The anisotropic oscillator on curved spaces: A new exactly solvable model

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1 The classical factorization method

Consider the unit mass 2D anisotropic oscillator classical Hamiltonian on the Euclidean plane

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2),$$

where $(x, y) \in \mathbb{R}^2$ are Cartesian coordinates, (p_x, p_y) their conjugate momenta and the frequencies (ω_x, ω_y) are arbitrary real numbers. If we introduce

$$\omega_x = \gamma\omega_y, \qquad \omega_y = \omega, \qquad \gamma \in \mathbb{R}^+/\{0\}, \qquad \xi = \gamma x, \qquad p_\xi = p_x/\gamma, \qquad \xi \in \mathbb{R}, \tag{1.1}$$

the Hamiltonian H can be rewritten as

$$H = \frac{1}{2}\, p_y^2 + \frac{\omega^2}{2}\, y^2 + \gamma^2 \left(\frac{1}{2}\, p_\xi^2 + \frac{\omega^2}{2\gamma^2}\, \xi^2 \right). \tag{1.2}$$

The two 1D Hamiltonians H^ξ and H^y given by

$$H^\xi = \frac{1}{2}\, p_\xi^2 + \frac{\omega^2}{2\gamma^2}\xi^2, \qquad H^y = \frac{1}{2}\, p_y^2 + \frac{\omega^2}{2}\, y^2, \qquad H = H^y + \gamma^2 H^\xi, \tag{1.3}$$

are two integrals of the motion for H: $\{H, H^\xi\} = \{H, H^y\} = \{H^\xi, H^y\} = 0$.

The factorization approach is based on the definition of "ladder functions" $B^\pm$, such that $H^\xi = B^+B^-$, and "shift functions" $A^\pm$, satisfying $H^y = A^+A^-$. These are

$$B^\pm = \mp \frac{i}{\sqrt{2}}\, p_\xi + \frac{1}{\sqrt{2}}\frac{\omega}{\gamma}\,\xi, \qquad \{H^\xi, B^\pm\} = \mp i\frac{\omega}{\gamma}\, B^\pm, \qquad \{B^-, B^+\} = -i\frac{\omega}{\gamma}.$$

$$A^\pm = \mp \frac{i}{\sqrt{2}}\, p_y - \frac{\omega}{\sqrt{2}}\, y, \qquad \{H^y, A^\pm\} = \pm i\omega A^\pm, \qquad \{A^-, A^+\} = i\omega.$$

The remarkable point now is that if we consider a rational value for γ,

$$\gamma = \frac{\omega_x}{\omega_y} = \frac{m}{n}, \qquad m, n \in \mathbb{N}^*, \tag{1.4}$$

we obtain two complex additional constants of motion $X^\pm$ and from them two real ones X and Y:

$$X^\pm = (B^\pm)^n (A^\pm)^m, \qquad X = \frac{1}{2}(X^+ + X^-), \qquad Y = \frac{1}{2i}(X^+ - X^-). \tag{1.5}$$

Thus we recover the (super)integrable anisotropic Euclidean oscillators [1, 2, 3]:

Theorem 1. (i) *The Hamiltonian H (1.2) is integrable for any value of the real parameter γ, since it is endowed with a quadratic constant of motion given by either H^ξ or H^y (1.3).*
(ii) *When $\gamma = m/n$ is a rational parameter (1.4), the Hamiltonian (1.2) defines a superintegrable anisotropic oscillator with commensurate frequencies $\omega_x : \omega_y$ and the additional constant of motion is given by either X or Y in (1.5). The sets (H, H^ξ, X) and (H, H^ξ, Y) are formed by three functionally independent functions.*

- **The 1 : 1 oscillator.** We set $\gamma = 1$ so $m = n = 1$, $\omega_x = \omega_y = \omega$, $\xi = x$ and $p_\xi = p_x$:

$$H^{1:1} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\omega^2}{2}(x^2 + y^2),$$

$$X = -\tfrac{1}{2}(p_x p_y + \omega^2 xy), \qquad Y = -\tfrac{1}{2}\omega(xp_y - yp_x).$$

The quadratic integral X is one of the components of the Demkov–Fradkin tensor, meanwhile Y is proportional to the angular momentum $J = xp_y - yp_x$.

- **The 2 : 1 oscillator.** Take $\gamma = 2$, $m = 2$ and $n = 1$, $\omega_x = 2\omega_y = 2\omega$, $\xi = 2x$ and $p_\xi = p_x/2$:

$$H^{2:1} = \frac{1}{2}\, p_y^2 + \frac{\omega^2}{2}\, y^2 + 4\left(\frac{1}{2}\, p_\xi^2 + \frac{\omega^2}{8}\, \xi^2 \right) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\omega^2}{2}\left(4x^2 + y^2\right),$$

$$X = -\frac{\omega}{4\sqrt{2}}\left(p_y(\xi p_y - 4yp_\xi) - \omega^2 \xi y^2\right) = -\frac{\omega}{2\sqrt{2}}\left(p_y J - \omega^2 xy^2\right),$$

$$Y = \frac{1}{2\sqrt{2}}\left(p_\xi p_y^2 + \omega^2 y(\xi p_y - yp_\xi)\right) = \frac{1}{4\sqrt{2}}\left(p_x p_y^2 + \omega^2 y(4xp_y - yp_x)\right).$$

Remark. Any $m : n$ oscillator (with γ) is equivalent to the $n : m$ one (with $1/\gamma$). This (trivial) fact from the Euclidean viewpoint will no longer hold when the curvature of the space is non-vanishing.

2 The quantum factorization method

The 2D anisotropic oscillator quantum Hamiltonian on the Euclidean plane reads

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2) = -\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2).$$

By introducing the frequency ω and the new variable ξ (1.1) we find that

$$\hat{H} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\omega^2}{2}\, y^2 + \gamma^2\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial \xi^2} + \frac{\omega^2}{2\gamma^2}\,\xi^2 \right), \tag{2.1}$$

and the corresponding eigenvalue equation is given by $\hat{H}\,\Psi(\xi, y) = E\,\Psi(\xi, y)$. From (2.1) we get the 1D Hamiltonian operators

$$\hat{H}^\xi = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial \xi^2} + \frac{\omega^2}{2\gamma^2}\,\xi^2, \qquad \hat{H}^y = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\omega^2}{2}\, y^2, \qquad \hat{H} = \hat{H}^y + \gamma^2 \hat{H}^\xi, \tag{2.2}$$

such that $[\hat{H}, \hat{H}^\xi] = [\hat{H}, \hat{H}^y] = [\hat{H}^\xi, \hat{H}^y] = 0$. We look for factorized solutions, $\Psi(\xi, y) = \Xi(\xi)\, Y(y)$:

$$\hat{H}^\xi\, \Xi(\xi) = E^\xi\, \Xi(\xi), \qquad \hat{H}^y\, Y(y) = E^y\, Y(y).$$

The factorizations of these systems in terms of ladder operators $\hat{B}^\pm$ and shift operators $\hat{A}^\pm$ read

$$\hat{H}^\xi = \hat{B}^+\hat{B}^- + \lambda^B, \qquad \hat{H}^y = \hat{A}^+\hat{A}^- + \lambda^A,$$

$$\hat{B}^\pm = \mp\frac{\hbar}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{\omega}{\sqrt{2}\gamma}\,\xi, \qquad \lambda^B = \frac{\hbar\omega}{2\gamma},$$

$$\hat{A}^\pm = \mp\frac{\hbar}{\sqrt{2}}\frac{\partial}{\partial y} - \frac{\omega}{\sqrt{2}}\, y, \qquad \lambda^A = -\frac{\hbar\omega}{2}.$$

The commutation rules of the two sets of operators $(\hat{H}^\xi, \hat{B}^\pm)$ and $(\hat{H}^y, \hat{A}^\pm)$ read

$$[\hat{H}^\xi, \hat{B}^\pm] = \pm\frac{\hbar\omega}{\gamma}\hat{B}^\pm, \qquad [\hat{B}^-, \hat{B}^+] = \frac{\hbar\omega}{\gamma},$$

$$[\hat{H}^y, \hat{A}^\pm] = \mp\hbar\omega\hat{A}^\pm, \qquad [\hat{A}^-, \hat{A}^+] = -\hbar\omega.$$

If $\gamma = m/n$, with $m, n \in \mathbb{N}^*$, as in the classical case, we obtain "additional" higher-order symmetries for $\hat{H}$ (2.1), beyond $\hat{H}^\xi$ and $\hat{H}^y$, since the operators

$$\hat{X}^\pm = (\hat{B}^\pm)^n\, (\hat{A}^\pm)^m \qquad \text{are such that} \qquad [\hat{H}, \hat{X}^\pm] = 0\,. \tag{2.3}$$

Summarizing, the quantum counterpart of Theorem 1 can be stated as follows.

Theorem 2. (i) *The Hamiltonian $\hat{H}$ (2.1) commutes with the operators $\hat{H}^\xi$ and $\hat{H}^y$ (2.2) and defines an integrable quantum system for any value of the real parameter γ.*
(ii) *Whenever $\gamma = m/n$ is a rational parameter, the Hamiltonian $\hat{H}$ commutes with $\hat{X}^\pm$ (2.3). The sets $(\hat{H}, \hat{H}^\xi, \hat{X}^+)$ and $(\hat{H}, \hat{H}^\xi, \hat{X}^-)$ are formed by three algebraically independent observables. The quantum anisotropic oscillator with commensurate frequencies $\omega_x : \omega_y$ is a superintegrable quantum model.*

Remark. From this the corresponding spectrum can be found which is degenerate for a rational γ [4].

3 Classical anisotropic curved oscillators

The curved analog of the Euclidean Hamiltonian H on the sphere $\mathbf{S}^2$ ($\kappa > 0$) and on the hyperbolic plane $\mathbf{H}^2$ ($\kappa < 0$) is given in terms of the Gaussian curvature κ by [4]

$$H_\kappa = \mathcal{T}_\kappa + U_\kappa^\gamma = \frac{1}{2}\left(\frac{p_x^2}{\mathrm{C}_\kappa^2(y)} + p_y^2 \right) + \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(\gamma x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y) \right) \tag{3.1}$$

$$\mathrm{C}_\kappa(u) = \begin{cases} \cos\sqrt{\kappa}\, u & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{-\kappa}\, u & \kappa < 0 \end{cases}, \quad \mathrm{S}_\kappa(u) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\, u & \kappa > 0 \\ u & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\, u & \kappa < 0 \end{cases}, \quad \mathrm{T}_\kappa(u) \equiv \frac{\mathrm{S}_\kappa(u)}{\mathrm{C}_\kappa(u)}.$$

After introducing $\xi = \gamma x$ (1.1) we find that

$$H_\kappa = \frac{p_y^2}{2} + \frac{\gamma^2 H_\kappa^\xi}{\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa}, \qquad H_\kappa^\xi = \frac{p_\xi^2}{2} + \frac{\omega^2}{2\kappa\gamma^2\,\mathrm{C}_\kappa^2(\xi)}, \qquad \{H_\kappa, H_\kappa^\xi\} = 0. \tag{3.2}$$

Next ladder and shift functions for H_κ^ξ are found to be

$$H_\kappa^\xi = B_\kappa^+ B_\kappa^- + \frac{\omega^2}{2\kappa\gamma^2}, \qquad B_\kappa^\pm = \mp\frac{i}{\sqrt{2}}\,\mathrm{C}_\kappa(\xi)\, p_\xi + \frac{\mathcal{E}_\kappa}{\sqrt{2}}\,\mathrm{S}_\kappa(\xi), \qquad \mathcal{E}_\kappa(p_\xi, \xi) = \sqrt{2\kappa H_\kappa^\xi}\,.$$

$$H_\kappa = A_\kappa^+ A_\kappa^- + \lambda_\kappa^A, \qquad A_\kappa^\pm = \mp\frac{i}{\sqrt{2}}\, p_y - \frac{\gamma\mathcal{E}_\kappa}{\sqrt{2}}\,\mathrm{T}_\kappa(y), \qquad \lambda_\kappa^A = \frac{1}{2\kappa}\left(\gamma^2\mathcal{E}_\kappa^2 - \omega^2\right).$$

These functions provide two additional integrals of the motion [4]:

$$\{H_\kappa, X_\kappa^\pm\} = 0, \qquad \text{where} \qquad X_\kappa^\pm = (B_\kappa^\pm)^n (A_\kappa^\pm)^m\,.$$

Theorem 3. (i) *For any γ, the Hamiltonian H_κ (3.1) defines an integrable anisotropic curved oscillator on $\mathbf{S}^2$ and $\mathbf{H}^2$, whose (quadratic) constant of motion is given by H_κ^ξ (3.2).*
(ii) *When $\gamma = m/n$ is a rational parameter, H_κ defines a superintegrable anisotropic curved oscillator and the additional constant of motion is given by either X_κ^+ or X_κ^-. The sets $(H_\kappa, H_\kappa^\xi, X_\kappa^+)$ and $(H_\kappa, H_\kappa^\xi, X_\kappa^-)$ are formed by three functionally independent functions.*

4 Quantum anisotropic curved oscillators

We define the quantum curved anisotropic oscillator Hamiltonian by [4]

$$\hat{H}_\kappa = -\frac{\hbar^2}{2}\left(\frac{1}{\mathrm{C}_\kappa^2(y)}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y} \right) + \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(\gamma x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y) \right). \tag{4.1}$$

After the change of variable $\xi = \gamma x$ (1.1) we write $\hat{H}_\kappa$ in terms of a 1D symmetry operator $\hat{H}_\kappa^\xi$

$$\hat{H}_\kappa = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\hbar^2}{2}\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y} + \frac{\gamma^2\hat{H}_\kappa^\xi}{\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa}, \qquad \kappa \neq 0,$$

$$\hat{H}_\kappa^\xi = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial \xi^2} + \frac{\omega^2}{2\kappa\gamma^2\,\mathrm{C}_\kappa^2(\xi)}, \qquad [\hat{H}_\kappa, \hat{H}_\kappa^\xi] = 0. \tag{4.2}$$

The eigenvalue equation for $\hat{H}_\kappa$ is $\hat{H}_\kappa\Psi_\kappa(\xi, y) = E_\kappa\Psi_\kappa(\xi, y)$ with factorizable solutions of the form $\Psi_\kappa(\xi, y) = \Xi_\kappa^\epsilon(\xi)\, Y_\kappa^{\gamma\epsilon}(y)$. Now ladder operators $\hat{B}_\kappa^\pm$ for $\hat{H}_\kappa^\xi$ turn out to be

$$\hat{B}_\kappa^- = \frac{\hbar}{\sqrt{2}}\,\mathrm{C}_\kappa(\xi)\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\,\mathrm{S}_\kappa(\xi)\,\hat{\mathcal{E}}_\kappa,$$

$$\hat{B}_\kappa^+ = -\frac{\hbar}{\sqrt{2}}\,\mathrm{C}_\kappa(\xi)\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\,\mathrm{S}_\kappa(\xi)\,\hat{\mathcal{E}}_\kappa,$$

$$\hat{\lambda}_\kappa^B = -\frac{\hat{\mathcal{E}}_\kappa}{2\kappa}(\hat{\mathcal{E}}_\kappa + \hbar\kappa), \qquad \hat{\mathcal{E}}_\kappa\,\Xi_\kappa^\epsilon(\xi) = \epsilon\,\Xi_\kappa^\epsilon(\xi)\,.$$

And the shift operators $\hat{H}_\kappa = \hat{A}_\kappa^+\hat{A}_\kappa^- + \hat{\lambda}_\kappa^A$ are given by

$$\hat{A}_\kappa^+ = -\frac{\hbar}{\sqrt{2}}\frac{\partial}{\partial y} - \frac{1}{\sqrt{2}}(\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa)\,\mathrm{T}_\kappa(y),$$

$$\hat{A}_\kappa^- = \frac{\hbar}{\sqrt{2}}\frac{\partial}{\partial y} - \frac{\gamma\hat{\mathcal{E}}_\kappa}{\sqrt{2}}\,\mathrm{T}_\kappa(y), \qquad \hat{\lambda}_\kappa^A = \frac{\gamma\hat{\mathcal{E}}_\kappa}{2\kappa}(\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa) - \frac{\omega^2}{2\kappa}.$$

Then the "additional" symmetry operators $\hat{X}_\kappa^\pm$ for the quantum Hamiltonian $\hat{H}_\kappa$ in the rational $\gamma = m/n$ case are defined as

$$\hat{X}_\kappa^\pm = (\hat{A}_\kappa^\pm)^m (\hat{B}_\kappa^\pm)^n, \qquad m, n \in \mathbb{N}^*. \tag{4.3}$$

Theorem 4. (i) *The quantum Hamiltonian $\hat{H}_\kappa$ (4.1) defines an integrable quantum system for any value of the parameter γ, since it commutes with the operator $\hat{H}_\kappa^\xi$ (4.2).*
(ii) *When γ is a rational parameter, $\hat{H}_\kappa$ defines a superintegrable anisotropic quantum curved oscillator with additional symmetry operators given by (4.3). The sets $(\hat{H}_\kappa, \hat{H}_\kappa^\xi, \hat{X}_\kappa^+)$ and $(\hat{H}_\kappa, \hat{H}_\kappa^\xi, \hat{X}_\kappa^-)$ are formed by three algebraically independent operators.*

Remark. The corresponding spectrum of $\hat{H}_\kappa$ on the sphere and on the hyperbolic case has been analytically solved in [4]. For a rational γ the spectrum is degenerate providing a new exactly solvable model. Furthermore, the spectrum of the quantum anisotropic oscillator on $\mathbf{S}^2$ is purely discrete (and has infinite values), whilst a (finite) discrete spectrum plus a continuous one arises for the system on $\mathbf{H}^2$.

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