Lie Groupoids and Algebroids applied to the study of Uniformity and Homogeneity of material bodies

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Purposes

- To characterize uniformity and homogeneity of a material body which properties of a special Lie groupoid (*material Lie groupoid*).
- To use the functor from Lie groupoids to Lie algebroids to characterize these properties over its associated Lie algebroid.
- To use the *Lie algebroid of derivations* to give a geometric characterization of these properties.

Definitions

- Let M be an $n-$dimensional manifold. $\Pi^1(M,M) \rightrightarrows M$ is the Lie groupoid where $\Pi^1(M,M)$ is the set of all 1-jets of isomorphisms over M and,
  (i) $\alpha\left(j^1_{x,y}\phi\right) = x$
  (ii) $\beta\left(j^1_{x,y}\phi\right) = y$
  (iii) $j^1_{y,z}\psi \cdot j^1_{x,y}\phi = j^1_{x,z}(\psi \circ \phi)$.
  This groupoid is called the *1-jets groupoid on* M.
- The associated Lie algebroid $A\Pi^1(M,M)$ is called *the 1-jets Lie algebroid*.

Let (x^i) and (y^j) be local coordinate systems on M. We can induce local coordinates on $\Pi^1(M,M)$ as follows

- $x^i\left(j^1_{x,y}\psi\right) = x^i(x)$.
- $y^j\left(j^1_{x,y}\psi\right) = y^j(y)$.
- $y^j_i\left(j^1_{x,y}\psi\right) = \dfrac{\partial\left(y^j \circ \psi\right)}{\partial x^i_{|x}}$.

Using the functor which turns Lie groupoids into Lie algebroids, we can consider local coordinates on $A\Pi^1(M,M)$ as follows

$$A\Pi^1(U,U) : \left(\left(x^i, x^i, \delta^i_j\right), 0, v^i, v^i_j\right) \cong$$
$$\cong \left(x^i, v^i, v^i_j\right).$$

Theorem

Let M be an $n-$dimensional manifold and $\mathfrak{D}(TM)$ be the algebroid of derivations on M. We can consider a map $\mathcal{D} : \Gamma\left(A\Pi^1(M,M)\right) \to Der(TM)$ given by

$$\mathcal{D}(\Lambda) \triangleq D^{\Lambda} = -\frac{\partial}{\partial t_{|t=0}}\left(Exp_t\Lambda\right),$$

which define a Lie algebroid isomorphism $\mathcal{D} : A\Pi^1(M,M) \to \mathfrak{D}(TM)$ over the identity map on M.

$\Lambda \in \Gamma\left(A\Pi^1(M,M)\right)$ be a section of the 1-jets algebroid with local expression

$$\Lambda\left(x^i\right) = \left(x^i, \Lambda^j, \Lambda^j_i\right).$$

The matrix Λ^j_i is (locally) the associated matrix to D^{Λ}, i.e.,

$$D^{\Lambda}\left(\frac{\partial}{\partial x^i}\right) = -\sum_j \Lambda^j_i \frac{\partial}{\partial x^j},$$

and the base vector field of D^{Λ} is given locally by $\left(x^i, \Lambda^j\right)$.

Integrability

Let $\Pi^1_G(M,M) \rightrightarrows M$ be a transitive Lie subgroupoid of $\Pi^1(M,M) \rightrightarrows M$. $\Pi^1_G(M,M)$ is said to be *integrable* if for each $x, y \in M$ there exist two open sets $U, V \subseteq M$ with $x \in U$, $y \in V$ and two local charts, $\psi_U : U \to \overline{U}$ and $\psi_V : V \to \overline{V}$, which induce a diffeomorphism

$$\Psi_{U,V} : \Pi^1_G(U,V) \to \overline{U} \times \overline{V} \times G,$$

such that
$\Psi_{U,V} = \left(\psi_U \circ \alpha, \psi_V \circ \beta, \overline{\Psi}_{U,V}\right)$,
where

$$\overline{\Psi}_{U,V}\left(j^1_{x,y}\phi\right) =$$
$$= j^1_{0,0}\left(\tau_{-\psi_V(y)} \circ \psi_V \circ \phi \circ \psi_U^{-1} \circ \tau_{\psi_U(x)}\right).$$

Equivalence

$\Pi_G(M,M)$ is integrable if and only if we can cover M by local charts (ψ_U, U) such that induce (Local) Lie groupoid isomorphisms from $\Pi^1_G(U,U)$ to the trivial Lie groupoid on G.

Proposition

A subgroupoid $\Pi^1_G(M,M)$ of $\Pi^1(M,M)$ is integrable if and only if for all $x, y \in M$ there exist coordinate systems (x^i) and (y^j) over x and y respectively such that

(i) The local section of (α, β)

$$P\left(x^i, y^j\right) = \left(x^i, y^j, \delta^j_i\right),$$

takes values into $\Pi^1_G(M,M)$.

(ii) The local section of $\sharp$

$$AP\left(x^i, \frac{\partial}{\partial x^i}\right) = \left(x^i, \frac{\partial}{\partial x^i}, 0\right),$$

takes values into $A\Pi^1_G(M,M)$.

(iii) The local covariant derivative with Christoffel symbols equal to zero respect to (x^i), i.e.,

$$\nabla^{\Lambda}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0,$$

take values into $\mathcal{D}\left(A\Pi^1_G(M,M)\right)$.

Material groupoid and algebroid

Let $\mathcal{B}$ be a material body with ϕ_0 the reference configuration and $W : \Pi^1(\mathcal{B},\mathcal{B}) \to V$ the response functional. We consider $\Omega(\mathcal{B}) \subseteq \Pi^1(\mathcal{B},\mathcal{B})$ as the set of 1-jets $j^1_{x,y}\psi$ on $\mathcal{B}$ such that

$$W\left(j^1_{y,\kappa(y)}\kappa \cdot j^1_{x,y}\psi\right) = W\left(j^1_{y,\kappa(y)}\kappa\right),$$

for each (local) infinitesimal deformation $j^1_{y,\kappa(y)}\kappa$.
Our assumption is that $\Omega(\mathcal{B})$ is a Lie subgroupoid of $\Pi^1(\mathcal{B},\mathcal{B})$. This groupoid is called *material subgroupoid*. The associated Lie algebroid $A\Omega(\mathcal{B})$ is called *material algebroid*.

Characterization

Let $\mathcal{B}$ be a material body,

- $\mathcal{B}$ is uniform iff $\Omega(\mathcal{B})$ is transitive.
- $\mathcal{B}$ is homogeneous iff $\Omega(\mathcal{B})$ is an integrable subgroupoid of $\Pi^1(\mathcal{B},\mathcal{B})$.

Conclusions

$\mathcal{B}$ is locally homogeneous if and only if for all $x, y \in \mathcal{B}$ there exist coordinate systems (x^i) and (y^j) over x and y respectively such that

- The local section of (α, β)

$$P\left(x^i, y^j\right) = \left(x^i, y^j, \delta^j_i\right),$$

  takes values into $\Omega(\mathcal{B})$.
- The local section of $\sharp$

$$AP\left(x^i, \frac{\partial}{\partial x^i}\right) = \left(x^i, \frac{\partial}{\partial x^i}, 0\right),$$

  takes values into $A\Omega(\mathcal{B})$.
- The local covariant derivative with Christoffel symbols equal to zero respect to (x^i), i.e.,

$$\nabla^{\Lambda}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0,$$

  take values into $\mathcal{D}\left(A\Omega(\mathcal{B})\right)$.

Remark

Let $W : \Pi^1(\mathcal{B},\mathcal{B}) \to V$ be the response functional which defines $\Omega(\mathcal{B})$. Then, we can define a new map $W^{-1} : \Pi^1(\mathcal{B},\mathcal{B}) \to V$ given by

$$W^{-1} = W \circ i,$$

where i is the inversion map of the 1-jets groupoid on $\mathcal{B}$. In this way, a section Θ of $A\Pi^1(\mathcal{B},\mathcal{B})$ is a section of the material algebroid if and only if

$$TW^{-1}\left(X_{\Theta}\right) = 0.$$

Equivalently, we can characterize the Lie subalgebroid $\mathcal{D}\left(A\Pi^1_G(\mathcal{B},\mathcal{B})\right)$ of $\mathcal{D}(T\mathcal{B})$ in the obvious way. In fact, Let (x^i) be a local coordinate system on $\mathcal{B}$ and D be a derivation on $\mathcal{B}$ with base vector field X. We denote

- $X\left(x^i\right) = \left(x^i, \Lambda^j\right)$.
- $D\left(\dfrac{\partial}{\partial x^i}\right) = \sum_j \Lambda^j_i \dfrac{\partial}{\partial x^j}$.

Then, D is in $\mathcal{D}\left(A\Pi^1_G(\mathcal{B},\mathcal{B})\right)$ if and only if over any (x^i) local coordinate system on $\mathcal{B}$ it is satisfied that

$$dW^{-1}_{|\left(x^i, x^i, g^j_i\right)}\left(0, \Lambda^j, -\Lambda^j_l \cdot g^l_i\right) = 0,$$

for all material symmetry $g \in G(x)$ which is locally written as follows

$$g \cong \left(x^i, x^i, g^j_i\right).$$