operators.

**A Paley–Wiener Theorem for a class of holomorphic function spaces on the Siegel upper half–space**

Monguzzi, Alessandro (Università degli Studi di Milano)

The Siegel upper half–space $U_{n+1}$ is the biholomorphic copy of the unit ball $B_{n+1}$ in $\mathbb{C}^{n+1}$ via the (multi–dimensional) Cayley transform. In this talk I will prove a Paley–Wiener theorem for a scale of holomorphic function spaces on $U_{n+1}$ that includes in particular the Drury–Arveson space on introduced by N.Arcozzi, R. Rochberg and E. Sawyer in “Two variations on the Drury-Arveson spaces”, Hilbert spaces of analytic functions, CRM Proc. Lecture Notes, vol. 51, 2010.

This is a joint work with N. Arcozzi, M. Salvatori and M. Peloso.
Polynomially stable $C_0$-semigroup and a system of two coupled strings

Rzepnicki, Lukasz (Nicolaus Copernicus University)

We study an energy decay problem in a system of two connected vibrating strings. It is known that the rate of the decay depends on boundary, coupling conditions and the ratio of wave speeds. Moreover, there are some cases for which the energy converges to zero but not exponentially. The natural question is if the decay can be polynomial. We use the $C_0$-semigroup approach and Borichev-Tomilov theorem to show that if the ratio of wave speeds is irrational and algebraic, then the answer for the above question is positive.

Presented results are joint work with Roland Schnaubelt.

Reflection principle for linear fractional composition operators

Schroderus, Riikka (University of Helsinki)

We consider linear fractional composition operators acting on the weighted Dirichlet spaces and present a reflection principle with respect to the classical Dirichlet space. This principle allows us to compute the spectra of certain linear fractional composition operators acting on the weighted Dirichlet spaces that are contained in the classical one. Moreover, we find a new universal operator by using the reflection principle and the celebrated universality result by Nordgren, Rosenthal and Wintrobe.

This is joint work with, respectively, Eva Gallardo-Gutiérrez and Hans-Olav Tylli.

Local form-subordination condition and Riesz basisness of root systems

Siegl, Petr (University of Bern)

We exploit the so called form-local subordination in the analysis of non-symmetric perturbations of unbounded self-adjoint operators with isolated simple positive eigenvalues. If the appropriate condition relating the size of gaps between the unperturbed eigenvalues and the strength of perturbation, measured by the form-local subordination, is satisfied, the root system of the perturbed operator contains a Riesz basis and usual asymptotic formulas
for perturbed eigenvalues and eigenvectors hold. The power of the abstract perturbation results is demonstrated particularly on Schrödinger operators with possibly unbounded or singular complex potential perturbations.

The talk is based on:

**Mixed BMO, Hankel operators and commutators**

Strouse, Elizabeth (Université Bordeaux)

It is a classical and well known fact that functions of bounded mean oscillation can be characterized as the symbols of bounded Hankel operators or as elements of the dual of the real Hardy space. The proof of this fact is a combines harmonic analysis and operator theory in a surprising but natural way. The extension of these results (by Cotlar, Sadosky, Ferguson, Lacey and others) gives nice relationships between different multivariable definitions of BMO and big and little Hankel operators. I will discuss the results of my recent article with Stefanie Petermichl an Yuneng Ou (Advances in Math, 2016) about a new kind of multivariable BMO and its characterization using all sorts of Hankel operators, commutators, and a wide range of Calderon-Zygmund operators.

**Interpolation and extrapolation of strictly singular operators between $L^p$ spaces**

Tradacete, Pedro (Universidad Carlos III de Madrid)

We study the interpolation and extrapolation properties of strictly singular operators between different $L^p$ spaces. To this end, the structure of strictly singular non-compact operators between $L^p - L^q$ spaces is analyzed. Among other things, we clarify the relation between strict singularity and the L-characteristic set of an operator. In particular, Krasnoselskii’s interpolation theorem for compact operators is extended to the class of strictly singular operators.

Joint work with F. L. Hernández and E. M. Semenov.
Essential Spectral Singularities and the Spectral Expansion for the Hill Operator

Veliev, Oktay (Dogus University)

I am going to give a talk about the construction the spectral expansion for the operator $L(q)$ with arbitrary complex-valued locally integrable and periodic potential $q$. In other word, we consider in detail the spectral expansion for the general case when the operator $L$ is not a spectral operator. Note that the construction of the complete spectral decomposition appears to have been open for about 50 years. To give a complete spectral decomposition we introduce new concepts as essential spectral singularities and singular quasimomenta.