Calabi-Bernstein results for complete maximal hypersurfaces in Robertson-Walker spacetimes with flat fiber

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The results shown in this poster have been obtained by José A. S. Pelegrín, Alfonso Romero and Rafael M. Rubio [7]

Abstract

In this work, under natural geometric and physical assumptions we provide new uniqueness and non-existence results for complete maximal hypersurfaces in spatially open Robertson-Walker spacetimes whose fiber is flat. Moreover, we apply our results to relevant spacetimes as the steady state spacetime, Einstein-de Sitter spacetime and radiation models.

1. Introduction

The importance in General Relativity of maximal and constant mean curvature spacelike hypersurfaces in spacetimes is well-known; a summary of several reasons justifying it can be found in [4]. Each maximal hypersurface can describe, in some relevant cases, the transition between the expanding and contracting phases of a relativistic universe. Moreover, the existence of constant mean curvature (and in particular maximal) hypersurfaces is necessary for the study of the structure of singularities in the space of solutions to the Einstein equations. Also, the deep understanding of this kind of hypersurfaces is essential to prove the positivity of the gravitational mass.

Among the uniqueness results, the so-called Calabi-Bernstein theorem stands out. It asserts that the only entire solutions to the maximal hypersurface equation in the Lorentz-Minkowski spacetime, $\mathbb{L}^{n+1}$, are the affine functions defining spacelike hyperplanes [2].

In this work we focus on the problems of uniqueness and non-existence of complete maximal hypersurfaces immersed in a spatially open Robertson-Walker spacetime with flat fiber. Note that these models have aroused a great deal of interest, since recent observations have shown that the current universe is very close to a spatially flat geometry [3].

2. Set up

Let (F, g_F) be an $n(\geq 2)$-dimensional (connected) Riemannian manifold, I an open interval in $\mathbb{R}$ endowed with the metric $-dt^2$ and f a positive smooth function defined on I. Then, the product manifold $I \times F$ endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g_F)\,, \tag{1}$$

where π_I and π_F denote the projections onto I and F, respectively, is called a *GRW spacetime* [1] with *fiber* (F, g_F), *base* $(I, -dt^2)$ and *warping function* f. If the fiber has constant sectional curvature, it is called a *Robertson-Walker spacetime*.

Given an n-dimensional manifold M, an immersion $\psi : M \to \overline{M}$ is said to be *spacelike* if the Lorentzian metric (1) induces, via ψ, a Riemannian metric g_M on M. In this case, M is called a spacelike hypersurface. We will denote by $\tau := \pi_I \circ \psi$ the restriction of π_I along ψ.

The time-orientation of $\overline{M}$ allows to take, for each spacelike hypersurface M in $\overline{M}$, a unique unitary timelike vector field $N \in \mathfrak{X}^\perp(M)$ globally defined on M with the same time-orientation as $\partial_t := \partial/\partial t$, i.e., such that $\overline{g}(N, \partial_t) \leq -1$ and $\overline{g}(N, \partial_t) = -1$ at a point $p \in M$ if and only if $N = \partial_t$ at p. For a spacelike hypersurface $\psi : M \to \overline{M}$ with Gauss map N, the *hyperbolic angle* φ, at any point of M, between the unit timelike vectors N and ∂_t, is given by $\cosh\varphi = -\overline{g}(N, \partial_t)$.

We will denote by A the shape operator associated to N. Then, the *mean curvature function* associated to N is given by $H := -(1/n)\mathrm{trace}(A)$. As it is well-known, the mean curvature is constant if and only if the spacelike hypersurface is, locally, a critical point of the n-dimensional area functional for compactly supported normal variations, under certain constraints of the volume. When the mean curvature vanishes identically, the spacelike hypersurface is called a *maximal* hypersurface.

In any GRW spacetime $\overline{M} = I \times_f F$ there is a remarkable family of spacelike hypersurfaces, namely its spacelike slices $\{t_0\} \times F$, $t_0 \in I$. It can be easily seen that a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the function τ is constant. Furthermore, a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the hyperbolic angle φ vanishes identically. The shape operator of the spacelike slice $\tau = t_0$ is given by $A = -f'(t_0)/f(t_0)\mathbb{I}$, where $\mathbb{I}$ denotes the identity transformation, and therefore its (constant) mean curvature is $H = f'(t_0)/f(t_0)$. Thus, a spacelike slice is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic).

2.1. Physical meaning of our mathematical assumptions

In any GRW spacetime $\overline{M}$ the integral curves of the vector field ∂_t are known as the comoving observers. In particular, for any point $p \in \overline{M}$, the integral curves of $\partial_t(p)$ are called the instantaneous comoving observers. We can easily see that the divergence on $\overline{M}$ of the reference frame ∂_t satisfies $\mathrm{div}(\partial_t) = n\frac{f'(t)}{f(t)}$. Therefore, the observers in ∂_t are spreading out if $f' > 0$ (resp. if $f' < 0$ they are coming together).

We also need to clarify the meaning of some energy conditions. A Lorentzian spacetime obeys the *Timelike Convergence Condition* (TCC) if its Ricci tensor satisfies

$$\overline{\mathrm{Ric}}(X, X) \geq 0,$$

for all timelike vectors X. It is usually argued that TCC is the mathematical way to express that gravity, on average, attracts (see [6]). Furthermore, if the spacetime satisfies the Einstein equation with a physically reasonable stress-energy tensor, then it must obey TCC. A weaker energy condition is the *Null Convergence Condition* (NCC), which reads

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for all lightlike vector Z, i.e. $Z \neq 0$ satisfying $\overline{g}(Z, Z) = 0$. A continuity argument shows that TCC implies NCC.

3. Non-existence results

Our first result will be based on the following lemma for maximal hypersurfaces in Robertson-Walker spacetimes with flat fiber.

Lemma 1 *Let $\psi : M \to \overline{M}$ be an n-dimensional maximal hypersurface immersed in a Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the Null Convergence Condition. Then, the Ricci curvature of M must be non-negative. In particular, it satisfies*

$$\mathrm{Ric}(Y, Y) \geq (n-1)\frac{f'(\tau)^2}{f(\tau)^2}|Y|^2 - (n-2)(\log f)''(\tau)\, g(Y, \nabla\tau)^2 - (\log f)''(\tau)|\nabla\tau|^2|Y|^2. \tag{2}$$

This lemma enables us to obtain our first non-existence result as well as some corollaries for relevant spacetimes.

Proposition 2 *There is no complete maximal hypersurface M in a spatially open Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the Null convergence Condition such that the restriction of the expanding/contracting function $\mathrm{div}(\partial_t)$ to M satisfies $\inf_M |\mathrm{div}(\partial_t)| > 0$.*

Corollary 3 *There are no complete maximal hypersurfaces in the $(n+1)$-dimensional steady state spacetime $\mathbb{R} \times_{e^t} \mathbb{R}^n$.*

Corollary 4 *There are no complete maximal hypersurfaces bounded away from future infinity in the $(n+1)$-dimensional Einstein-de Sitter spacetime $\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$.*

Corollary 5 *There are no complete maximal hypersurfaces bounded away from future infinity in the $(n+1)$-dimensional Roberson-Walker Radiation Model spacetime $\mathbb{R}^+ \times_{(2at)^{1/2}} \mathbb{R}^n$, with $a > 0$.*

4. Uniqueness Results

The following lemma will be the key to obtain our main results in these ambiences.

Lemma 6 *Let $\psi : M \to \overline{M}$ be an n-dimensional maximal hypersurface immersed in a GRW spacetime $\overline{M} = I \times_f F$ with Ricci-flat fiber that obeys the Null Convergence Condition, then*

$$\frac{1}{2}\Delta \sinh^2\varphi \geq \left((n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)}\right)\sinh^4\varphi. \tag{3}$$

Lemma 6 and Lemma 1 allow us to use the Liouville type result given by Nishikawa in [5] to prove the following theorem.

Theorem 7 *Let $\overline{M} = I \times_f F$ be a Robertson-Walker spacetime with flat fiber that obeys the Null Convergence Condition. Then, the only complete maximal hypersurfaces immersed in $\overline{M}$ satisfying $\inf\left\{(n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)}\right\} > 0$ are the spacelike slices $\{t_0\} \times F$ with $f'(t_0) = 0$.*

Observe that the assumption on the function $(n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)}$ defined on the hypersurface is scarcely restrictive, even if combined with the NCC. In fact, if we consider its extension $(n+1)\frac{f'(t)^2}{f(t)^2} - n\frac{f''(t)}{f(t)}$ defined on the spacetime, we have from the NCC that $(n+1)\frac{f'(t)^2}{f(t)^2} - n\frac{f''(t)}{f(t)} = \frac{f'(t)^2}{f(t)^2} - n(\log f)''(t) \geq 0$.

We will give now two models where Theorem 7 holds.

Example 1 Consider the Robertson-Walker spacetime $\overline{M} = \mathbb{R} \times_f \mathbb{R}^n$ with warping function $f(t) = e^{-t^2}$. This spacetime obeys NCC and the inequality on the infimum holds for any maximal hypersurface immersed in $\overline{M}$. Therefore, the only complete maximal hypersurface in $\overline{M}$ is the spacelike slice $\{0\} \times \mathbb{R}^n$.

This spacetime models a relativistic universe without singularities (in the sense of [6, Def. 12.16]) that goes from an expanding phase to a contracting one. The physical space in this transition of phase is represented by the spacelike slice $\{0\} \times \mathbb{R}^n$.

Example 2 We obtain another example of a Robertson-Walker spacetime satisfying the assumptions in Theorem 7 by considering $\overline{M} = I \times_f \mathbb{R}^n$. Where $I =]-a, a[$ and the warping function is $f(t) = \sqrt{a^2 - t^2}$, being a a positive constant. This spacetime has a big bang singularity at $t = -a$ as well as a big crunch at $t = a$ [6, Def. 12.16].

This spacetime satisfies NCC as well as the required inequality in Theorem 7. Hence, the only complete maximal hypersurface in this spacetime is the spacelike slice $\{0\} \times \mathbb{R}^n$, which represents the physical space in the transition from an expanding phase of the spacetime to a contracting one.

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