Jacobi Multipliers and Hamel's Formalism

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1. Introduction

In this work we establish the relation between the Jacobi last multiplier [2, 3], that is a geometrical tool in the solution of problems in mechanics and that provides Lagrangians descriptions and constants of motion for systems of ODEs, and nonholonomic Lagrangian mechanics where the dynamics is determined by Hamel's equations [4].

2. Jacobi multipliers and quasi-velocities

Let (M, Ω) be an oriented n-dimensional manifold. The divergence of a vector field X on M is determined by $\mathcal{L}_X \Omega = \operatorname{div}(X)\Omega$. A non-vanishing function μ on M is called a Jacobi multiplier if $\mathcal{L}_{(\mu X)}\Omega = 0$, that is, $\operatorname{div}(\mu X) = 0$, and that is equivalent to $\mathcal{L}_X(\mu\Omega) = 0$.

Consider another volume form Ω' on M, then there exist a non-vanishing function η such that $\Omega = \eta\Omega'$. Hence, if μ is a Jacobi multipler for Ω then $\mu' = \mu/\eta$ is a Jacobi multiplier for Ω'. This particular relation will be crucial to deduce the integrating factor of a system with nonholonomic constraints.

We can choose a local basis of vector fields (X_j) on M and the dual basis (α^i). Any tangent vector $v \in T_xM$ can be expressed uniquely as $v = w^j X_j(x)$, where the fibre coordinates (w^i) are called quasi-velocities of v in the given basis and (x^i, w^i) are the quasi-coordinates of $v \in TM$. If $X_j = \beta_j^k(x)\partial_{x^k}$ is the coordinate expression of X_j then $dx^j = \beta_k^j(x)\alpha^k$ and it follows that $v^i = w^j \beta_j^i(x)$ (see e.g. [1]).

The equations of motion in quasi-coordinates of a conservative system, defined by a regular Lagrangian $L \in C^\infty(TM)$, are called Boltzmann-Hamel equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial w^l}\right) - \beta_l^k \frac{\partial L}{\partial x^k} - w^m \gamma_{ml}^k \frac{\partial L}{\partial w^k} = 0, \quad l = 1, \ldots, n, \tag{1}$$

where γ_{ml}^k are the Hamel symbols. Equations (1) determine the solution vector field $\Gamma_L = w^j \beta_j^i \frac{\partial}{\partial x^i} + \dot{w}^i \frac{\partial}{\partial w^i}$, with

$$\dot{w}^i = \mathcal{W}^{il}\left(\beta_l^k \frac{\partial L}{\partial x^k} + w^m \gamma_{ml}^k \frac{\partial L}{\partial w^k} - w^m \beta_m^k \frac{\partial^2 L}{\partial x^k \partial w^l}\right), \tag{2}$$

where $\mathcal{W}^{ik}$ are the elements of the inverse matrix $\mathcal{W} = \left(\frac{\partial^2 L}{\partial w^i \partial w^j}\right)$. Notice that, the solution Γ_L satisfies $\mathcal{L}_{\Gamma_L}(\omega_L^{\wedge n}) = 0$, where ω_L is the Cartan 2-form.

Let M **be the Euclidean space** $\mathbb{R}^n$. On TM we can consider the volume form

$$\Omega = dx^1 \wedge \ldots \wedge dx^n \wedge dw^1 \wedge \ldots \wedge dw^n. \tag{3}$$

Theorem 1 *Let α be the transformation matrix defined by the 1-forms α^i, i.e. $\alpha = (\alpha_j^i)$. If the conservative system is defined by a regular Lagrangian L, then the determinant of the product $\alpha\mathcal{W}$ is a Jacobi multiplier for (Γ_L, Ω).*

3. Jacobi multipliers and nonholonomic systems

Consider a regular nonholonomic system with a set of linear constraints, which defines a rank r vector subbundle $\mathcal{D}$ of $\tau : TM \to M$ called the constraint submanifold. In quasi-coordinates $(x^i, w^j) = (x^i, w^a, w^A)$ on TM the equations that defines the constraint manifold $\mathcal{D}$ are simply $w^A = 0$, with $A = n - r + 1, \ldots, n$, hence (x^i, w^a) are the coordinates for $\mathcal{D}$. The annihilator $\mathcal{D}^\circ$ is generated by the set of 1-forms $\{\alpha^A \mid A = m + 1, \ldots, n\}$, with $m = n - r$.

The evolution of the nonholonomic system is determined by the Lagrange–d'Alembert principle, which states that the dynamics of the system is given by the integral curves of a vector field Γ tangent to $\mathcal{D}$ that satisfies the Lagrange–d'Alembert equation:

$$i_\Gamma \omega_L - dE_L = -\lambda_A \, \tau^* \alpha^A, \tag{4}$$

where $\lambda_A \in C^\infty(TM)$ are the Lagrangian multipliers of the system, determined by the tangency conditions $\mathcal{L}_\Gamma w^A = 0$, for all $A = m + 1, \ldots, n$.

Notice that,

$$\mathcal{L}_\Gamma \omega_L = -d(\lambda_A \tau^* \alpha^A). \tag{5}$$

So, in general, $\mathcal{L}_\Gamma(\omega_L^{\wedge n}) \neq 0$.

Theorem 2 *Let $I_1, I_2, \ldots, I_s$ be $s = 2n - r - 2$ independent constants of motion of the nonholonomic system, and let J be the determinant of the Jacobian matrix of the function $I = (I_1, \ldots, I_s)$ w.r.t. the non free coordinates. If μ is a Jacobi multiplier for (Γ, Ω), then, $\widehat{\mu} = \mu/J$ is the last Jacobi multiplier, that is, $\widehat{\mu}$ is an integrating factor for the reduced 2-dimensional system.*

Remark as a corollary that Theorem 2 can be applied even if the system has zero constraints, i.e. $r = 0$.

References

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