$SO_0(p, p+1)$-Higgs bundles

Brian Collier

University of Maryland

Plan

- Motivation, background and $p = 1$
- $p = 2$, low dimensional isomorphisms and Zariski closures
- $p \geq 3$

- $X = (S, J)$ will be a Riemann surface of genus $g \geq 2$
- $\mathrm{H} \subset \mathrm{G}$ maximal compact of a connected semisimple Lie group
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition of $\mathfrak{g}$

Definition

A G-Higgs bundle is a pair $(\mathcal{E}_{\mathrm{H}_\mathbb{C}}, \varphi)$ where $\mathcal{E}_{\mathrm{H}_\mathbb{C}}$ is a holomorphic $\mathrm{H}_\mathbb{C}$ bundle and $\varphi \in H^0(X, \mathcal{E}_{\mathrm{H}_\mathbb{C}}[\mathfrak{m}_\mathbb{C}] \otimes K)$.

$$\{(\mathcal{E}_{\mathrm{H}_\mathbb{C}}, \varphi) \textit{ polystable}\}/\mathcal{G}_{\mathrm{H}_\mathbb{C}} = \mathcal{M}(\mathrm{G})$$

For $\mathrm{G} = \mathrm{SL}(n, \mathbb{C})$ .... $(E, \Phi : E \to E \otimes K)$ with $det(E) = \mathcal{O}$ and $Tr(\Phi) = 0$
Unstable if $\exists F \subset E$ with $\Phi(F) \subset F \otimes K$ and $deg(F) > 0$.

Topological invariants

Topological H bundles over X are classified by $\pi_1(\mathrm{H}) = \pi_1(\mathrm{G}) = \pi_1(\mathrm{H}_{\mathbb{C}})$.

$$\mathcal{M}(\mathrm{G}) = \bigsqcup_{\omega \in \pi_1(\mathrm{H})} \mathcal{M}^{\omega}(\mathrm{G})$$

If G is compact or complex $\mathcal{M}^{\omega}(\mathrm{G})$ is nonempty and connected (Ramanathan, Li). For a real groups this is more complicated

Theorem (Hitchin)

If G is a split real group (such as $\mathrm{PSL}(n, \mathbb{R}), \mathrm{Sp}(2n, \mathbb{R}), \mathrm{SO}_0(p, p+1)$) there is a connected component of $\mathcal{M}(\mathrm{G})$ which is diffeomorphic to a vector space of holomorphic differentials and not distinguished by $\pi_1(\mathrm{H})$ if $\mathit{rank}(\mathrm{G}) \geq 2$.

$$\mathrm{Hit}(\mathrm{SO}_0(p, p+1)) = \bigoplus_{j=1}^{p} H^0(K^{2j})$$

$\mathrm{SO}_0(p, p+1)$-Higgs bundles

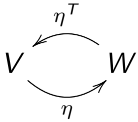
Definition

An $\mathrm{SO}_0(p, p+1)$-Higgs bundle is a triple (V, W, η) where V and W are rank p and $p+1$ orthogonal vector bundles with trivial determinant and $\eta \in H^0(V^* \otimes W \otimes K)$... $\eta : V \to W \otimes K$.

Orthogonal structures $Q_V : V \xrightarrow{\cong} V^*$ and $Q_W : W \xrightarrow{\cong} W^*$
The corresponding $\mathrm{SL}(2p+1, \mathbb{C})$-Higgs bundle is

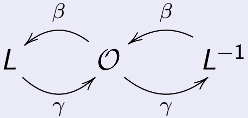
$$(E, \Phi) = \left(V \oplus W, \begin{pmatrix} 0 & \eta^T \\ \eta & 0 \end{pmatrix} \right)$$

where $\eta^T : W \to V \otimes K$ is given by $Q_V^{-1} \circ \eta^* \circ Q_W$.



$$p = 1$$

$$V = \mathcal{O}, \quad (W, Q_W) = \left(L \oplus L^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$
$$\eta = (\beta, \gamma) \in H^0(LK) \oplus H^0(L^{-1}K).$$



If $deg(L) > 0$ then (by stability) then $\gamma \neq 0 \in H^0(L^{-1}K)$, thus $d = deg(L) \leq 2g-2$, (similarly $d = deg(L) \geq -2g+2$)

$$\mathcal{M}(\mathrm{SO}_0(1,2)) = \bigsqcup_{-2g+2 \leq d \leq 2g-2} \mathcal{M}_d(\mathrm{SO}_0(1,2))$$

Theorem (Hitchin)

For $d \neq 0$, the space $\mathcal{M}_d(\mathrm{SO}_0(1,2))$ is smooth and diffeomorphic to a rank $|d| + g - 1$ vector bundle $\mathcal{F}_d$ over $Sym^{-|d|+2g-2}(X)$.

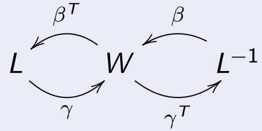
Corollaries

- $|\pi_0(\mathcal{M}(\mathrm{SO}_0(1,2))| = 1 + 2(2g-2)$
- If $d \neq 0$, $H^*(\mathcal{M}_d(\mathrm{SO}_0(1,2))) \cong H^*(Sym^{-|d|+2g-2}(X))$
- (maximal case) $L = K$ and $\mathcal{M}_{2g-2}(\mathrm{SO}_0(1,2)) \cong H^0(K^2)$

$p = 2$

$$(V, W, \eta): \quad V, Q_V = L \oplus L^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad W, Q_W$$

$$\eta = (\beta, \gamma) \in H^0(LK \otimes W) \oplus H^0(L^{-1}K \otimes W)$$



Topological invariants: $deg(L)$ and $sw_2 \in H^2(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.
$deg(L) > 0 \Rightarrow \gamma \neq 0$.

$$LK^{-1} \xrightarrow[\gamma]{} W \xrightarrow[Q_W]{} W^* \xrightarrow[\gamma^*]{} L^{-1}K$$

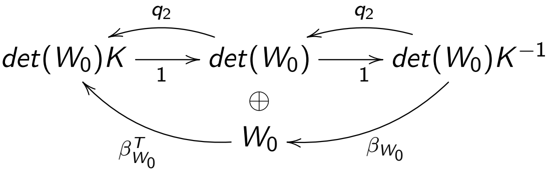
is an element of $H^0((L^{-1}K)^2) \setminus \{0\}$, thus $deg(L) \leq 2g - 2$

$$\mathcal{M}(\mathrm{SO}_0(2,3)) = \coprod_{2g-2 \leq b \leq 2g-2} \mathcal{M}^{b, sw_2}(\mathrm{SO}_0(2,3))$$

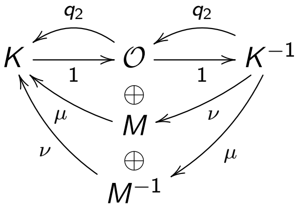
When $deg(L) = 2g - 2$, $(L^{-1}K)^2 = \mathcal{O}$ (i.e. $L^{-1}K$ is an $\mathrm{O}(1, \mathbb{C})$-bundle).

$$W = im(\gamma) \oplus im(\gamma)^{\perp} \cong LK^{-1} \oplus W_0 \cong det(W_0) \oplus W_0$$

where W_0 is a holomorphic $\mathrm{O}(2, \mathbb{C})$-bundle. In this case, the Higgs bundle is given by:



New topological invariant: $sw_1(W_0, Q_0) \in H^1(X, \mathbb{Z}/2\mathbb{Z})$, if $sw_1 = 0$



$deg(M) > 0$ then $\mu \in H^0(M^{-1}K^2) \setminus \{0\} \Rightarrow deg(M) \leq 4g - 4$

$$\mathcal{M}^{2g-2}(\mathrm{SO}_0(2,3)) = \coprod_{sw_1\neq 0} \mathcal{M}^{2g-2,sw_2}_{sw_1} \sqcup \coprod_{0\leq d\leq 4g-4} \mathcal{M}^{2g-2}_d$$

$2(2^{2g}-1)+1+4g-4$ components (Bradlow–Garcia-Prada–Gothen)
For $|b| < 2g-2$, $\mathcal{M}^{b,sw_2}(\mathrm{SO}_0(2,3))$ is connected (Gothen-Oliviera)

$$|\pi_0(\mathcal{M}(\mathrm{SO}_0(2,3))| = 1 + 2(2g-1+2(2^{2g}-1)+4g-3).$$

Theorem (C.)

For $d>0$, the space $\mathcal{M}^{2g-2}_d(\mathrm{SO}_0(2,3))$ is smooth and diffeomorphic to the product $\mathcal{F}_d \times H^0(K^2)$ where $\mathcal{F}_d$ is a rank $d+3g-3$ vector bundle over $Sym^{-d+4g-4}(X)$.

Corollaries

- For $d>0$, $H^*(\mathcal{M}^{2g-2}_d(\mathrm{SO}_0(2,3))) \cong H^*(Sym^{-d+4g-4}(X))$
- $\mathcal{M}^{2g-2}_{4g-4}(\mathrm{SO}_0(2,3)) \cong H^0(K^4) \oplus H^0(K^2)$ (Hitchin component)

$$\mathcal{M}^{2g-2}(\mathrm{SO}_0(2,3)) = \bigsqcup_{sw_1\neq 0} \mathcal{M}^{2g-2,sw_2}_{sw_1} \sqcup \bigsqcup_{0\leq d\leq 4g-4} \mathcal{M}^{2g-2}_d$$

Theorem (Alessandrini-C.)

The spaces $\mathcal{M}^{2g-2,sw_2}_{sw_1}(\mathrm{SO}_0(2,3))$ is an orbifold 'diffeomorphic' to

$$\mathcal{F}/(\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}) \times H^0(K_X^2)$$

where $\mathcal{F}\to\mathrm{Prym}^{sw_2}(X_{sw_1}, X)$ is the rank $6g-6$ vector bundle over the connected component of the Prym variety associated to sw_2 with $\pi^{-1}(M) = H^0(MK^2_{X_{sw_1}})$. Here $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ acts as

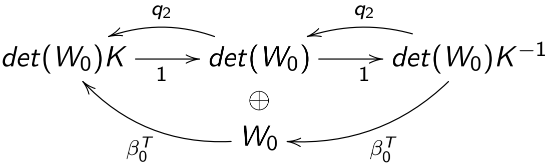
$$\tau\cdot(M,\mu) = (\iota^*M, \iota^*\mu) \qquad \text{and} \qquad \sigma\cdot(M,\mu) = (\iota^*M, -\iota^*\mu)$$

Corollary

$\mathcal{M}^{2g-2,sw_2}_{sw_1}(\mathrm{SO}_0(2,3))$ deformation retracts onto $\mathrm{Prym}^{sw_2}(X_{sw_1}, X)/\mathbb{Z}/2\mathbb{Z}$

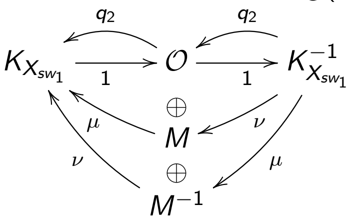
Proof

A Higgs bundle in $\mathcal{M}^{2g-2,sw_2}_{sw_1}(\mathrm{SO}_0(2,3))$ is given by



Parameterize points of $\mathcal{M}^{2g-2}_0(X_{sw_1}, \mathrm{SO}_0(2,3))$ which are ι^*-invariant.

$\pi^* W_0 = M \oplus M^{-1}$ with $deg(M) = 0$ and $\iota^*(M \oplus M^{-1}) = M^{-1} \oplus M$



$$\iota^*(M, \mu, \nu, q_2) = (M^{-1}, \nu, \mu, \iota^* q_2)$$

So (M, μ, ν, q_2) is ι^* invariant if $M \in \mathrm{Prym}(X_{sw_1}, X)$, $\iota^*\mu = \nu$, $\iota^* q_2 = q_2$
Set $\pi : \mathcal{F} \to \mathrm{Prym}(X_{sw_1}, X)$ with $\pi^{-1}(M) = H^0(MK^2_{X_{sw_1}}) \cong \mathbb{C}^{3g_{X_{sw_1}}-3}$.
This gives a surjection

$$\mathcal{F} \times H^0(K^2) \twoheadrightarrow \mathcal{M}^{2g-2, sw_2}_{sw_1}(\mathrm{SO}_0(2,3))$$

The ι^*-invariant gauge transformations of $(K \oplus K^{-1}, M \oplus M^{-1} \oplus \mathcal{O})$ are generated by $(g_1, g_2^{\pm}) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)$

$$(g_1, g_2^{\pm}) \cdot (M, \mu, \nu, q_2) = (M^{-1}, \pm\nu, \pm, \mu, q_2).$$

This gives a difffeomorphism

$$\mathcal{F}/(\mathbb{Z}/2\mathbb{Z})^2 \times H^0(K^2) \longleftrightarrow \mathcal{M}^{2g-2, sw_2}_{sw_1}(\mathrm{SO}_0(2,3)$$

Dimension check:

$$dim(\mathrm{Prym}(X_{sw_1}, X)) = g_{X_{sw_1}} - g = 2g - 1 - g = g - 1$$

$$\text{fiber dimension: } 3g_{X_{sw_1}} - 3 = 3(2g-1) - 3 = 6g - 6.$$

Mod(S)-invariant parameterizations of $\mathcal{X}^{2g-2}(\mathsf{SO}_0(2,3))$

$$\mathcal{M}(X,\mathsf{G}) \xleftrightarrow{\cong_X} \mathcal{X}(\pi_1(S),\mathsf{G}) = \mathsf{Hom}^{red}(\pi_1(S),\mathsf{G})/\mathsf{G}$$

$$h_\rho : \widetilde{X} \to \mathsf{G}/\mathsf{H}$$

The mapping class group acts on $\mathcal{X}(\pi_1(S),\mathsf{G})$ but not on $\mathcal{M}(\mathsf{G})$.

Theorem (Labourie, C, Alessandrini-C)

For each maximal $\mathsf{SO}_0(2,3)$-rep $\rho \in \mathcal{X}^{2g-2}(\pi_1(S),\mathsf{SO}_0(2,3))$ there is a unique Riemann surface structure X_ρ in which the harmonic metric $h_\rho : \widetilde{X}_\rho \to \mathsf{SO}_0(2,3)/\mathsf{SO}(2)\times\mathsf{SO}(3)$ is a minimal immersion.

Corollary (Labourie, C., Alessandrini-C.)

There is a mapping class group invariant projection

- $\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3)) \to \mathsf{Teich}(S)$
- $\mathcal{X}_{sw_1}^{2g-2,sw_2}(\mathsf{SO}_0(2,3)) \to \mathsf{Teich}(S)$

where $\pi^{-1}(X) = \mathcal{F}_d$ and $\pi^{-1}(X) = \mathcal{F}/(\mathbb{Z}/2\mathbb{Z})^2$

If $h_\rho : \widetilde{X} \to (\mathrm{G}/\mathrm{K}, g)$ is harmonic, the Hopf differential is given by

$$(h_\rho^* g)^{(2,0)} = Tr(dh_\rho^{1,0} \otimes dh_\rho^{1,0}) \in H^0(K_X^2)$$

Measures the failure of h_ρ to be conformal. If h_ρ is harmonic and conformal then h_ρ is a (branched) minimal immersion (Sacks-Uhlenbeck).
Consider the energy function on $\mathcal{X}(\mathrm{G}) \times \mathrm{Teich}(S)$

$$\mathcal{E} : \mathcal{X}(\mathrm{G}) \times \mathrm{Teich}(S) \to \mathbb{R} \qquad \mathcal{E}(\rho, X) = \int_X |dh_\rho^{1,0}|^2$$

- For each $X \in \mathrm{Teich}(S)$, $\mathcal{E}_X : \mathcal{X}(\mathrm{G}) \to \mathbb{R}$ gives the Morse function on $\mathcal{M}(\mathrm{G})$, it has variations of Hodge structures as critical points.
- For each $\rho \in \mathcal{X}(\mathrm{G})$, $\mathcal{E}_\rho : \mathrm{Teich}(S) \to \mathbb{R}$ has minimal immersions as critical points, (Euler-Lagrange Eq $Tr(dh_\rho^{(1,0)} \otimes dh_\rho^{(1,0)}) = 0$).

Theorem (Labourie)

If ρ is a maximal representation then $\mathcal{E}_\rho$ is smooth and proper.

Low dimensional isomorphisms and Zariski closures

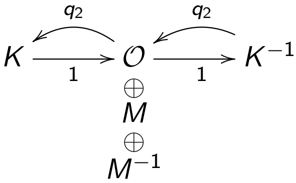
$\mathrm{SO}_0(1,2) \cong \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PSp}(2,\mathbb{R})$ and $\mathrm{SO}_0(2,3) \cong \mathrm{PSp}(4,\mathbb{R})$

$\mathcal{X}(\mathrm{PSL}(n,\mathbb{R}))$ has 3-components, only the Hitchin component does not contain compact representations. (Hitchin)

Maximal $\mathrm{Sp}(4,\mathbb{R})$ reps are special.

Theorem (Bradlow–Garcia-Prada–Gothen, C.)

Each of the components $\mathcal{X}_{sw_1}^{2g-2,sw_2}(\mathrm{SO}_0(2,3))$, $\mathcal{X}_0^{2g-2}(\mathrm{SO}_0(2,3))$ and $\mathcal{X}_{4g-4}^{2g-2}(\mathrm{SO}_0(2,3))$ contain reps Fucshian $\mathrm{SO}_0(1,2)$ times $\mathrm{SO}(2)$ or irreducibly embedded Fuchsian Zariski closure. For $0 < d < 4g-4$, all representations in $\mathcal{X}_d^{2g-2}(\mathrm{SO}_0(2,3))$ are Zariski dense.



Theorem (C.)

For each integer $d \in (0, p(2g-2)]$ there is a smooth connected component $\mathcal{M}_d(\mathrm{SO}_0(p,p+1))$ of $\mathcal{M}(\mathrm{SO}_0(p,p+1))$ which is diffeomorphic to the product

$$\mathcal{F}_d \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$$

where $\mathcal{F}_d \to \mathrm{Sym}^{-d+p(2g-2)}(X)$ is a rank $d+(2p-1)(g-1)$ vector bundle.

Remark: $dim(\mathcal{F}_d) = dim(H^0(K^{2p}))$

Corollaries

- $H^*(\mathcal{M}_d(\mathrm{SO}_0(p,p+1)) \cong \mathrm{Sym}^{-d+p(2g-2)}(X)$.
- $|\pi_0(\mathcal{M}(\mathrm{SO}_0(p,p+1))| \geq p(2g-2)+4$
- maximal case $\mathcal{M}_d(\mathrm{SO}_0(p,p+1)) = \mathrm{Hit}(\mathrm{SO}_0(p,p+1))$.

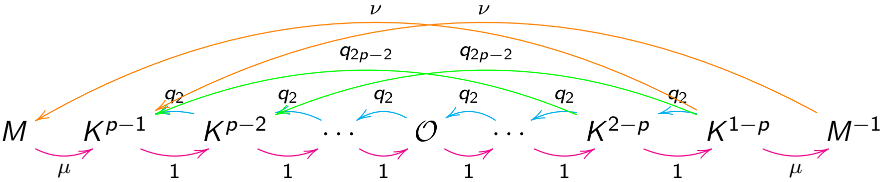
Proof

The Higgs bundles is this component are given by (V, W, η)

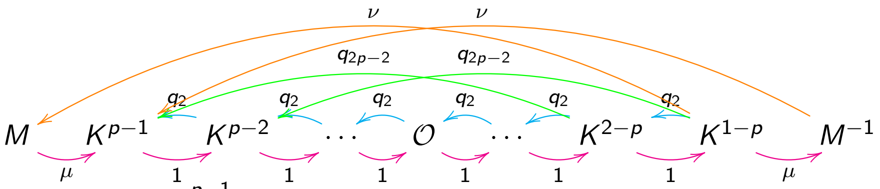
$$(K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{3-p} \oplus K^{1-p} \;\;,\;\; M \oplus K^{p-2} \oplus \cdots \oplus K^{2-p} \oplus M^{-1})$$

$$\eta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \nu \\ & 1 & q_2 & q_4 & \cdots & q_{2p-2} \\ & & 1 & q_2 & \cdots & q_{2p-4} \\ & & & \ddots & \ddots & \\ & & & & 1 & q_2 \\ & & & & & \mu \end{pmatrix} : V \longrightarrow W \otimes K$$

$M \in \mathrm{Pic}^d(X)$, $\mu \in H^0(M^{-1}K^p) \setminus \{0\}$, $\nu \in H^0(MK^p)$, $(\mathcal{E}, \Phi)$ is given by



$M \in \mathrm{Pic}^d(X)$, $\mu \in H^0(M^{-1}K^p) \setminus \{0\}$, $\nu \in H^0(MK^p)$, $q_{2j} \in H^0(K^{2j})$.



$$\{M, \mu, \nu\} \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \longrightarrow \mathcal{M}(\mathrm{SO}_0(p, p+1)) \qquad \text{surjective???}$$

Lemma

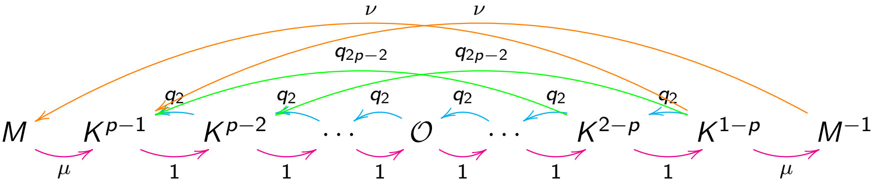
For $(M, \mu, \nu, q_2, \cdots, q_{2p-2})$ the only gauge transformations which act on this data is $(M, \mu, \nu, q_2, \cdots, q_{2p-2}) \longrightarrow (M, \lambda\mu, \lambda^{-1}\nu, q_2, \cdots, q_{2p-2})$.

$$\mathcal{F}_d \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \longrightarrow \mathcal{M}(\mathrm{SO}_0(p, p+1)) \qquad \text{open}$$

$tr(\Phi^{2j}) = q_{2j}$ for $j < p$ and $tr(\Phi^{2p}) = \mu \otimes \nu$. Since $\mathrm{Sym}^a(X)$ is compact, and the Hitchin fibration is proper, a divergent sequence in

There is also a component $\mathcal{M}_0(\mathrm{SO}_0(p, p+1))$

$M \in \mathrm{Pic}^0(X)$, $\mu \in H^0(M^{-1}K^p)$, $\nu \in H^0(MK^p)$, $\nu = 0 \Leftrightarrow \mu = 0$



$$\{(M, \mu, \nu) | \mu = 0 \Leftrightarrow \nu = 0\} \times \bigoplus_{j=1}^{p-1} (H^0(K^{2j}) \xrightarrow{\Psi} \mathcal{M}(\mathrm{SO}_0(p, p+1))$$

Lemma

For $(M, \mu, \nu, q_2, \cdots, q_{2p-2})$ the only gauge transformations which act on this data is $(M, \mu, \nu, q_2, \cdots, q_{2p-2}) \longrightarrow (M, \lambda\mu, \lambda^{-1}\nu, q_2, \cdots, q_{2p-2})$ and $(M, \mu, \nu, q_2, \cdots, q_{2p-2}) \longrightarrow (M^{-1}, \lambda\nu, \lambda^{-1}\mu, q_2, \cdots, q_{2p-2})$.

- Image of Ψ has dimension $dim(\{M, \mu, \nu\}/\mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z})$ which is the expected dimension
- Properness of the Hitchin fibration implies the closure of the image of Ψ is $\mu = 0$ or $\nu = 0$

Connected componets of $\mathcal{M}(\mathrm{SO}_0(p, p+1))$

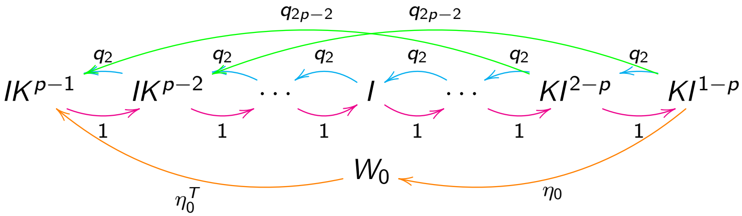
Theorem (Arroyo–Bradlow–C.–Gothen–Garcia-Prada–Oliveira)

For p-odd, $|\pi_0(\mathcal{M}(\mathrm{SO}_0(p, p+1)))| = 1 + p(2g-2) + 4.$
For p-even, $|\pi_0(\mathcal{M}(\mathrm{SO}_0(p, p+1)))| = 2(2^{2g}-1) + 1 + p(2g-2) + 4.$

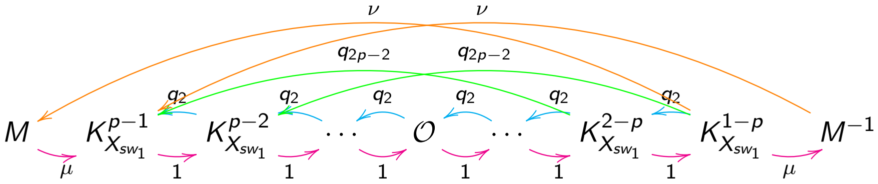
Parameterizations

If is p-even then for each $(sw_1, sw_2) \in H^1(X, \mathbb{Z}/2\mathbb{Z}) \setminus \{0\} \times H^2(X, \mathbb{Z}/2\mathbb{Z})$ there is a component $\mathcal{M}(\mathrm{SO}_0(p, p+1))$ which is an orbifold diffeomorphic to $\mathcal{F}/(\mathbb{Z}/2\mathbb{Z})^2 \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$ where $\pi : \mathcal{F} \to \mathrm{Prym}(X_{sw_1}, X)$ is the rank $(2p-1)(2g-2)$ vector bundle with $\pi^{-1}(M) = H^0(MK^p_{X_{sw_1}})$.

The Higgs bundles are given by

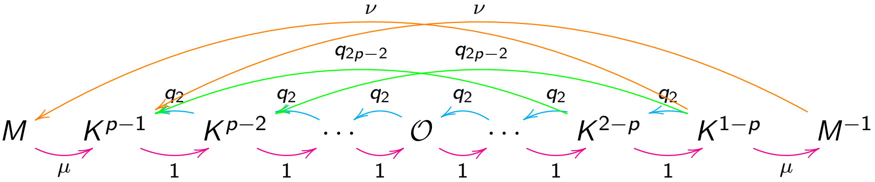


W_0 is a holomorphic $\mathrm{O}(2,\mathbb{C})$ bundle with $det(W_0) = I$.
Pulling back to the orientation double cover...



with $deg(M) = 0$.

Remarks on Positivity



- If $deg(M) = 0$ and $\mu = \nu = 0$, then the representation has Zariski closure in $\mathrm{SO}(p-1,p) \times \mathrm{SO}(2)$ and it is Anosov.
- Better, it is POSITIVE.
- Similarly for $\mathcal{M}^{2g-2,sw_2}_{sw_1}(\mathrm{SO}_0(p,p+1))$ if $\eta_0 = 0$ then the Zariski closure is in $\mathrm{S}(\mathrm{O}(p,p-1) \times \mathrm{O}(2))$ and is POSITIVE.
- If Guichard-Wienhard conjecture is true (positivity is a closed condition) then the whole components $\mathcal{M}^{2g-2}_0(\mathrm{SO}_0(p,p-1))$ and $\mathcal{M}^{2g-2,sw_2}_{sw_1}(\mathrm{SO}_0(p,p+1))$ correspond to positive representations.
- If $deg(M) \neq 0$ the components are smooth, so there are no preferred (Fuchsian/Hitchin) points.

Thank you

HAPPY BIRTHDAY NIGEL