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DRINFEL'D DOUBLES FOR (2+1)-POINCARÉ GRAVITY

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Abstract

By starting from the complete classification of three-dimensional Lie bialgebra structures given in [1], all possible Drinfel'd double structures for the Poincaré algebra $\mathfrak{iso}(2,1)$ are constructed in a physically adapted basis. The corresponding classical r-matrices are given, together with the resulting pairing between basis generators. We explore the relations between the associated quantum Poincaré groups and (2+1) gravity with vanishing cosmological constant, where the former appear naturally as symmetries of the corresponding non-commutative spacetimes.

1 Quantum groups and Poisson-Lie groups in (2+1) gravity

Quantum group symmetries play an important role in the quantisation of gravity. They occur:

- In **Hamiltonian quantisation formalisms** such as combinatorial quantisation and in path integral approaches (state sum models, spin foams).
- In **phenomenological approaches** in three and higher dimensions, in **non-commutative geometry models** such as κ-Poincaré models and related 'doubly special relativity' theories.

For a given Lie algebra/group, there are **many possible quantum deformations**. In **(2+1)-gravity**, the classical counterpart of quantum groups **(Poisson-Lie groups)** arise naturally:

- Poisson-Lie (PL) structures on the isometry groups of (2+1) spaces with constant curvature play a relevant role as **phase spaces when (2+1) gravity is considered as a Chern-Simons (CS) gauge theory** [2, 3].
- These PL structures are given by certain **classical r-matrices** that **have to be 'compatible' with the CS formalism**, in the sense that the **symmetric component of** r has to be directly related with the Ad-invariant symmetric bilinear form in the CS action.

Therefore, it seems natural that the appropriate **quantum groups** in (2+1) gravity should be the **quantizations of these admissible r-matrices and PL symmetries**.

2 The (2+1) gravity ↔ Drinfel'd double relation

But such admissible r-matrices have to be identified. As it has been shown in [4]:

All the classical r-matrices coming from a Drinfel'd double structure of the (2+1) Lorentzian isometry groups -(A)dS and Poincaré- fulfill the Fock-Rosly condition and are compatible with the CS formalism.

In this contribution we present in detail the results for the Poincaré case:

- We give the **5 possible Drinfel'd double structures for the Poincaré** Lie algebra $\mathfrak{iso}(2,1)$.
- We obtain **5 candidates for quantum deformations of the Poincaré symmetries** that would be appropriate in a (2+1)-gravity setting.
- The new **Poincaré non-commutative spacetimes** coming from such quantum groups should be relevant in the corresponding (2+1) quantum gravity theory.

3 The Poincaré isometry group and its Lie algebra

Any solution of the 3d vacuum Einstein equations is of **constant curvature** which **cosmological constant** Λ and is locally isometric to one of the **six standard spacetimes**:

$\Lambda>0$	$\Lambda=0$	$\Lambda<0$
$\mathbf{dS}^{2+1}=SO(3,1)/SO(2,1)$ $\mathrm{Isom}(\mathbf{dS}^{2+1})=SO(3,1)$	$\mathbf{M}^{2+1}=ISO(2,1)/SO(2,1)$ $\mathrm{Isom}(\mathbf{M}^{2+1})=ISO(2,1)$	$\mathbf{AdS}^{2+1}=SO(2,2)/SO(2,1)$ $\mathrm{Isom}(\mathbf{AdS}^{2+1})=SO(2,2)$
$\mathbf{S}^3=SO(4)/SO(3)$ $\mathrm{Isom}(\mathbf{S}^3)=SO(4)$	$\mathbf{E}^3=ISO(3)/SO(3)$ $\mathrm{Isom}(\mathbf{E}^3)=ISO(3)$	$\mathbf{H}^3=SO(3,1)/SO(3)$ $\mathrm{Isom}(\mathbf{H}^3)=SO(3,1)$

- **Euclidean signature**: the **three-sphere** $\mathbf{S}^3$ ($\Lambda>0$), 3d **hyperbolic** space $\mathbf{H}^3$ ($\Lambda<0$) and 3d **Euclidean** space $\mathbf{E}^3$ ($\Lambda=0$).
- **Lorentzian signature**: the 3d **de Sitter** space $\mathbf{dS}^{2+1}$ ($\Lambda>0$) [4], **Anti-de Sitter** space $\mathbf{AdS}^{2+1}$ ($\Lambda<0$) [4] and **Minkowski** space $\mathbf{M}^{2+1}$ ($\Lambda=0$) [6].

The **Lie algebra** of the **Poincaré isometry group of the (2+1) Minkowski spacetime** can be written in a **kinematical basis** in terms of generators J (**rotation**), K_1,K_2 (**boosts**) and P_0,P_1,P_2 (**translations**), where the latter commute.
The explicit commutation relations are:

$$\begin{array}{lll} [J,K_1]=K_2, & [J,K_2]=-K_1, & [K_1,K_2]=-J,\\ [J,P_0]=0, & [J,P_1]=P_2, & [J,P_2]=-P_1,\\ [K_1,P_0]=P_1, & [K_1,P_1]=P_0, & [K_1,P_2]=0,\\ [K_2,P_0]=P_2, & [K_2,P_1]=0, & [K_2,P_2]=P_0,\\ [P_0,P_1]=0, & [P_0,P_2]=0, & [P_1,P_2]=0. \end{array} \tag{3.1}$$

We have **two quadratic Casimir elements** given by

$$C_1=P_0^2-P_1^2-P_2^2,$$
$$C_2=\tfrac{1}{2}\left(J\,P_0+P_0\,J+K_2\,P_1+P_1\,K_2-(K_1\,P_2+P_2\,K_1)\right).$$

and associated to the second one, **the following non-degenerate symmetric Ad-invariant bilinear form** which is the relevant one in (2+1)-gravity:

$$\langle J,P_0\rangle_t=1,\qquad \langle K_1,P_2\rangle_t=-1,\qquad \langle K_2,P_1\rangle_t=1$$

4 Classical r-matrices from Drinfel'd doubles

Definition [5]: **A $2d$-dimensional Lie algebra $\mathfrak{a}$ has the structure of a (classical) Drinfel'd double if there exists a basis $\{X_1,\dots,X_d,x^1,\dots,x^d\}$ of $\mathfrak{a}$ in which the Lie bracket takes the form**

$$[X_i,X_j]=c_{ij}^k X_k \qquad [x^i,x^j]=f_k^{ij}x^k \qquad [x^i,X_j]=c_{jk}^i x^k-f_j^{ik}X_k.$$

As a consequence:

- An **Ad-invariant symmetric bilinear form on $\mathfrak{a}$** is given by
$$\langle X_i,X_j\rangle=0 \qquad \langle x^i,x^j\rangle=0 \qquad \langle x^i,X_j\rangle=\delta_j^i \qquad \forall i,j.$$
- A **quadratic Casimir operator for $\mathfrak{a}$** is always given by
$$C=\tfrac{1}{2}\sum_i (x^i\,X_i+X_i\,x^i).$$
- **If $\mathfrak{a}$ is a DD Lie algebra, its corresponding Lie group can be always endowed with a PL structure** generated by the **canonical classical r-matrix**
$$r=\sum_i x^i\otimes X_i=\tfrac{1}{2}\sum_i x^i\wedge X_i+\tfrac{1}{2}\sum_i (x^i\otimes X_i+X_i\otimes x^i)$$
which is a (constant) solution of the Classical Yang-Baxter equation $[[r,r]]=0$.

Since the **symmetric component of** r coincides with the **tensorized form of the canonical quadratic Casimir element in $\mathfrak{a}$**, then the **Fock-Rosly condition is automatically fulfilled**.

Therefore, in (2+1)-Poincaré gravity, any DD structure on $so(2,1)\ltimes\mathbb{R}^3$ will provide an admissible r-matrix.

5 Admissible Poincaré r-matrices from $so(2,1)\ltimes\mathbb{R}^3$ DDs

The **complete classification of the six-dimensional DD Lie algebras is known** [1]. In particular, the **Poincaré Lie algebra $so(2,1)\ltimes\mathbb{R}^3$** admits only **five DD structures $(\mathfrak{g},\mathfrak{g}^*)$**:

1. $(so(2,1),\mathfrak{r}_3(1))\equiv$ Case 3 from [1]
2. $(\mathfrak{r}_3(1),\mathfrak{n}_3)\equiv$ Case 10 from [1]
3. $(\mathfrak{r}'_3(1),\mathfrak{n}_3)\equiv$ Case 13 from [1]
4. $(\mathfrak{r}'_3(1),\mathfrak{s}_3(0))\equiv$ Case 14′ from [1]
5. $(\mathfrak{r}_3(-1),\mathfrak{r}'_3(1))\equiv$ Case 14 from [1]

The corresponding **admissible r-matrices** coming from them are given below. In all cases we give the most general admissible r-matrix.

1. $r'_{\mathrm{A}}=\alpha_1\,(J\wedge K_1+K_2\wedge K_1)+\beta_1\,(P_0\wedge J+K_1\wedge P_2+P_1\wedge K_2)$
2. $r'_{\mathrm{A}}=\alpha_2\,(P_0\wedge K_2+P_2\wedge K_2)+\beta_2\left(J\wedge P_2+\tfrac{1}{2}\,(P_1\wedge K_2+J\wedge P_0+K_1\wedge P_2)\right)$
3. $r'_{\mathrm{A}}=\alpha_3\,(P_0\wedge K_2+P_2\wedge K_2)-\beta_3\left(J\wedge P_2+\tfrac{1}{2}\,(P_1\wedge K_2+J\wedge P_0+K_1\wedge P_2)\right)+$
$+\delta_3 P_1\wedge P_0+\gamma_3 2\,(P_2\wedge P_0)+\epsilon_3 2P_1\wedge P_2$
4. $r'_{\mathrm{A}}=\alpha_4\,(2P_1\wedge J+J\wedge P_0+K_1\wedge P_2+P_1\wedge K_2)$
5. $r'_{\mathrm{A}}=\alpha_5\left(P_1\wedge J+\tfrac{1}{2}\,(J\wedge P_0+K_1\wedge P_2+P_1\wedge K_2)\right)+\beta_5 P_0\wedge P_1$

It is worth noting that the second and fourth r-matrices are just particular cases of the third and fifth ones, respectively. This fact will be translated to the resulting noncommutative space-times below.

6 Five noncommutative (2+1) Poincaré spacetimes

These five admissible r-matrices give rise to five new quantum Poincaré groups, whose explicit construction is in progress [6]. We present here the **expressions for the PL structure corresponding to these five cases**.

The Poisson brackets that define the PL structure on $\mathcal{C}^\infty(ISO(2,1))$ associated to a given classical r-matrix $r=r^{ij}X_i\otimes X_j$ are given by the **Sklyanin bracket**

$$\{f,g\}=r^{ij}(X_i^L f\,X_j^L g-X_i^R f\,X_j^R g) \qquad f,g\in\mathcal{C}^\infty(ISO(2,1)).$$

Under a **suitable parametrization of the $ISO(2,1)$ group in terms of geodesic parallel coordinates**, and after computing the left and right invariant vector fields on it, the **Poisson-Lie brackets among the spacetime coordinates read** [6]

Case	$\{x_0,x_1\}$	$\{x_0,x_2\}$	$\{x_1,x_2\}$
1.	$-\alpha_1(x_0x_2+x_1x_2)+2\beta_1x_2$	$\alpha_1(x_0x_1+x_1^2)-2\beta_1x_1$	$\alpha_1(x_0x_1+x_0^2)-2\beta_1x_0$
2.	0	$\alpha_2\,(-x_0+x_2)$	$\beta_2\,(-x_0+x_2)$
3.	$0+f_1(\xi_1,\xi_2,\theta)$	$\alpha_3\,(-x_0+x_2)+f_2(\xi_1,\xi_2,\theta)$	$\beta_3\,(-x_0+x_2)+f_3(\xi_1,\xi_2,\theta)$
4.	0	0	$-2\alpha_4(x_0+x_1)$
5.	$0+f_4(\xi_1,\xi_2,\theta)$	$0+f_5(\xi_1,\xi_2,\theta)$	$-2\alpha_5(x_0+x_1)+f_6(\xi_1,\xi_2,\theta)$

Note that:

- The quantization of these algebras will give **new non-commutative Poincaré spacetimes**.
- As stated above, 2 and 4 are particular cases of 3 and 5, respectively. **The commutation relations for each case are recovered when the appropriate parameters vanish.**
- Cases 1, 2 and 4 present **commutation relations among spacetime coordinates where only these same coordinates appear**. This fact is in agreement with **the idea of a (physical) noncommutative spacetime**.

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References

[1] X. Gomez, Classification of three-dimensional Lie bialgebras, *J. Math. Phys.* 41 (2000) 4939.
[2] A.Y. Alekseev and A.Z. Malkin A Z, *Commun. Math. Phys.* **169**, 99 (1995)
V.V. Fock and A.A. Rosly, *Am. Math. Soc. Transl.*, **191** 67 (1999)
C. Meusburger and B.J. Schroers, *Nucl. Phys. B* **806**, 462 (2009)
[3] A. Achucarro and P.K. Townsend, *Phys. Lett. B* **180**, 89 (1986)
E. Witten, *Nucl. Phys. B* **311**, 46 (1988)
[4] A. Ballesteros, F.J. Herranz and C. Meusburger, *Class. Quantum Grav.* **30**, 155012 (2013)
[5] V.G. Drinfel'd, *Proc. Int. Congress of Math.*, *AMS*, 798 (1987)
[6] A. Ballesteros, I. Gutiérrez-Sagredo, F.J. Herranz, Drinfel'd doubles for (2+1) Poincaré gravity, in preparation (2016).