Dimensional analysis of relativistic field equations: applications to multi-scalar-tensor and bi-metric field theories

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1. Introduction

Recently, there is renewed interest in elucidating all possible field equations of modified theories of gravity (v. gr. [1], [2], [4]).

In this poster, we scketch how to apply the ideas of [3] in order to refine the current knowledge of these field equations. These ideas esentially consist on taking care of the dimensional analysis of the fields under consideration to simplify the computations of the mathematical analysis of the problem.

To begin with, we explain in the second Section the notion of homogeneous natural tensor, that corresponds to that of "tensorial quantity with units" in the physical language. We prove a result (Theorem 1) that describes the –usually finite dimensional– vector space of homogeneous natural tensors with a fixed weight. This theorem is the main result that will be applied to several examples later on.

Finally, in the third Section, we pay special attention to the scalar-tensor theories and the theory of electromagnetic p-branes.

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2. Homogeneous natural tensors

Let X be a smooth manifold –all the considerations that follow are local, so that we may assume without loss of generality that $X = \mathbb{R}^n$.

Let

$$\textit{Metrics}\ ,\ k\text{-}\textit{Tensors}\ ,\ \textit{Orient}$$

denote the sheaves of pseudo-Riemannian metrics, k-covariant tensors and orientations, respectively.

Definition. *A* **natural tensor***, associated to a triple (g, ϕ, or) of a pseudo-Riemannian metric, a k-tensor and an orientation on X, is a regular and natural morphism of sheaves:*

$$T\colon \textit{Metrics} \times k\text{-}\textit{Tensors} \times \textit{Orient} \longrightarrow p\text{-}\textit{Tensors}\ .$$

A natural tensor is **homogeneous** *of weight $(a; w)$, with $a \in \mathbb{Z}$, $w \in \mathbb{R}$, if for any metric g, any k-tensor ϕ, any orientation or and any real number $\lambda > 0$ it holds*

$$T(\lambda^2 g, \lambda^a \phi, or) \;=\; \lambda^w T(g, \phi, or)\ .$$

The main result that allows the explicit description of homogeneous tensors in certain situations is the following theorem:

Theorem 1. *There exists an $\mathbb{R}$-linear isomorphism between the vector space of natural tensors*

$$T\colon \textit{Metrics} \times k\text{-}\textit{Tensors} \times \textit{Orient} \longrightarrow p\text{-}\textit{Tensors}$$

homogeneous of weight $(a; w)$, with $a < k$, and

$$\bigoplus_{d_i, \bar{d}_j} \mathrm{Hom}_{SO}\left(S^{d_2}N_2 \otimes \cdots \otimes S^{d_r}N_r \otimes S^{\bar{d}_0}V_0 \otimes \ldots \otimes S^{\bar{d}_s}V_s\ ,\ \otimes^p T^*_x X\ \right)$$

where $V_r := \otimes^k T^*_x X \otimes S^r T^*_x X$, $N_r \subset S^2 T^*_x X \otimes S^r T^*_x X$ and the summation is over all sequences $\{d_2, \ldots, d_r\}$ and $\{\bar{d}_0, \ldots, \bar{d}_s\}$, of non-negative integers satisfying:

$$2d_2 + \ldots + r\, d_r + (k-a)\bar{d}_0 + \ldots + (k+s-a)\bar{d}_s = p - w\ . \qquad (1)$$

If this equation has no solutions, the above vector space reduces to zero.

3. Homogeneous lagrangians on relativistic theories

General Relativity

The fundamental field is the time metric g, and we look for natural morphisms:

$$\mathcal{L} : \textit{Metrics} \times \times \textit{Orient} \;\longrightarrow\; \mathcal{C}^\infty_X$$

that are homogeneous of weight $w = -2$.

Corollary. *If $\mathcal{L}(g, or)$ is a natural lagrangian homogeneous of weight -2, then it is a multiple of the scalar curvature r_g; i.e., there exists $\lambda \in \mathbb{R}$ such that, for any metric g and any orientation or,*

$$\mathcal{L}(g, or) \;=\; \lambda\, r_g\ .$$

Electromagnetism of p-branes

In this case, the electromagnetic field is a closed $(p+2)$-form F that, under a change of λ^{-1} in the fundamental unit, changes as $F \rightsquigarrow \lambda^{p+1}F$. Therefore, if $\textit{ClosedForms}_{p+2}$ denotes the sheaf of closed $(p+2)$-forms on X, now we look for natural morphisms:

$$\mathcal{L} : \textit{Metrics} \times \textit{ClosedForms}_{p+2} \times \textit{Orient} \;\longrightarrow\; \mathcal{C}^\infty_X$$

that are homogeneous of weight $(a = p+1; w = -2)$.

Example. *The square of the modulus of the closed form,*

$$\|F\|^2_g = F_{i_1 \ldots i_{p+2}} F^{i_1 \ldots i_{p+2}}$$

is a natural lagrangian homogeneous of weight $(p+1, -2)$.

Corollary. *Any natural lagrangian $\mathcal{L}(g, F, or)$ homogeneous of weight $(p+1, -2)$ can be written as:*

$$\mathcal{L}(g, F, or) \;=\; \lambda\, r_g \;+\; \mu\, \|F\|^2_g \qquad \lambda, \mu \in \mathbb{R}.$$

Scalar-tensor theories

In this case, the extra field is a smooth function ϕ that, under a change in the fundamental unit, we assume that it remains unchaged $\phi \rightsquigarrow \phi$. Therefore, for natural morphisms:

$$\mathcal{L} : \textit{Metrics} \times \textit{Fuctions} \times \textit{Orient} \;\longrightarrow\; \mathcal{C}^\infty_X$$

that are homogeneous of weight $(a = 0; w = -2)$.

A trivial corollary of Theorem 1 is the following:

Corollary. *Any natural lagrangian $\mathcal{L}(g, \phi, or)$ homogeneous of weight $(0, -2)$ can be uniquely written as:*

$$\mathcal{L}(g, \phi, or) \;=\; f_1(\phi)\, r_g \;+\; f_2(\phi)\, \|\mathrm{grad}\, \phi\|^2_g \;+\; f_3(\phi)\, \mathrm{tr}_g\, (\mathrm{Hess}\, \phi)\ ,$$

for some smooth functions $f_1, f_2, f_3\colon \mathbb{R} \to \mathbb{R}$.

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